A universal property of odd degree real Fermat curves

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Abstract. Fix an odd integer $d \geq 1$ and let $F_d \subset \mathbb{R}^2$ denote the real Fermat curve defined by the equation $x^d + y^d = 1$. Here we prove that F_d has the following property: Let $X \subseteq \mathbb{R}^n$ be any real algebraic variety. Then there exists a “one-parameter isotopic Nash modification” $\tilde{X}_t$ of X such that $\tilde{X}_0 = X$ and each real algebraic variety $\tilde{X}_t$, $t \neq 0$ may be biregularly embedded into $(F_d)^n$.

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For the standard definitions of real algebraic varieties, regular maps between two real algebraic varieties and Nash maps, see [1]. For all real algebraic varieties X, Y , let $\mathcal{N}(X, Y)$ denote the set of all Nash maps $f : X \rightarrow Y$. Here we just recall the following definitions heavily used in [2] and [3] (see [3], Def. 1.6 and 1.7).

Definition 1. Let X and $\tilde{X}$ be real algebraic varieties and $\phi : \tilde{X} \rightarrow X$ a regular map. We will say that ϕ is a weak change of the algebraic structure of X if it is bijective and ϕ^{-1} is a Nash map.

Definition 2. Let X, Z, X^* be real algebraic varieties, $z_0 \in Z$ and $\pi : X^* \rightarrow Z$ a regular map. A map $\beta : X^* \rightarrow X$ is called a weak deformation of X parametrized by (π, z_0) if it is regular, $\pi^{-1}(z_0)$ is biregularly isomorphic to X , while, for each $z \in Z \setminus \{z_0\}$, the map $\beta|_{\pi^{-1}(z)} : \pi^{-1}(z) \rightarrow X$ is a weak change of the algebraic structure of X .

Hence a regular map $\phi : \tilde{X} \rightarrow X$ between real algebraic varieties is a weak change of the algebraic structure of X if and only if it is a Nash isomorphism. In particular, it must be a homeomorphism for the euclidean topology. Notice that, if ϕ is a weak change of the algebraic structure of X , then ϕ^{-1} maps nonsingular points to nonsingular points. Hence if X is nonsingular, then $\tilde{X}$ is nonsingular also.

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Definition 3. Let M be an affine Nash manifold. We say that a Nash map $F : M \times \mathbb{R} \rightarrow M$ is a Nash diffeotopy of M if $F|_{M \times \{t\}}$ is a Nash isomorphism for all $t \in \mathbb{R}$ and $F|_{M \times \{0\}} : M \times \{0\} \rightarrow M$ is just the isomorphic projection. Let $\mathcal{U}$ be a neighborhood of the identity map Id_M in $\mathcal{N}(M, M)$, equipped with the C^∞ compact-open topology. We say that F is in $\mathcal{U}$ if $F|_{M \times \{t\}} \in \mathcal{U}$ for all $t \in \mathbb{R}$.

Following the proof of [3], Th. 1.8, we are able to obtain the following result which surprised us.

Theorem. For each odd integer $d \geq 1$, the real Fermat curve $F_d \subset \mathbb{R}^2$ defined by the equation $x^d + y^d = 1$ has the following universal property. Let $X \subseteq \mathbb{R}^n$ be any real algebraic variety. Identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{R}^n$ with the subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^{2n}$. Let $\pi : \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R}$ denote the projection onto the last factor. Choose a neighborhood $\mathcal{U}$ of $Id_{\mathbb{R}^{2n}}$ in $\mathcal{N}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ with respect to the C^∞ compact-open topology. Then there exist a real algebraic variety $X^* \subseteq \mathbb{R}^{2n} \times \mathbb{R}$ and a weak deformation $\beta : X^* \rightarrow X$ of X parametrized by $(\pi|_{X^*}, 0)$ such that:

- (a) $X^* \cap \pi^{-1}(0) = X$ and β extends to a Nash diffeotopy of $\mathbb{R}^{2n}$ in $\mathcal{U}$;
- (b) for each $t \in \mathbb{R} \setminus \{0\}$, the real algebraic variety $\tilde{X}_t := X^* \cap \pi^{-1}(t)$ may be biregularly embedded into $(F_d)^n$.

Notice that, in the statement of the Theorem, not only we may embed $\tilde{X}_t$, $t \neq 0$, in a product of genus $(d-1)(d-2)/2$ real algebraic curves all isomorphic to F_d , but that we may embed it in $(F_d)^n$ where the exponent n is the same for all real algebraic varieties which may be embedded in $\mathbb{R}^n$. singular ", see [3], Remark 2.3, and use the proof of Theorem given below.

Proof of the Theorem. The proof is just a typographical modification of the proof of [3], Th. 1.8: it is the result that it is surprising, not its proof! Fix an odd integer $d \geq 1$ and $\epsilon > 0$. Consider the polynomial $G(x, y, t) \in \mathbb{R}[x, y, t]$ defined by the formula:

$$G(x, y, t) := (\epsilon tx + y(1 + t^2))^d + (-\epsilon tx + y(1 + t^2))^d - \epsilon^d t^d.$$

For each $t_0 \in \mathbb{R}$, set $D[t_0] := \{(x, y) \in \mathbb{R}^2 \mid G(x, y, t_0) = 0\}$. Notice that $D[0]$ is equal to the axis $\{y = 0\}$ and, for all $t', t'' \in \mathbb{R} \setminus \{0\}$, the plane curves $D[t']$ and $D[t'']$ are biregularly isomorphic: just make a dilatation of the y -coordinate. Call $x_1, \dots, x_n, y_1, \dots, y_n, t$ the coordinates of $\mathbb{R}^{2n+1}$. Let $G^* \subset \mathbb{R}^{2n+1}$ denote the algebraic subset defined by the n equations $G(x_i, y_i, t) = 0$, $i = 1, \dots, n$. Set $X^* := G^* \cap (X \times \mathbb{R}^{n+1})$. Fix $t \neq 0$. Up to scaling, we may assume that $D[t]$ has $(x+y)^d + (-x+y)^d = 1$ as equation. Notice that the linear isomorphism $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $\varphi(x, y) := (x + y, -x + y)$ induces a biregular isomorphism from $D[t]$ to F_d . In particular, by the Implicit Function Theorem for Nash maps, there exists a Nash function on $\mathbb{R}$ whose graph coincides with $D[t]$. If ϵ is sufficiently small, then the proof of [3], Th. 1.8, gives verbatim the proof of the Theorem.

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