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ALGEBRAS AND THEIR DUAL IN RIGID TENSOR CATEGORIES

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Abstract

It is known that if H is an algebra (resp. a coalgebra) in a rigid tensor category, then its dual H^* is a coalgebra (resp. an algebra) in the same category. We have reproved this result by using specific definitions in terms of diagrams. These definitions are also used to prove such results for braided Hopf algebra in a braided rigid tensor category. Moreover, under certain conditions, an algebra A in a rigid tensor category is proved to be isomorphic to A^*. Finally, a formula for the comultiplication on A^* is derived.

1 Introduction

This paper will make continual use of formulas and ideas from [4] which is itself based on the papers [3, 5], but is mostly self contained in terms of notation and definitions.

It is well known that for every factorization $X = GM$ of a group into two subgroups G and M, a Hopf algebra $H = kM \triangleright\!\!\blacktriangleleft k(G)$ can be constructed, where kM is the group Hopf algebra of M and $k(G)$ is the Hopf algebra of functions on G. In the symbol $kM \triangleright\!\!\blacktriangleleft k(G)$, the $\triangleright$ part means that kM acts on $k(G)$, and the $\blacktriangleleft$ part means that $k(G)$ coacts on kM.

The comultiplication on a Hopf algebra means that a tensor product can be defined for representations. The idea of tensor product is formalized in the definition of a tensor or monoidal category. If the Hopf algebra has a quasitriangular structure, there is a map of representations from $V \otimes W$ to $W \otimes V$, making the category of representations into a braided tensor category. These categories have a description in terms of diagrams of crossing lines, giving a direct calculation of the associated knot invariants. Another application is the differential structures on bicrossproduct Hopf algebras (see[5]).

It is possible to construct a non-trivially associated tensor category $\mathcal{C}$ from data which is a choice of left coset representatives M for a subgroup G of a finite group X (see[3, 4]).

Throughout the paper we assume that all groups mentioned, unless otherwise stated, are finite, and that all vector spaces are finite dimensional over a field k, which will be denoted by $\underline{1}$ as an object in the category.

2 Preliminaries

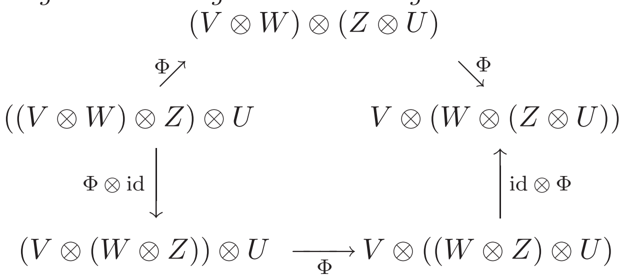
The idea of Hopf algebras in braided categories goes back to Milnor and Moore [13]. The notion of braided category plays an important role in quantum group theory. Majid (see [11]) studies Hopf algebras in braided categories under the name "braided groups" with an algebraic motivation from biproduct construction as well as many motivations from physics [17]. The book [11] has been used as a standard reference for tensor and braided tensor categories.

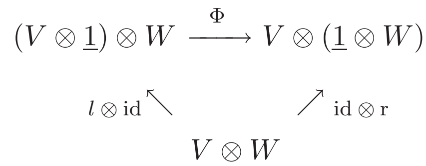
Definition 2.1 *A* **tensor** (**or monoidal**) **category** *is the datum* $(\mathcal{C}, \otimes, \underline{1}, \Phi, l, r)$*, where* $\mathcal{C}$ *is a category and* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is a functor which is associative in the sense that there is a natural equivalence* $\Phi : (\ \otimes\ )\otimes \to \otimes(\ \otimes\ )$ *which just means that there are given functorial isomorphisms*

$$\Phi_{V,W,Z} : (V \otimes W) \otimes Z \cong V \otimes (W \otimes Z), \quad \text{for all} \quad V, W, Z \in \mathcal{C},$$

obeying the pentagon condition given in the diagram below . A unit object $\underline{1}$ *is also required and natural equivalences between the functors* $(\ ) \otimes \underline{1}$, $\underline{1} \otimes (\ )$ *and the identity functor* $\mathcal{C} \to \mathcal{C}$*, i.e.*

there should be given functorial isomorphisms $l_V : V \cong V \otimes \underline{1}$ *and* $r_V : V \cong \underline{1} \otimes V$, *obeying the triangle condition given in the diagram below.*



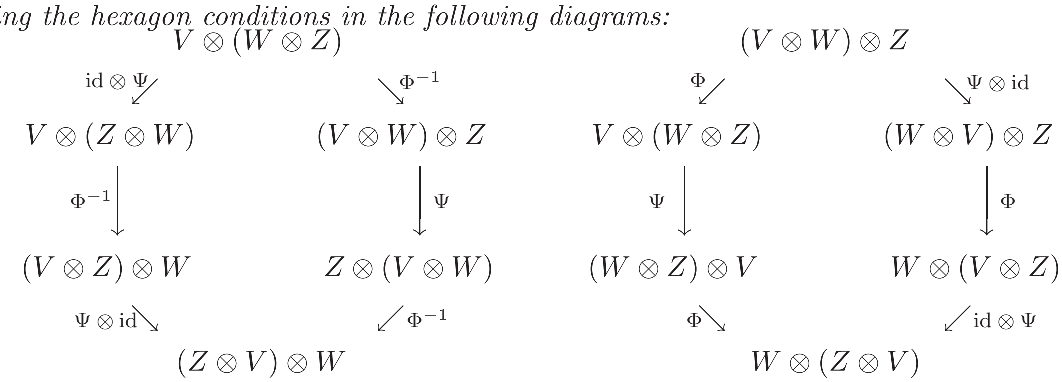


a) The pentagon condition b) The triangle condition

Definition 2.2 *A* **braided tensor** *(* **or quasitensor** *)* **category** $(\mathcal{C}, \otimes, \Psi)$ *is a tensor (monoidal) category which is commutative in the sense that there is a natural equivalence between the two functors* $\otimes, \otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, *which just means that there are given functorial isomorphisms*

$$\Psi_{V,W} : V \otimes W \to W \otimes V , \quad \text{for all} \quad V, W \in \mathcal{C} ,$$

obeying the hexagon conditions in the following diagrams:



If we suppress Φ, the hexagon conditions can be given by the following formulas:

$$\Psi_{V \otimes W,Z} = \Psi_{V,Z} \circ \Psi_{W,Z} , \quad \Psi_{V,W \otimes Z} = \Psi_{V,Z} \circ \Psi_{V,W} , \quad \text{for all} \quad V, W, Z \in \mathcal{C} .$$

Definition 2.3 *An object* V *in a tensor category* $\mathcal{C}$ *has a left dual or is rigid if there is an object* V^* *and morphisms* $\mathrm{ev_V : V^* \otimes V \to \underline{1}}$, $\mathrm{coev_V : \underline{1} \to V \otimes V^*}$ *such that*

$$V \cong \underline{1} \otimes V \stackrel{\mathrm{coev}\otimes\mathrm{id}}{\longrightarrow} (V \otimes V^*) \otimes V \stackrel{\Phi}{\longrightarrow} V \otimes (V^* \otimes V) \stackrel{\mathrm{id}\otimes\mathrm{ev}}{\longrightarrow} V \otimes \underline{1} \cong V ,$$
$$V^* \cong V^* \otimes \underline{1} \stackrel{\mathrm{id}\otimes\mathrm{coev}}{\longrightarrow} V^* \otimes (V \otimes V^*) \stackrel{\Phi^{-1}}{\longrightarrow} (V^* \otimes V) \otimes V^* \stackrel{\mathrm{ev}\otimes\mathrm{id}}{\longrightarrow} \underline{1} \otimes V^* \cong V^* ,$$

compose to $\mathrm{id_V}$ *and* $\mathrm{id_{V^*}}$, *respectively. Also if* V *and* W *are rigid in* $\mathcal{C}$ *and* $\phi : V \to W$ *is a morphism in the category, then* $\phi^* = (\mathrm{ev_V} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev_W}) : \mathrm{W^* \to V^*}$, *is called the dual or the adjoint morphism of* ϕ [11].

