Nonnegative Periodic Solutions for an Integral Equation Modeling Infectious Disease with Latency Periods

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Abstract

Based on the theory of Leray-Schauder degree and Schauder's fixed point theorem, we find nontrivial nonnegative periodic solutions for an integral equation that models infectious disease with latency periods.

Keywords: infectious disease model, multiple delays, integral equation, nonnegative periodic solution

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1 Introduction

In [1], an integral equation of the form

$$x(t) = \int_{t-\tau}^t f(s, x(s)) ds, \quad t \in R. \quad (1)$$

is introduced which models the spread of some infectious diseases. More specifically, in (1), $x(t)$ represents the proportion of infectives in a population at time t , f is a periodic function of t and $f(t, x(t))$ is the proportion of new infectives per unit time and $\tau > 0$ is the length of time an individual remains infectious.

Since there usually are different latency periods for different classes (e.g. age groups) of infected persons to be able to pass the disease to other persons, it is therefore of interest to consider nonlinear integral equations with delays:

$$x(t) = \int_{t-\tau}^t f(s, x(s), x(s - \sigma_1(s)), \dots, x(s - \sigma_n(s))) ds, \quad t \in R, \quad (2)$$

where τ is positive, $\sigma_1(t), \dots, \sigma_n(t)$ are real, continuous functions on R and ω -periodic, $f(t, x_0, x_1, \dots, x_n)$ is a real, nonnegative continuous function on R^{n+2} such that $f(t, 0, \dots, 0) = 0$ and

$$f(t + \omega, x_0, \dots, x_n) = f(t, x_0, \dots, x_n), \quad (t, x_0, \dots, x_n) \in R^{n+2}.$$

Here the number ω is assumed to be positive.

There are several techniques (including Krasnolselskii's fixed point theorem, Leggett and Williams fixed point theorem, etc. see e.g. [1,2,3,4] and others) for deriving nonnegative periodic solutions of integral equations such as (1) and

$$x(t) = \int_{t-\tau}^t k(t, s) f(s, x(s)) ds, \quad t \in R.$$

In this paper, based on the theory of Leray-Schauder degree and Schauder fixed point theorem, we are able to prove the existence of one or more nontrivial nonnegative ω -periodic solutions of (2). These results are new and appear to be relative simple as compared with those found elsewhere.

2 Main Results

Let X be the set of all continuous ω -periodic real functions defined on R endowed with the usual linear structure as well as the norm $\|x\| = \max_{0 \leq t \leq \omega} |x(t)|$. Then X is a Banach space. Set

$$(\Phi x)(t) = \int_{t-\tau}^t f(s, x(s), x(s - \sigma_1(s)), \dots, x(s - \sigma_n(s))) ds, \quad (3)$$

it is easy to see that $\Phi : X \rightarrow X$ is completely continuous.

THEOREM 1. Suppose there exist positive constants ρ_1, ρ_2 and α and nonnegative, continuous and ω -periodic function $b : R \rightarrow R$ such that $0 < \rho_1 < \alpha < \rho_2$,

(a₁) $f(t, x_0, x_1, \dots, x_n) < \rho_1/\tau$ for $t \in R$ and $|x_0|, |x_1|, \dots, |x_n| \leq \rho_1$;

(b₁) $f(t, x_0, x_1, \dots, x_n) < \rho_2/\tau$ for $t \in R$ and $|x_0|, |x_1|, \dots, |x_n| \leq \rho_2$; and

(c₁) $\min_{0 \leq t \leq \omega} \int_{t-\tau}^t b(s) ds > \alpha$ and $f(t, x_0, x_1, \dots, x_n) \geq b(t)$ for $0 \leq t \leq \omega$ and $x_0, \dots, x_n \geq \alpha$.

Then (2) has two nontrivial, nonnegative ω -periodic solutions $\bar{x}(t)$ and $\bar{\bar{x}}(t)$ such that $\min_{0 \leq t \leq \omega} \bar{x}(t) > \alpha$.

