A universal property of odd degree
real Fermat curves

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Abstract. Fix an odd integer \( d \geq 1 \) and let \( F_d \subset \mathbb{R}^2 \) denote the real Fermat curve defined by the equation \( x^d + y^d = 1 \). Here we prove that \( F_d \) has the following property: Let \( X \subseteq \mathbb{R}^n \) be any real algebraic variety. Then there exists a “one-parameter isotopic Nash modification” \( \tilde{X}_t \) of \( X \) such that \( \tilde{X}_0 = X \) and each real algebraic variety \( \tilde{X}_t, \ t \neq 0 \) may be biregularly embedded into \((F_d)^n\).

Mathematics Subject Classification: 14P05; 14P20
Keywords: Nash variety; real algebraic variety; Nash map

For the standard definitions of real algebraic varieties, regular maps between two real algebraic varieties and Nash maps, see [1]. For all real algebraic varieties \( X, Y \), let \( \mathcal{N}(X, Y) \) denote the set of all Nash maps \( f : X \rightarrow Y \). Here we just recall the following definitions heavily used in [2] and [3] (see [3], Def. 1.6 and 1.7).

Definition 1. Let \( X \) and \( \tilde{X} \) be real algebraic varieties and \( \phi : \tilde{X} \rightarrow X \) a regular map. We will say that \( \phi \) is a weak change of the algebraic structure of \( X \) if it is bijective and \( \phi^{-1} \) is a Nash map.

Definition 2. Let \( X, Z, X^* \) be real algebraic varieties, \( z_0 \in Z \) and \( \pi : X^* \rightarrow Z \) a regular map. A map \( \beta : X^* \rightarrow X \) is called a weak deformation of \( X \) parametrized by \((\pi, z_0)\) if it is regular, \( \pi^{-1}(z_0) \) is biregularly isomorphic to \( X \), while, for each \( z \in Z \setminus \{z_0\} \), the map \( \beta|\pi^{-1}(z) : \pi^{-1}(z) \rightarrow X \) is a weak change of the algebraic structure of \( X \).

Hence a regular map \( \phi : \tilde{X} \rightarrow X \) between real algebraic varieties is a weak change of the algebraic structure of \( X \) if and only if it is a Nash isomorphism. In particular, it must be a homeomorphism for the euclidean topology. Notice that, if \( \phi \) is a weak change of the algebraic structure of \( X \), then \( \phi^{-1} \) maps nonsingular points to nonsingular points. Hence if \( X \) is nonsingular, then \( \tilde{X} \) is nonsingular also.

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1The author was partially supported by MIUR and GNSAGA of INdAM (Italy)
2The author was partially supported by MIUR and GNSAGA of INdAM (Italy)
Definition 3. Let $M$ be an affine Nash manifold. We say that a Nash map $F : M \times \mathbb{R} \to M$ is a Nash diffeotopy of $M$ if $F|_M \times \{t\}$ is a Nash isomorphism for all $t \in \mathbb{R}$ and $F|M \times \{0\} : M \times \{0\} \to M$ is just the isomorphic projection. Let $\mathcal{U}$ be a neighborhood of the identity map $Id_M$ in $\mathcal{N}(M, M)$, equipped with the $C^\infty$ compact-open topology. We say that $F$ is in $\mathcal{U}$ if $F|M \times \{t\} \in \mathcal{U}$ for all $t \in \mathbb{R}$.

Following the proof of [3], Th. 1.8, we are able to obtain the following result which surprised us.

Theorem. For each odd integer $d \geq 1$, the real Fermat curve $F_d \subseteq \mathbb{R}^2$ defined by the equation $x^d + y^d = 1$ has the following universal property. Let $X \subseteq \mathbb{R}^n$ be any real algebraic variety. Identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{R}^{2n}$ and $\mathbb{R}^n$ with the subspace $\mathbb{R}^n \times \{0\}$ of $\mathbb{R}^{2n}$. Let $\pi : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ denote the projection onto the last factor. Choose a neighborhood $\mathcal{U}$ of $Id_{\mathbb{R}^{2n}}$ in $\mathcal{N}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ with respect to the $C^\infty$ compact-open topology. Then there exist a real algebraic variety $X^* \subseteq \mathbb{R}^{2n} \times \mathbb{R}$ and a weak deformation $\beta : X^* \to X$ of $X$ parametrized by $\pi(X^*, 0)$ such that:

(a) $X^* \cap \pi^{-1}(0) = X$ and $\beta$ extends to a Nash diffeotopy of $\mathbb{R}^{2n}$ in $\mathcal{U}$;
(b) for each $t \in \mathbb{R}\backslash\{0\}$, the real algebraic variety $X_t := X^* \cap \pi^{-1}(t)$ may be biregularly embedded into $(F_d)^n$.

Notice that, in the statement of the Theorem, not only we may embed $\tilde{X}_t$, $t \neq 0$, in a product of genus $(d - 1)(d - 2)/2$ real algebraic curves all isomorphic to $F_d$, but that we may embed it in $(F_d)^n$ where the exponent $n$ is the same for all real algebraic varieties which may be embedded in $\mathbb{R}^n$. singular ”, see [3], Remark 2.3, and use the proof of Theorem given below.

Proof of the Theorem. The proof is just a typographical modification of the proof of [3], Th. 1.8: it is the result that it is surprising, not its proof! Fix an odd integer $d \geq 1$ and $\epsilon > 0$. Consider the polynomial $G(x, y, t) \in \mathbb{R}[x, y, t]$ defined by the formula:

$$G(x, y, t) := (\epsilon tx + y(1 + t^2))^d + (-\epsilon tx + y(1 + t^2))^d - \epsilon^d t^d.$$ 

For each $t_0 \in \mathbb{R}$, set $D[t_0] := \{(x, y) \in \mathbb{R}^2 \mid G(x, y, t_0) = 0\}$. Notice that $D[0]$ is equal to the axis $\{y = 0\}$ and, for all $t', t'' \in \mathbb{R}\backslash\{0\}$, the plane curves $D[t']$ and $D[t'']$ are biregularly isomorphic: just make a dilatation of the $y$-coordinate. Call $x_1, \ldots, x_n, y_1, \ldots, y_n, t$ the coordinates of $\mathbb{R}^{2n+1}$. Let $G^* \subseteq \mathbb{R}^{2n+1}$ denote the algebraic subset defined by the $n$ equations $G(x_i, y_i, t) = 0$, $i = 1, \ldots, n$. Set $X^* := G^* \cap (X \times \mathbb{R}^{n+1})$. Fix $t \neq 0$. Up to scaling, we may assume that $D[t]$ has $(x+y)^d + (-x+y)^d = 1$ as equation. Notice that the linear isomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(x, y) := (x + y, -x + y)$ induces a biregular isomorphism from $D[t]$ to $F_d$. In particular, by the Implicit Function Theorem for Nash maps, there exists a Nash function on $\mathbb{R}$ whose graph coincides with $D[t]$. If $\epsilon$ is sufficiently small, then the proof of [3], Th. 1.8, gives verbatim the proof of the Theorem.

References


MA, 1969.


Received: October 14, 2005