Definition 2.4 *If every object in the tensor category $\mathcal{C}$ has a dual, then we say that $\mathcal{C}$ is a* **rigid** *tensor category.*

We will use the following specific definitions in our work. A morphism $T : V \to W$, a tensor product $F : V \otimes W \to Y$, the braid $\Psi_{V,W} : V \otimes W \to W \otimes V$ and the maps $\mathrm{ev_V} : \mathrm{V^*} \otimes \mathrm{V} \to \underline{1}$ and $\mathrm{coev_V} : \underline{1} \to \mathrm{V} \otimes \mathrm{V^*}$ in tensor categories are represented in terms of diagrams as the following in order:

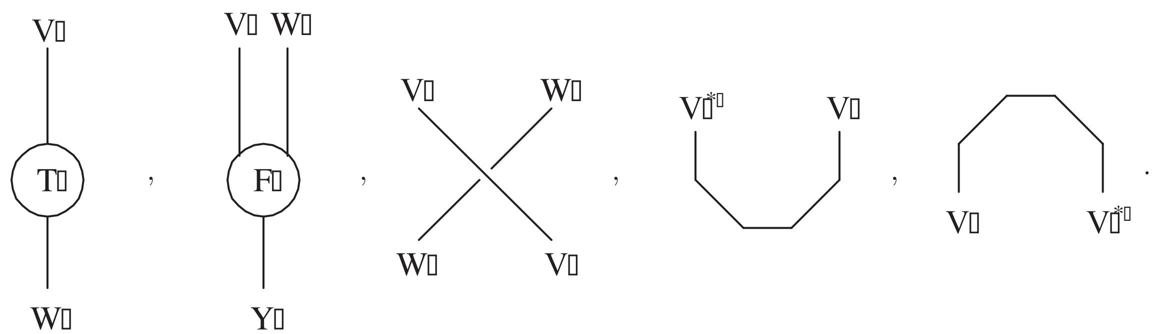


Figure 1

The definition of dual and the adjoint morphism can be given in terms of diagrams as the following, in order, read from top to bottom:

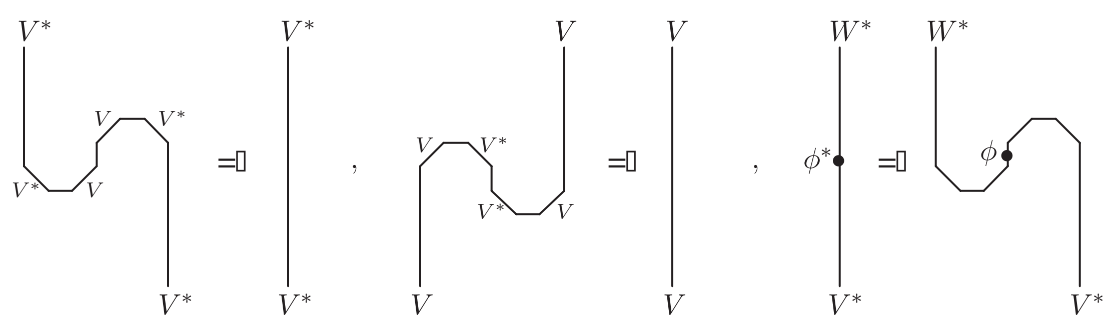


Figure 2

Some axioms of a Hopf algebra in a tensor category can be illustrated for the unit, counit, associativity, coassociativity and the antipode, in order, by the following diagrams:

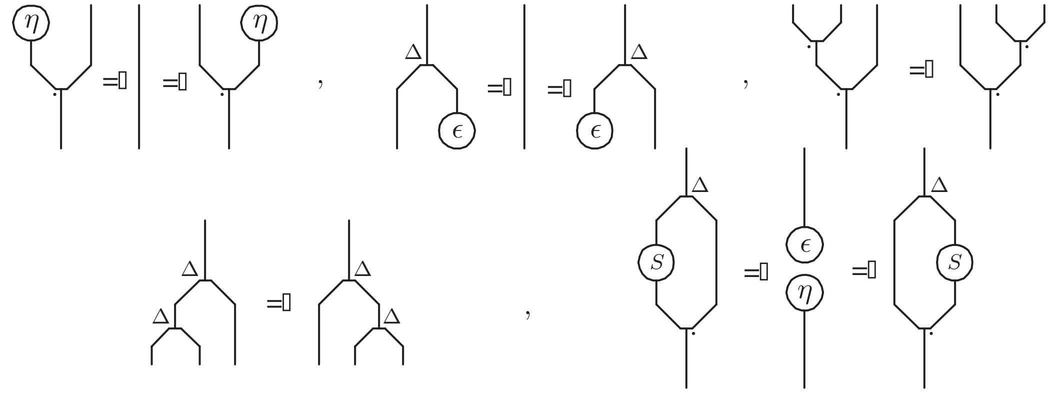


Figure 3

In the following diagrams we give the homomorphism property for a braided coproduct (or compatibility axiom between multiplication and comultiplication) and the braided antihomomorphism property of S, in order, where H is any braided Hopf algebra:

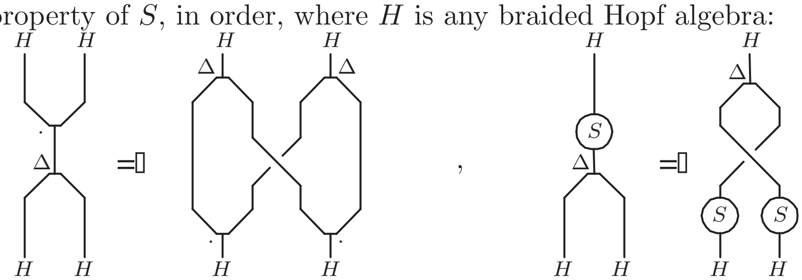


Figure 4

Definition 2.5 *For a group X and a subgroup G, we call $M \subset X$ a set of left coset representatives if for every $x \in X$ there is a unique $s \in M$ so that $x \in Gs$. The decomposition $x = us$ for $u \in G$ and $s \in M$ is called the unique factorization of x* [4].

In what follows, $M \subset X$ is assumed to be a fixed set of left coset representatives for the subgroup $G \subset X$. In addition, the identity in X will be denoted by e.

Definition 2.6 *For $s, t \in M$ we define $\tau(s,t) \in G$ and $s \cdot t \in M$ by the unique factorization $st = \tau(s,t)(s \cdot t)$ in X. The functions $\triangleright : M \times G \to G$ and $\triangleleft : M \times G \to M$ are also defined by the unique factorization $su = (s \triangleright u)(s \triangleleft u)$ for $s, s \triangleleft u \in M$ and $u, s \triangleright u \in G$* [4].

From [4] we know that for $t, s, p \in M$ and $u, v \in G$, the following identities hold:

$$s \triangleright (t \triangleright u) = \tau(s,t)\big((s \cdot t) \triangleright u\big)\tau\big(s \triangleleft (t \triangleright u), t \triangleleft u\big)^{-1} \quad \text{and} \quad (s \cdot t) \triangleleft u = \big(s \triangleleft (t \triangleright u)\big) \cdot (t \triangleleft u)\,,$$

$$s \triangleright uv = (s \triangleright u)\big((s \triangleleft u) \triangleright v\big) \quad \text{and} \quad s \triangleleft uv = (s \triangleleft u) \triangleleft v\,,$$

$$\tau(p,s)\tau(p \cdot s, t) = \big(p \triangleright \tau(s,t)\big)\tau\big(p \triangleleft \tau(s,t), s \cdot t\big) \quad \text{and} \quad \big(p \triangleleft \tau(s,t)\big) \cdot (s \cdot t) = (p \cdot s) \cdot t\,,$$

$$e \triangleleft v = e\,,\;\; e \triangleright v = v\,,\;\; t \triangleright e = e\,,\;\; t \triangleleft e = t\,.$$

The category $\mathcal{C}$ is defined as the following [4]: Take a category $\mathcal{C}$ of finite dimensional vector spaces over a field k, whose objects are right representations of the group G and have M-gradings, i.e. an object V can be written as $\bigoplus_{s\in M} V_s$. The action for the representation is written as $\bar{\triangleleft} : V \times G \to V$. In addition it is supposed that the action and the grading satisfy the compatibility condition, i.e. $\langle \xi \bar{\triangleleft} u\rangle = \langle \xi\rangle \triangleleft u$. The morphisms in the category $\mathcal{C}$ is defined to be linear maps which preserve both the grading and the action, i.e. for a morphism $\vartheta : V \to W$ we have $\langle \vartheta(\xi)\rangle = \langle \xi\rangle$ and $\vartheta(\xi)\bar{\triangleleft} u = \vartheta(\xi \bar{\triangleleft} u)$ for all $\xi \in V$ and $u \in G$. $\mathcal{C}$ can be made into a tensor category by taking $V \otimes W$ to be the usual vector space tensor product, with actions and gradings given by

$$\langle \xi \otimes \eta\rangle = \langle \xi\rangle \cdot \langle \eta\rangle \quad \text{and} \quad (\xi \otimes \eta)\bar{\triangleleft} u = \xi \bar{\triangleleft}(\langle \eta\rangle \triangleright u) \otimes \eta \bar{\triangleleft} u\,.$$

There is an associator $\Phi_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ given by

$$\Phi((\xi \otimes \eta) \otimes \zeta) = \xi \bar{\triangleleft} \tau(\langle \eta\rangle, \langle \zeta\rangle) \otimes (\eta \otimes \zeta)\,.$$