Proof. Let

$$\Omega_1 = \{x \in X \mid \|x\| < \rho_1\},$$

$$\Omega_2 = \{x \in X \mid \|x\| < \rho_2\},$$

and

$$\Omega_3 = \left\{ x \in X \mid \|x\| < \rho_2, \min_{0 \leq t \leq \omega} x(t) > \alpha \right\}.$$

It is easy to see that Ω_1, Ω_2 and Ω_3 are nonempty, bounded and open subsets of X . Since $\rho_1 < \alpha < \rho_2$, $\Omega_1 \subset \Omega_2$, $\Omega_3 \subset \Omega_2$ and $\Omega_1 \cap \Omega_3 = \emptyset$. Furthermore,

$$\bar{\Omega}_1 = \{x \in X \mid \|x\| \leq \rho_1\},$$

$$\bar{\Omega}_2 = \{x \in X \mid \|x\| \leq \rho_2\},$$

and

$$\bar{\Omega}_3 = \left\{ x \in X \mid \|x\| \leq \rho_2, \min_{0 \leq t \leq \omega} x(t) \geq \alpha \right\}.$$

By the condition (a_1) , for $x \in \bar{\Omega}_1$, we have

$$\|\Phi x\| < \max_{0 \leq t \leq \omega} \int_{t-\tau}^t \frac{1}{\tau} \rho_1 ds = \rho_1.$$

Hence

$$\Phi(\bar{\Omega}_1) \subset \Omega_1. \quad (4)$$

Similarly, by (b_1) , for $x \in \bar{\Omega}_2$, we have

$$\|\Phi x\| < \max_{0 \leq t \leq \omega} \int_{t-\tau}^t \frac{1}{\tau} \rho_2 ds = \rho_2 \quad (5)$$

and

$$\Phi(\bar{\Omega}_2) \subset \Omega_2. \quad (6)$$

For $x \in \overline{\Omega}_3$, since $\overline{\Omega}_3 \subset \overline{\Omega}_2$, (5) holds and since $\min_{0 \leq t \leq \omega} x(t) \geq a$, from (c_1) , we have

$$\begin{aligned} \min_{0 \leq t \leq \omega} (\Phi x)(t) &= \min_{0 \leq t \leq \omega} \int_{t-\tau}^t f(s, x(s), x(s - \sigma_1(s)), \dots, x(s - \sigma_n(s))) ds \\ &\geq \min_{0 \leq t \leq \omega} \int_{t-\tau}^t b(s) ds > \alpha. \end{aligned} \quad (7)$$

By (5) and (7), we see that

$$\Phi(\overline{\Omega}_3) \subset \Omega_3. \quad (8)$$

From (5), (6) and (8),

$$\deg(I - \Phi, \Omega_i, \theta) = 1, \quad i = 1, 2, 3, \quad (9)$$

where $I : X \rightarrow X$ is the identity mapping, θ is the zero-element in X and $\deg$ is the Leray-Schauder degree. Hence Φ has a fixed point $\bar{x}$ in Ω_3 . Clearly, $\bar{x} = \bar{x}(t)$ is a nonnegative ω -periodic solution of (2) and

$$\min_{0 \leq t \leq \omega} \bar{x}(t) > \alpha.$$

From (9), we have

$$\begin{aligned} &\deg(I - \Phi, \Omega_2 \setminus (\overline{\Omega}_1 \cap \overline{\Omega}_3), \theta) \\ &= \deg(I - \Phi, \Omega_2, \theta) - \deg(I - \Phi, \Omega_1, \theta) - \deg(I - \Phi, \Omega_3, \theta) \\ &= -1. \end{aligned}$$

It follows that Φ has a fixed point $\bar{\bar{x}}$ in $\Omega_2 \setminus (\overline{\Omega}_1 \cap \overline{\Omega}_3)$. It is easy to see that $\bar{\bar{x}} \neq \bar{x}$ and $\bar{\bar{x}} \neq \theta$. Furthermore since f is a nonnegative continuous function and

$$\bar{\bar{x}}(t) = \int_{t-\tau}^t f(s, \bar{\bar{x}}(s), \bar{\bar{x}}(s - \sigma_1(s)), \dots, \bar{\bar{x}}(s - \sigma_n(s))) ds \geq 0,$$

we know that $\bar{\bar{x}}$ is a nontrivial, nonnegative and ω -periodic solution of (2). The proof is complete.

THEOREM 2. Suppose there are nonnegative constants $\alpha_0, \dots, \alpha_n$, positive constants α, β and γ , and nonnegative, continuous and ω -periodic function $b : R \rightarrow R$ such that $\sum_{i=0}^n \alpha_i = 1$; $0 < \gamma < 1$;

(c_1) $\min_{0 \leq t \leq \omega} \int_{t-\tau}^t b(s) ds > \alpha$ and $f(t, x_0, x_1, \dots, x_n) \geq b(t)$ for $0 \leq t \leq \omega$ and $x_0, \dots, x_n \geq \alpha$;

(a_2) $\limsup_{x_0, \dots, x_n \rightarrow 0} \max_{0 \leq t \leq \omega} \frac{f(t, x_0, x_1, \dots, x_n)}{\prod_{i=0}^n |x_i|^{\alpha_i}} < \frac{1}{\tau}$;

(b₂) $f(t, x_0, \dots, -x_i, \dots, x_n) = f(t, x_0, \dots, x_i, \dots, x_n)$ for $x_0, \dots, x_n \in R$ and $f(t, x_0, \dots, x_n) \leq \beta (\prod_{i=0}^n |x_i|^{\alpha_i})^\gamma$ for $(t, x_0, \dots, x_n) \in R^{n+2}$. Then (2) has two nontrivial nonnegative ω -periodic solutions $\bar{x}(t)$ and $\bar{\bar{x}}(t)$ such that $\min_{0 \leq t \leq \omega} \bar{x}(t) > \alpha$.

Indeed, from (a₂), we know that there is a positive constant ρ_1 such that $\rho_1 < \alpha$ and

$$f(t, x_0, x_1, \dots, x_n) < \frac{1}{\tau} \rho_1 \quad (10)$$

for $t \in R$ and $|x_0|, \dots, |x_n| \leq \rho_1$. Choose ρ_2 sufficiently large so that $\rho_2 > \alpha$ and $\frac{1}{\tau} \rho_2^{1-\gamma} > \beta$. In view of condition (b₂),

$$f(t, x_0, x_1, \dots, x_n) < \frac{1}{\tau} \rho_2 \quad (11)$$

for $t \in R$ and $|x_0|, \dots, |x_n| \leq \rho_2$. We now see that all conditions of Theorem 1 are satisfied. Therefore the same conclusion in Theorem 1 is true.

Remark 1. One implication of Theorems 1 and 2 is that under favorable conditions, two or more recurring patterns of the spreading of the infectious disease may be possible.

Example 1. Consider the following equation

$$\begin{aligned} x(t) = & \int_{t-\tau}^t p(s) \left(|x(s)|^{\alpha_0} \prod_{i=1}^n |x(s - \sigma_i(s))|^{\alpha_i} \right)^{1/2} \\ & \times \ln \left(1 + |x(s)|^{\alpha_0} \prod_{i=1}^n |x(s - \sigma_i(s))|^{\alpha_i} \right) ds, \end{aligned} \quad (12)$$

where $\alpha_0, \dots, \alpha_n$ are nonnegative constants such that $\sum_{i=0}^n \alpha_i = 1$. The functions $\sigma_1(t), \dots, \sigma_n(t)$ and $p(t)$ are real, continuous on R and ω -periodic. Let

$$f(t, x_0, x_1, \dots, x_n) = p(t) \left(\prod_{i=0}^n |x_i|^{\alpha_i} \right)^{\frac{1}{2}} \ln \left(1 + \prod_{i=0}^n |x_i|^{\alpha_i} \right).$$