Next for the **rigidity** of $\mathcal{C}$, suppose that $(M,\cdot)$ has right inverses, i.e. for every $s \in M$ there is an $s^R \in M$ so that $s \cdot s^R = e$ and consider $V = \bigoplus_{s\in M} V_s$, where $\xi \in V_s$ corresponds to $\langle \xi\rangle = s$. Now take the dual vector space V^*, and set $V^*_{s^L} = \{\alpha \in V^* : \alpha|_{V_t} = 0 \quad \forall t \neq s\}$. Then $V^* = \bigoplus_{s\in M} V^*_{s^L}$, and we define $\langle \alpha\rangle = s^L$ when $\alpha \in V^*_{s^L}$. The evaluation map $\mathrm{ev} : V^* \otimes V \to k$ is defined by $\mathrm{ev}(\alpha, \xi) = \alpha(\xi)$. Considering the action $\bar{\triangleleft} u$, if we apply evaluation to $\alpha \bar{\triangleleft}(\langle \xi\rangle \triangleright u) \otimes \xi \bar{\triangleleft} u$ we should get $\alpha(\xi)\bar{\triangleleft} u = \alpha(\xi)$. So we define $\big(\alpha \bar{\triangleleft}(\langle \xi\rangle \triangleright u)\big)(\xi \bar{\triangleleft} u) = \alpha(\xi)$, or if we put $\eta = \xi \bar{\triangleleft} u$ we get $\big(\alpha \bar{\triangleleft}\big((\langle \eta\rangle \triangleleft u^{-1}) \triangleright u\big)\big) = \alpha(\eta \bar{\triangleleft} u^{-1}) = \big(\alpha \bar{\triangleleft}(\langle \eta\rangle \triangleright u^{-1})^{-1}\big)(\eta)$. If this is rearranged to give $\alpha \triangleleft v$, we get the following formula:

$$(\alpha \bar{\triangleleft} v)(\eta) = \alpha\big(\eta \bar{\triangleleft} \tau(\langle \eta\rangle^L, \langle \eta\rangle)^{-1} (\langle \eta\rangle^L \triangleright v^{-1}) \tau(\langle \eta\rangle^L \triangleleft v^{-1}, (\langle \eta\rangle^L \triangleleft v^{-1})^R)\big). \tag{1}$$

For the coevaluation map to be defined, a basis $\{\xi\}$ of each V_s is taken and a corresponding dual basis $\{\hat{\xi}\}$ of each $V^*_{s^L}$, i.e. $\hat{\eta}(\xi) = \delta_{\xi,\eta}$. Then these bases are put together for all $s \in M$ to get the following definition, which is a morphism in $\mathcal{C}$ [4]:

$$\mathrm{coev}(1) = \sum_{\xi \in basis} \xi \bar{\triangleleft} \tau(\langle\xi\rangle^L, \langle\xi\rangle)^{-1} \otimes \hat{\xi}\,.$$

The algebra A in the tensor category $\mathcal{C}$ is constructed so that the group action and the grading in the definition of $\mathcal{C}$ can be combined. We consider a single object A, a vector space spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. For any object V in $\mathcal{C}$ define a map $\bar{\triangleleft} : V \otimes A \to V$ by $\xi \bar{\triangleleft} (\delta_s \otimes u) = \delta_{s,\langle\xi\rangle} \xi \bar{\triangleleft} u$. This map is a morphism in $\mathcal{C}$ only if $\langle\xi\rangle \cdot \langle\delta_s \otimes u\rangle = \langle\xi \bar{\triangleleft} u\rangle$ i.e. $s \cdot \langle\delta_s \otimes u\rangle = s \triangleleft u$ if $\langle\xi\rangle = s$. If we put $a = \langle\delta_s \otimes u\rangle$, the action of $v \in G$ is given by $(\delta_s \otimes u) \bar{\triangleleft} v = \delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} uv$.

3 Dual of an algebra, coalgebra and Hopf algebra

In this section we prove the well known results that are: if H is an algebra in a rigid tensor category, then its dual H^* is a coalgebra in that category and if H is a coalgebra in a rigid tensor category, then its dual H^* is an algebra in the same category, using specific definitions in terms of diagrams (see [12]). In the same way, it is proved that if H is a braided Hopf algebra in a rigid braided tensor category, then we can make H^* into a braided Hopf algebra.

Proposition 3.1 *If H is an algebra in a rigid tensor category, then its dual H^* is a coalgebra in the category using the following definitions:*

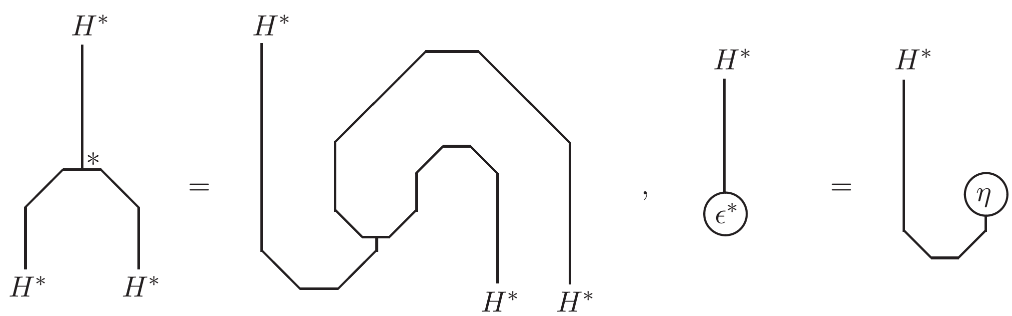
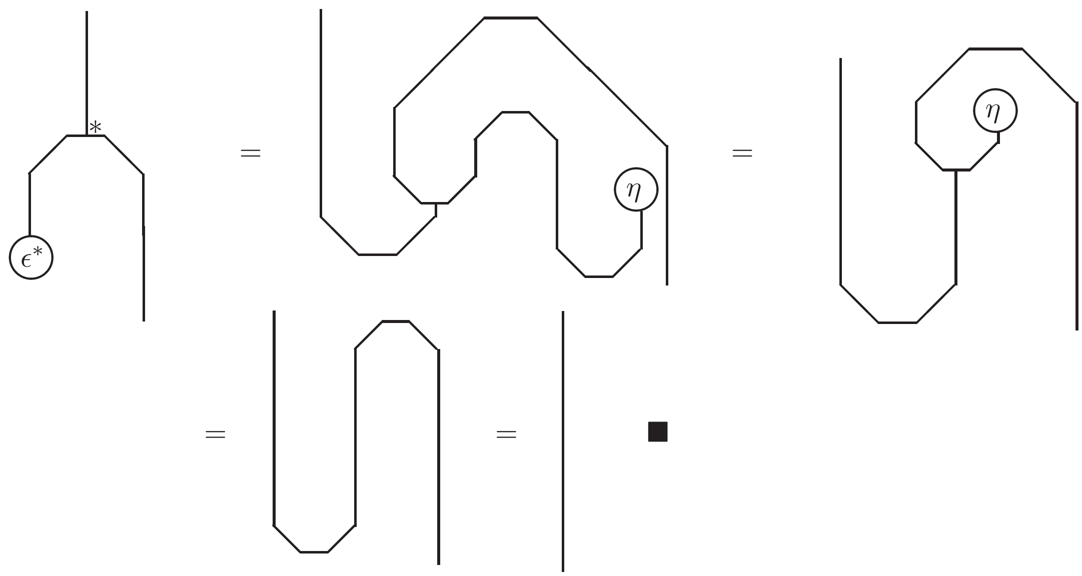
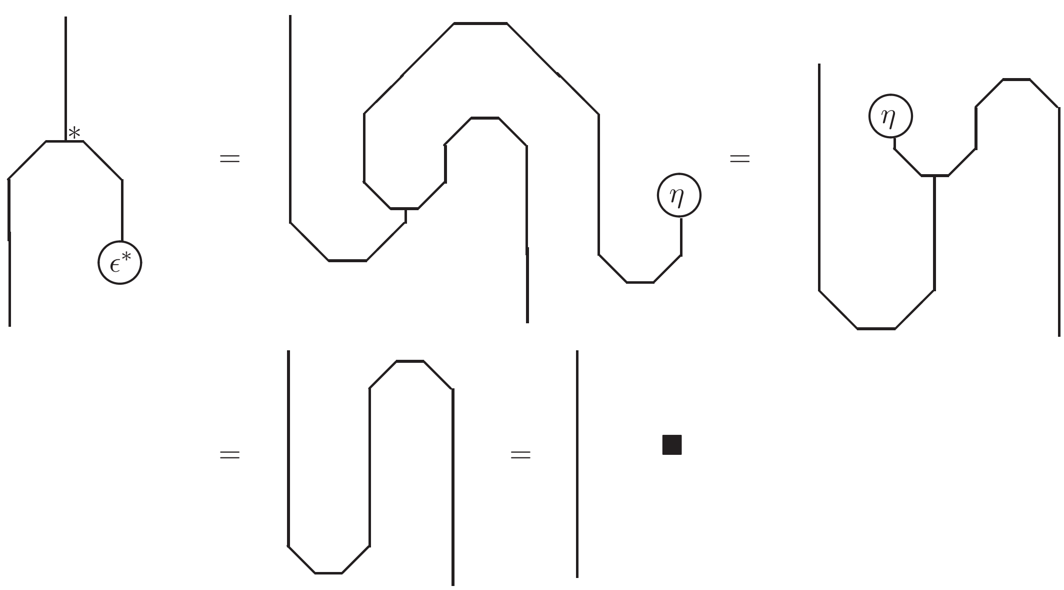


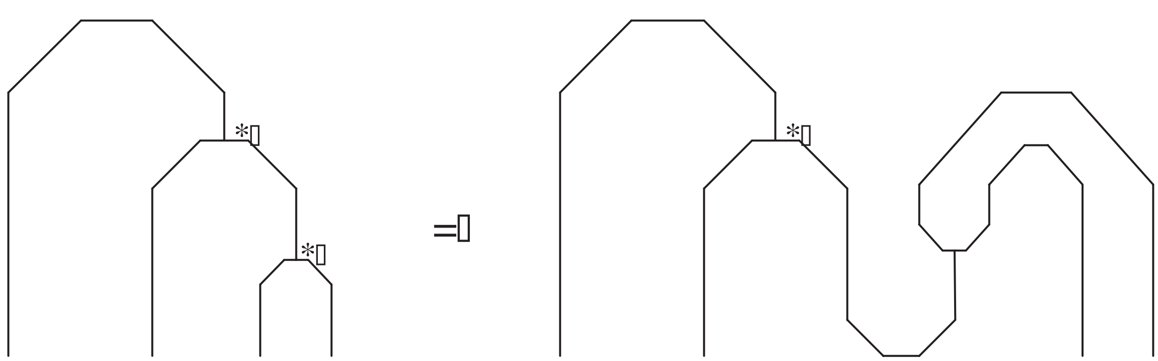
Figure 5: a) comultiplication b) counit

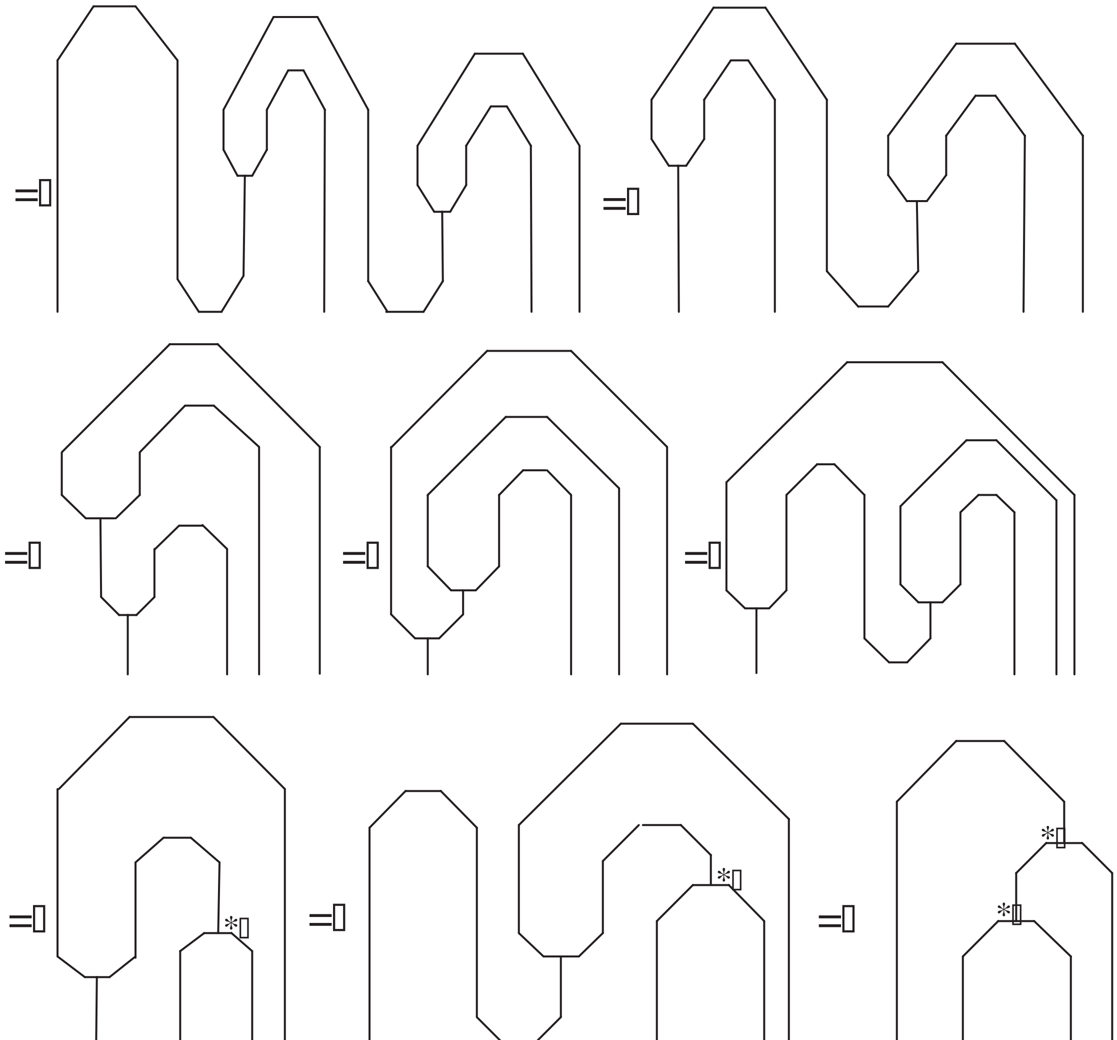
Proof. First we check the counit property for ϵ^*.





Now we check the coassociativity of the comultiplication.





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Proposition 3.2 *If H is a coalgebra in a rigid tensor category, then its dual H^* is an algebra in the category using the following definitions:*

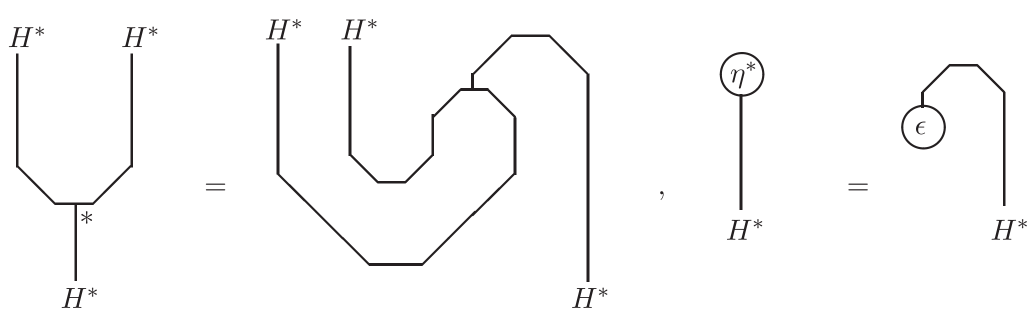
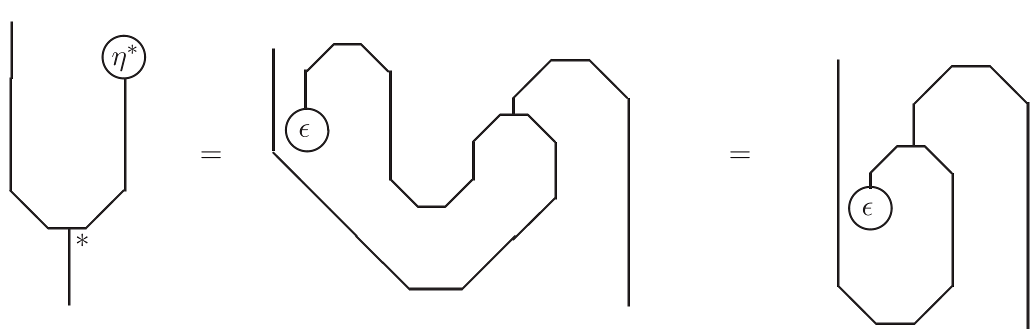
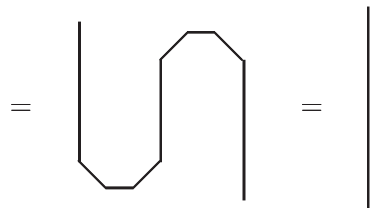
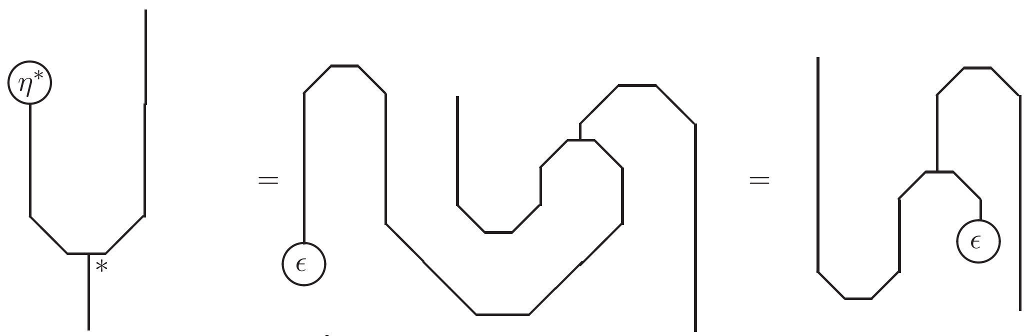


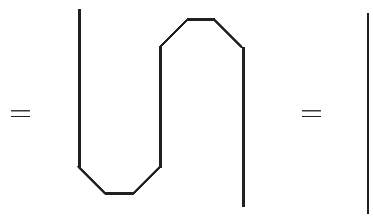
Figure 6: a) multiplication b) unit

Proof. First we check the unit property for η^*.