If we set $p(t) \geq 0$ and $\min_{0 \leq t \leq \omega} \int_{t-\tau}^t p(s) ds = \frac{\varepsilon}{2}$, $\alpha = e^2 - 1$ and $b(t) = (e^2 - 1)p(t)$, then the condition (c₁) of Theorem 1 holds. Since

$$\limsup_{x_0, \dots, x_n \rightarrow 0} \max_{0 \leq t \leq \omega} \frac{f(t, x_0, x_1, \dots, x_n)}{\prod_{i=0}^n |x_i|^{\alpha_i}} = 0,$$

we know that the condition (a_2) of Theorem 2 holds. Finally, if we choose $\beta = 4$ and $\gamma = 1/4$, it is then easy to see that the condition (b_2) of Theorem 2 is true. By Theorem 2, (12) has two nontrivial nonnegative ω -periodic solutions.

THEOREM 3. Suppose there are positive constants ρ_2 and α such that $\rho_2 > \alpha$,

(b_1) $f(t, x_0, x_1, \dots, x_n) < \rho_2/\tau$ for $t \in R$ and $|x_0|, |x_1|, \dots, |x_n| \leq \rho_2$; and

(c_2) $b(t)$ is a nonnegative, continuous and ω -periodic function on R such that $\min_{0 \leq t \leq \omega} \int_{t-\tau}^t b(s) ds > \alpha$ and $f(t, x_0, x_1, \dots, x_n) \geq b(t)$ for $0 \leq t \leq \omega$ and $\alpha \leq x_0, \dots, x_n \leq \rho_2$.

Then (2) has a positive ω -periodic solutions $\bar{x}(t)$ and

$$\alpha \leq \min_{0 \leq t \leq \omega} x(t) \leq \max_{0 \leq t \leq \omega} x(t) \leq \rho_2.$$

Proof. Let Ω_3 be defined as in the proof of Theorem 1. Then

$$\bar{\Omega}_3 = \{x \in X \mid \alpha \leq x(t) \leq \rho_2\}$$

is a nonempty, bounded and closed subset in X . From conditions (b_1) and (c_2) , we know that for $x \in \bar{\Omega}_3$,

$$\|\Phi x\| < \max_{0 \leq t \leq \omega} \int_{t-\tau}^t \frac{1}{\tau} \rho_2 ds = \rho_2,$$

and

$$\min_{0 \leq t \leq \omega} (\Phi x) \geq \min_{0 \leq t \leq \omega} \int_{t-\tau}^t b(s) ds > \alpha.$$

Thus $\Phi x \in \bar{\Omega}_3$. Since it is easy to see that $\Phi : X \rightarrow X$ is completely continuous, by Schauder's fixed point theorem, we know that Φ has a fixed point in $\bar{\Omega}_3$. The proof is complete.

Remark 2. One possible implication of Theorem 3 is that the spread of some infectious diseases may have a cyclical behavior which changes as the environmental factors vary.

Example 2. Consider the following equation

$$x(t) = \int_{t-\tau}^t \left(b_0(t) |x(s)|^{\alpha_0} + \sum_{i=1}^n b_i(t) |x(s - \sigma_i(s))|^{\alpha_i} \right) ds, \quad (13)$$

where $0 < \alpha_0, \dots, \alpha_n < 1$, $b_0(t), \dots, b_n(t)$ are nonnegative, continuous and ω -periodic functions on R such that $\min_{0 \leq t \leq \omega} \int_{t-\tau}^t \sum_{i=0}^n b_i(s) ds \geq 1$, $\sigma_1(t), \dots, \sigma_n(t)$ are real, continuous and ω -periodic functions on R . Let

$$f(t, x_0, x_1, \dots, x_n) = \sum_{i=0}^n b_i(t) |x_i|^{\alpha_i}$$

and choose $\alpha = 1$, $\rho_2 > \alpha$ such that $\max_{0 \leq t \leq \omega} \sum_{i=0}^n b_i(t) < \frac{1}{\tau} \rho_2^{1-\delta}$ where $\delta = \max_{0 \leq i \leq n} \{\alpha_i\}$. It is then easy to check that all conditions of Theorem 3 are satisfied. By Theorem 3, we know that (13) has a positive ω -periodic solution.

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