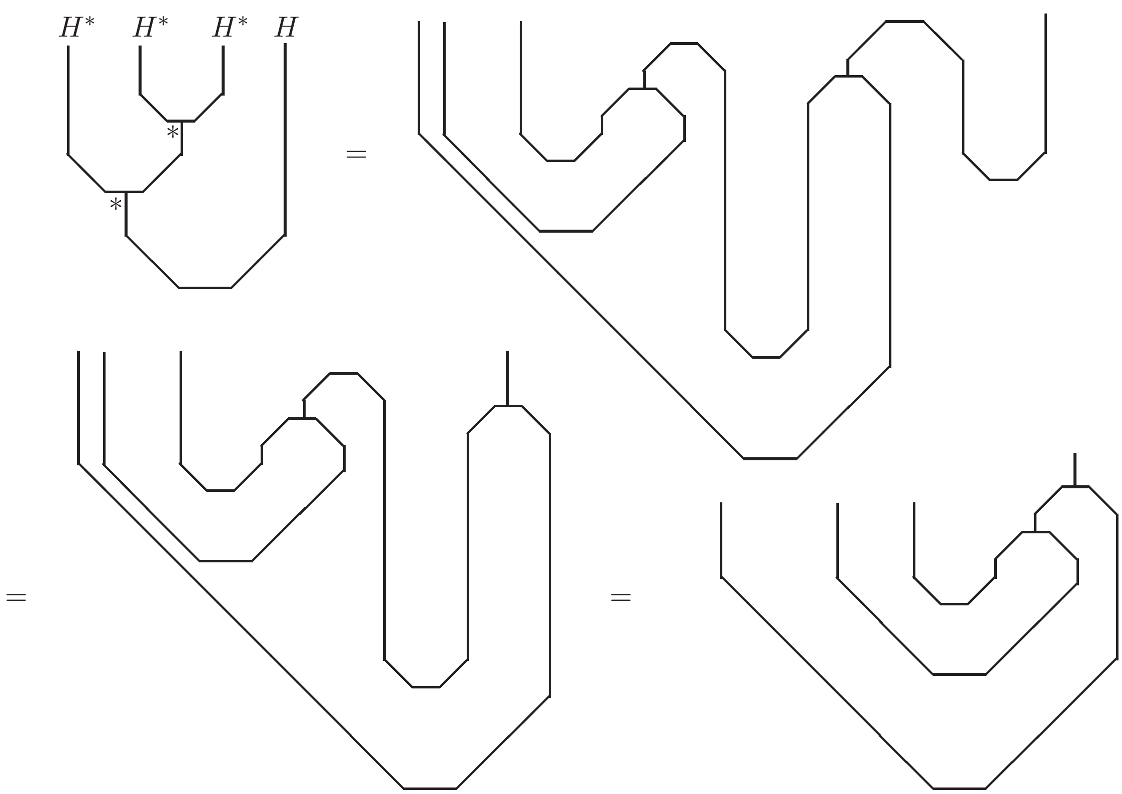


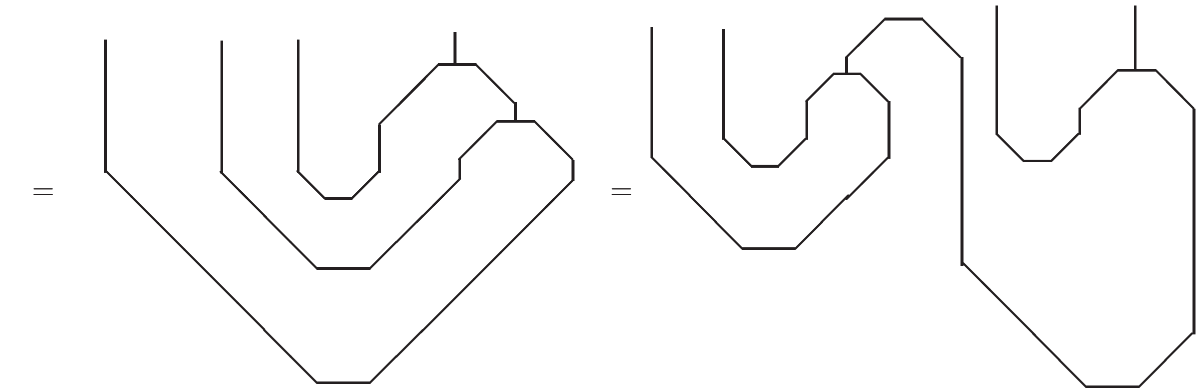
 ■

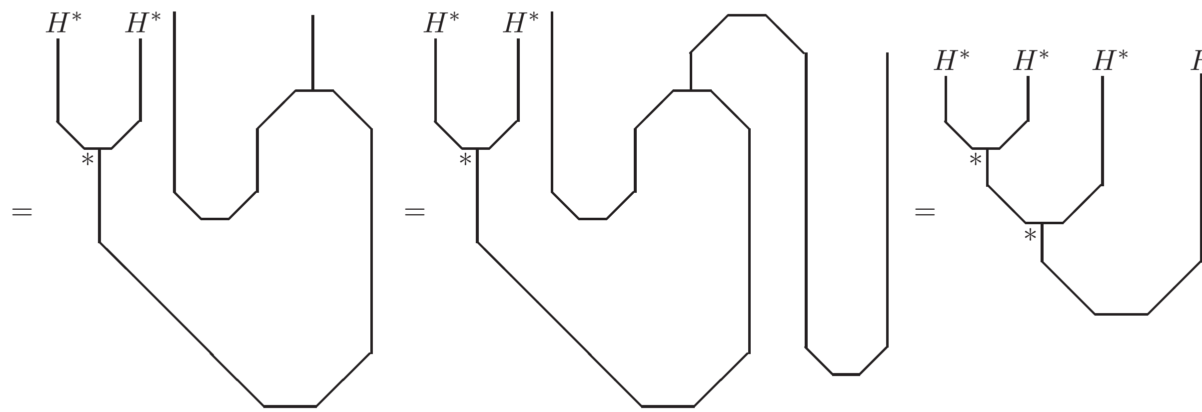


 ■

Now we check the associativity property for the multiplication.







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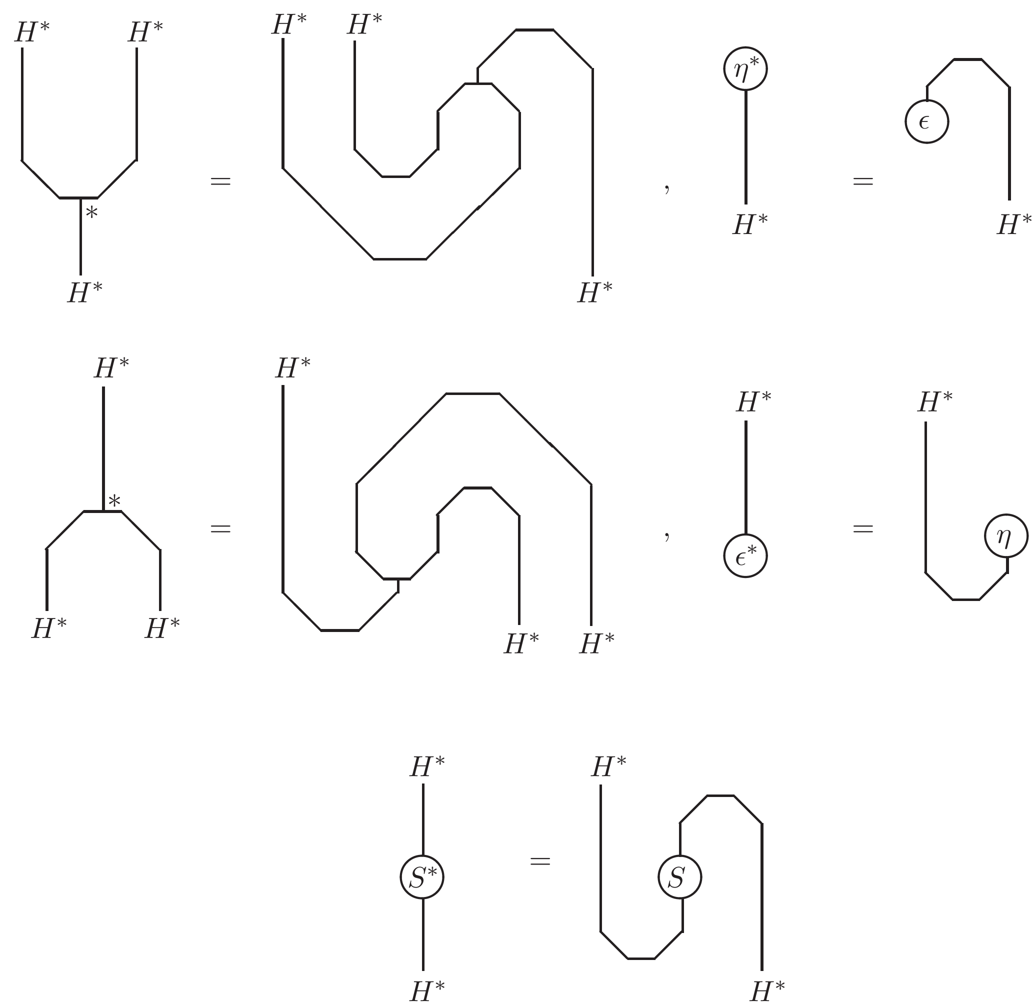
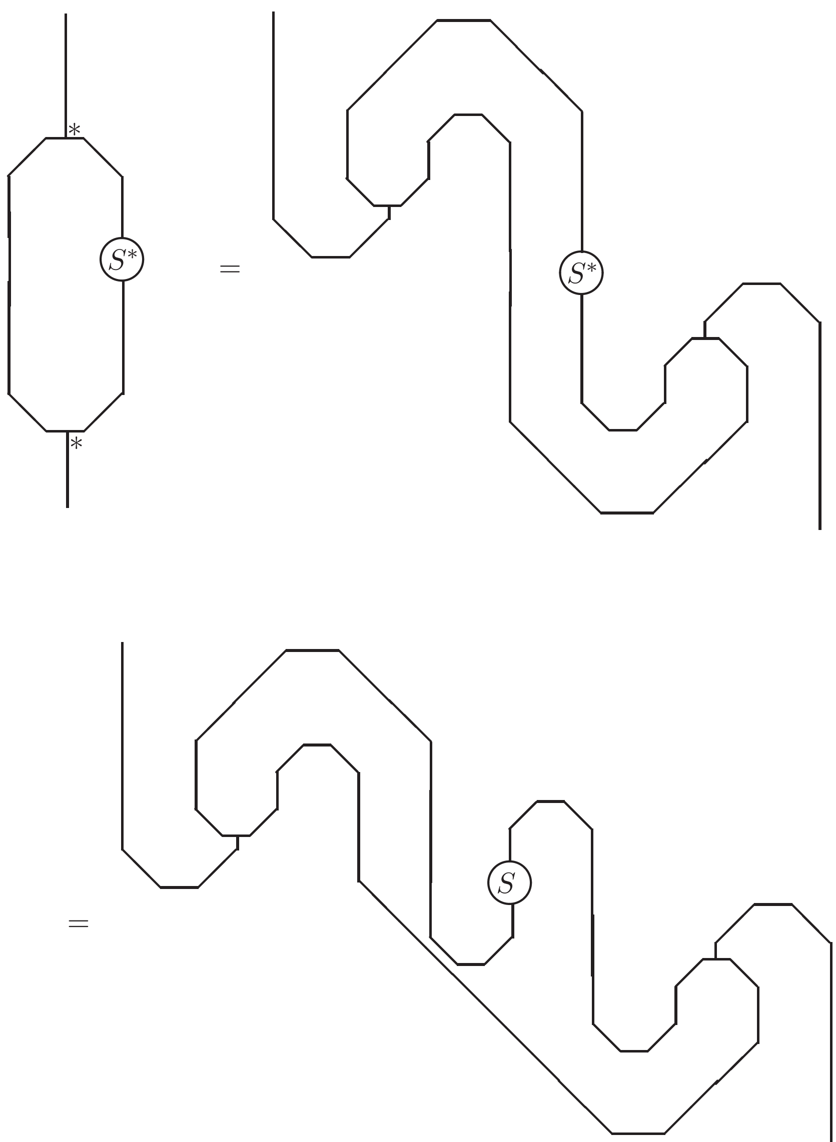
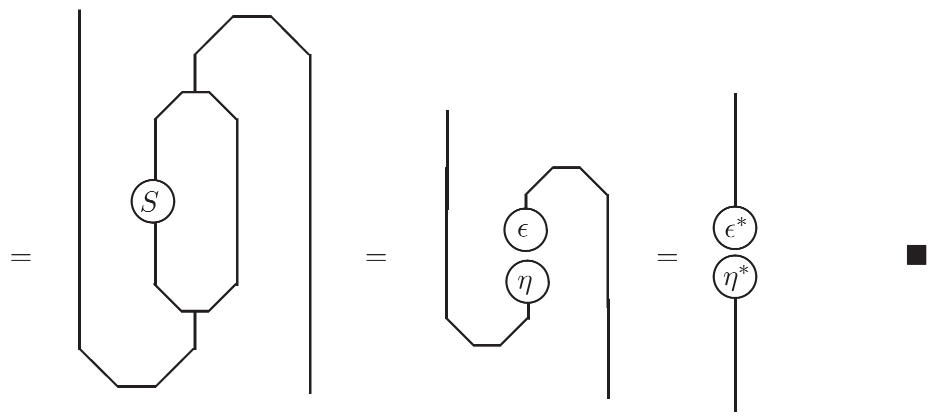
Proposition 3.3 *If H is a braided Hopf algebra in a rigid braided category, then we can make H^* into a braided Hopf algebra by the following definitions:*



Proof. We have already proved in the previous two propositions that unit, counit, associativity and coassociativity properties are held. So we only need to prove the property for S^* and the compatibility condition. We now check the property for S^* as the following:

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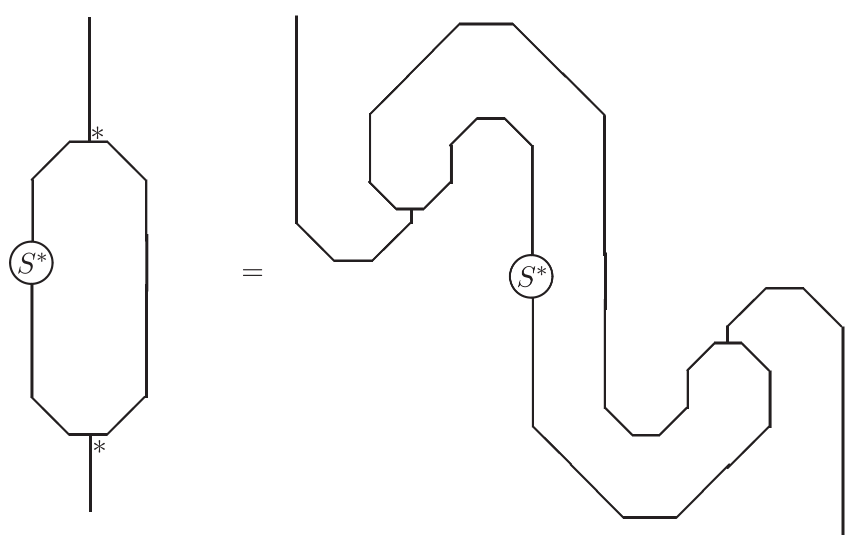
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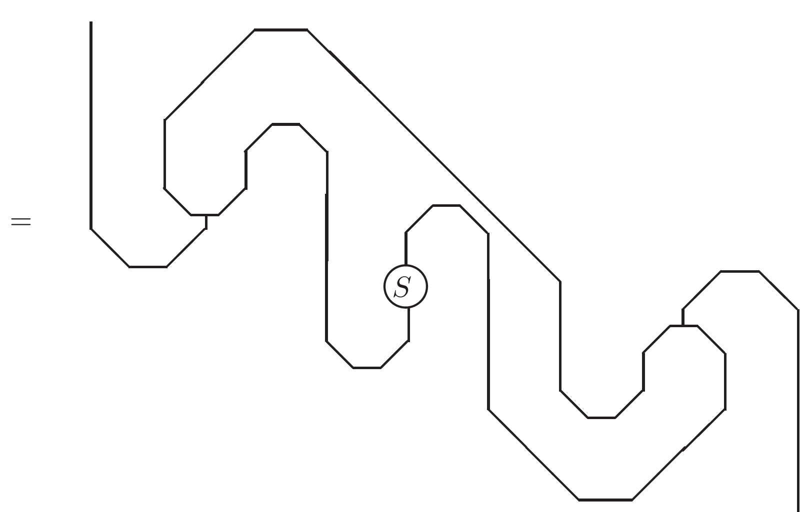
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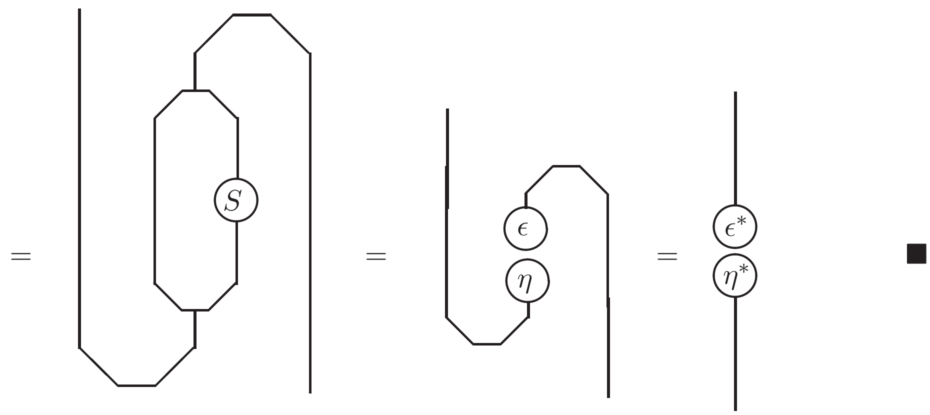
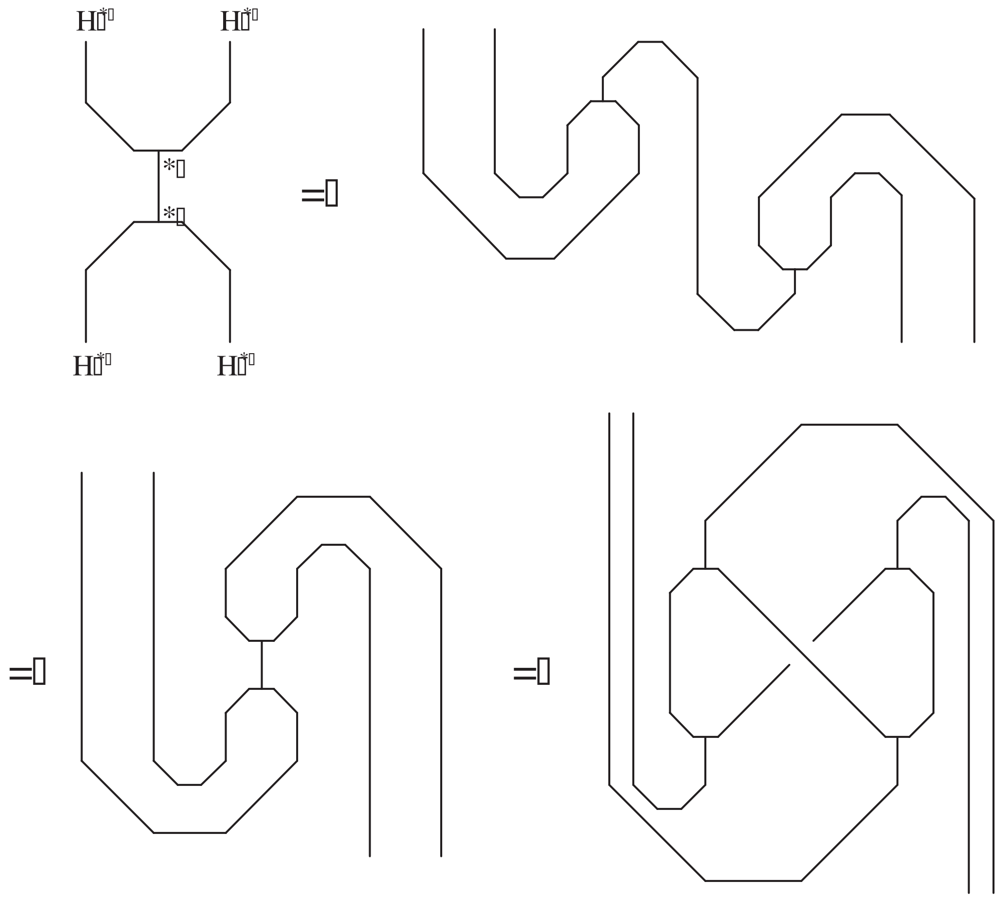
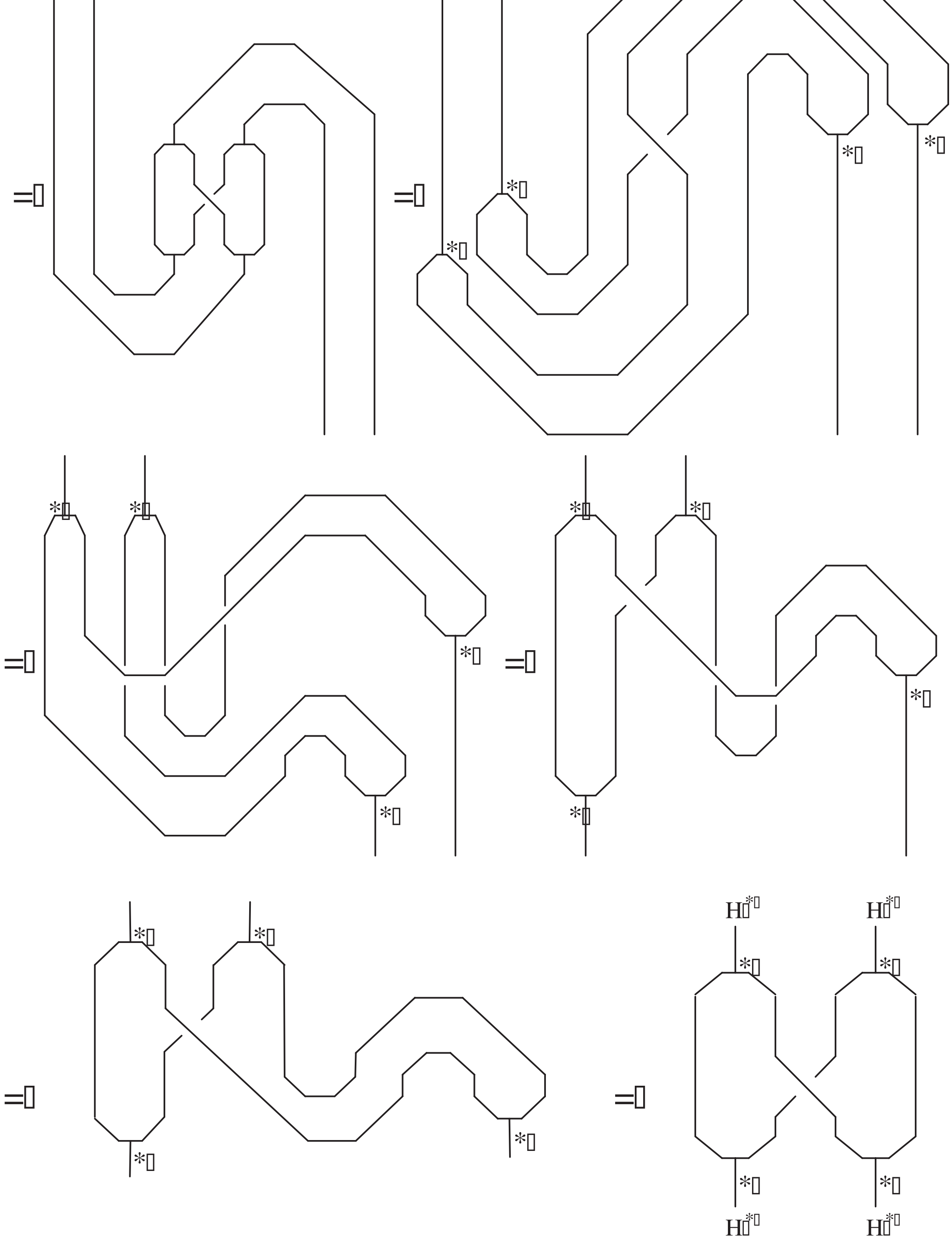


Lastly, we check the compatibility condition between the multiplication and the comultiplication.





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4 The dual of the algebra A in $\mathcal{C}$

Proposition 4.1 *Define a basis $\delta_u \otimes s$ of A^* with evaluation map given by*

$$\mathrm{ev}\big((\delta_u \otimes s) \otimes (\delta_t \otimes v)\big) = \delta_{s,t}\, \delta_{u,v}\,,$$

for $s, t \in M$ and $u, v \in G$. Then the M-grade and the G-action on A^* are defined as follows: $\langle \delta_u \otimes s \rangle = \langle \delta_s \otimes u \rangle^L$, and for any $w \in G$

$$(\delta_u \otimes s) \bar{\triangleleft} (\langle \delta_u \otimes s \rangle^R \triangleright w) = \delta_{(\langle \delta_u \otimes s \rangle^R \triangleright w)^{-1} u w} \otimes s \triangleleft (\langle \delta_u \otimes s \rangle^R \triangleright w)\,.$$

Proof. Take the algebra $A \in \mathcal{C}$. For $(\delta_s \otimes u) \in A$, $(\delta_u \otimes s) \in A^*$, then for $(\delta_t \otimes v) \in A$ the evaluation map is given by

$$\mathrm{ev}\big((\delta_u \otimes s) \otimes (\delta_t \otimes v)\big) = \delta_{s,t}\, \delta_{u,v}\,,$$

where $s, t \in M$ and $u, v \in G$. The evaluation map will not be affected if for any $w \in G$ we apply $\bar{\triangleleft} w$ as follows

$$\mathrm{ev}\Big(\big((\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_t \otimes v\rangle \triangleright w)\big) \otimes \big((\delta_t \otimes v)\bar{\triangleleft} w\big)\Big) = \delta_{s,t}\, \delta_{u,v}\,.$$

Using the definition of the action of G on the elements of the algebra A, the last equation can be rewritten as

$$\mathrm{ev}\Big(\big((\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_t \otimes v\rangle \triangleright w)\big) \otimes \big(\delta_{t \triangleleft (\langle \delta_t \otimes v\rangle \triangleright w)} \otimes (\langle \delta_t \otimes v\rangle \triangleright w)^{-1} v w\big)\Big) = \delta_{s,t}\, \delta_{u,v}\,.$$

If we take $\delta_t \otimes v = \delta_s \otimes u$ then we get the following equation

$$(\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_s \otimes u\rangle \triangleright w) = \delta_{(\langle \delta_s \otimes u\rangle \triangleright w)^{-1} u w} \otimes s \triangleleft (\langle \delta_s \otimes u\rangle \triangleright w)\,. \tag{2}$$

As $\langle \delta_u \otimes s\rangle \cdot \langle \delta_s \otimes u\rangle = e$, so $\langle \delta_u \otimes s\rangle = \langle \delta_s \otimes u\rangle^L$ or $\langle \delta_s \otimes u\rangle = \langle \delta_u \otimes s\rangle^R$. Hence the equation (2) can be rewritten as

$$(\delta_u \otimes s)\bar{\triangleleft}(\langle \delta_u \otimes s\rangle^R \triangleright w) = \delta_{(\langle \delta_u \otimes s\rangle^R \triangleright w)^{-1} u w} \otimes s \triangleleft (\langle \delta_u \otimes s\rangle^R \triangleright w)\,. \quad \blacksquare$$

Proposition 4.2 *There is a morphism $T : A \to A^*$ in the category $\mathcal{C}$ defined by*

$$T(\delta_s \otimes u) = \delta_{u^{-1}\tau(b, b^R)} \otimes s \triangleleft u\,,$$

where $b = \langle \delta_s \otimes u\rangle$.

Proof. We have defined a linear map $T : A \to A^*$. If this map is to be a morphism in the category it should preserve the grading and the action in $\mathcal{C}$ i.e. $\langle T(\delta_s \otimes u)\rangle = \langle \delta_s \otimes u\rangle$ and

$$\big(T(\delta_s \otimes u)\big)\bar{\triangleleft}(b^R \triangleright w) = T\big((\delta_s \otimes u)\bar{\triangleleft}(b^R \triangleright w)\big)\,. \tag{3}$$

To prove this we start with

$$T\big((\delta_s\otimes u)\bar{\triangleleft}(b^R{\triangleright}w)\big)=T\Big(\delta_{s\triangleleft\left(b\triangleright(b^R\triangleright w)\right)}\otimes\big(b{\triangleright}(b^R{\triangleright}w)\big)^{-1}u(b^R{\triangleright}w)\Big).$$

If we put $T(\delta_s\otimes u)=\delta_v\otimes t$ for some $v\in G$ and $t\in M$, then (4.1) implies

$$\big(T(\delta_s\otimes u)\big)\bar{\triangleleft}(b^R{\triangleright}w)=(\delta_v\otimes t)\bar{\triangleleft}(b^R{\triangleright}w)=\delta_{(b^R\triangleright w)^{-1}vu}\otimes t{\triangleleft}(b^R{\triangleright}w).$$

Now we need to find t and v. We know that $\langle\delta_v\otimes t\rangle=\langle\delta_s\otimes u\rangle=b=\langle\delta_t\otimes v\rangle^L$, so $b^R=\langle\delta_t\otimes v\rangle$, then we have

$$t\cdot b^R=t{\triangleleft}v\quad\text{and}\quad s\cdot b=s{\triangleleft}u\,,$$

thus $(t\cdot b^R)\cdot b^{RR}=(t{\triangleleft}v)\cdot b^{RR}$, or $\big(t{\triangleleft}\tau(b^R,b^{RR})\big)\cdot(b^R\cdot b^{RR})=(t{\triangleleft}v)\cdot b^{RR}$ which implies

$$t{\triangleleft}\tau(b^R,b^{RR})=(t{\triangleleft}v)\cdot b^{RR}.\tag{4}$$

Now as $b\cdot b^R=e$ and $b^R\cdot b^{RR}=e$, then $(b\cdot b^R)\cdot b^{RR}=b{\triangleleft}\tau(b^R,b^{RR})\cdot(b^R\cdot b^{RR})$ implies that $b^{RR}=b{\triangleleft}\tau(b^R,b^{RR})$. Also we know that

$$b(b^Rb^{RR})=b\tau(b^R,b^{RR})=\big(b{\triangleright}\tau(b^R,b^{RR})\big)\big(b{\triangleleft}\tau(b^R,b^{RR})\big),$$

but on the other hand $b(b^Rb^{RR})=(bb^R)b^{RR}=\tau(b,b^R)b^{RR}$, so by the uniqueness of the factorization $b{\triangleright}\tau(b^R,b^{RR})=\tau(b,b^R)$, hence (4) can be rewritten as $t{\triangleleft}\tau(b^R,b^{RR})=(t{\triangleleft}v)\cdot\big(b{\triangleleft}\tau(b^R,b^{RR})\big)$, which implies that

$$\begin{aligned}t&=\Big((t{\triangleleft}v)\cdot\big(b{\triangleleft}\tau(b^R,b^{RR})\big)\Big){\triangleleft}\tau(b^R,b^{RR})^{-1}\\&=\Big(t{\triangleleft}v\big(\big(b{\triangleleft}\tau(b^R,b^{RR})\big){\triangleright}\tau(b^R,b^{RR})^{-1}\big)\Big)\cdot b\\&=\Big(t{\triangleleft}v\big(b{\triangleright}\tau(b^R,b^{RR})\big)^{-1}\Big)\cdot b=\big(t{\triangleleft}v\tau(b,b^R)^{-1}\big)\cdot b.\end{aligned}$$

Let $s=t{\triangleleft}v\tau(b,b^R)^{-1}$, then $u=\tau(b,b^R)v^{-1}$, or $v=u^{-1}\tau(b,b^R)$ and $t=s{\triangleleft}u$. Therefore,

$$T(\delta_s\otimes u)=\delta_{u^{-1}\tau(b,b^R)}\otimes s{\triangleleft}u.\tag{5}$$

Now we want to show that (3) is satisfied. Start with the grade as

$$\langle(\delta_s \otimes u)\bar{\triangleleft}(b^R{\triangleright}w)\rangle = \langle\delta_s \otimes u\rangle{\triangleleft}(b^R{\triangleright}w) = b{\triangleleft}(b^R{\triangleright}w) = c\,,$$

now to check the action staring with the right hand side of (3) as follows

$$\begin{aligned} T\big((\delta_s \otimes u)\bar{\triangleleft}(b^R{\triangleright}w)\big) &= T\Big(\delta_{s\triangleleft(b\triangleright(b^R\triangleright w))} \otimes \big(b{\triangleright}(b^R{\triangleright}w)\big)^{-1} u(b^R{\triangleright}w)\Big) \\ &= \delta_{(b^R\triangleright w)^{-1}u^{-1}(b\triangleright(b^R\triangleright w))\tau(c,c^R)} \otimes s{\triangleleft}u(b^R{\triangleright}w) \\ &= \delta_{(b^R\triangleright w)^{-1}u^{-1}\tau(b,b^R)w} \otimes s{\triangleleft}u(b^R{\triangleright}w)\,, \end{aligned}$$

the last equality because

$$b{\triangleright}(b^R{\triangleright}w) = \tau(b,b^R)w\tau\big(b{\triangleleft}(b^R{\triangleright}w),(b^R{\triangleleft}w)\big)^{-1} = \tau(b,b^R)w\tau(c,c^R)^{-1}.$$

Finally,

$$\begin{aligned} \big(T(\delta_s \otimes u)\big)\bar{\triangleleft}(b^R{\triangleright}w) &= (\delta_{u^{-1}\tau(b,b^R)} \otimes s{\triangleleft}u)\bar{\triangleleft}(b^R{\triangleright}w) \\ &= \delta_{(b^R\triangleright w)^{-1}u^{-1}\tau(b,b^R)w} \otimes s{\triangleleft}u(b^R{\triangleright}w)\,, \end{aligned}$$

as required. ■

Proposition 4.3 *Let the morphism $T : A \to A^*$ be as defined in proposition (4.2). Then there is an inverse morphism $T^{-1} : A^* \to A$ in the category $\mathcal{C}$ defined by*

$$T^{-1}(\delta_v \otimes t) = \delta_{t\triangleleft v\tau(b,b^R)^{-1}} \otimes \tau(b,b^R)\,v^{-1}\,,$$

for $(\delta_v \otimes t) \in A^*$, where $b = \langle\delta_v \otimes t\rangle$.

Proof. From proposition (4.2) we know that $T(\delta_s \otimes u) = \delta_{u^{-1}\tau(b,b^R)} \otimes s{\triangleleft}u$, where $b = \langle\delta_s \otimes u\rangle$. Put $T(\delta_s \otimes u) = \delta_v \otimes t$, then also $\langle\delta_v \otimes t\rangle = b$ as the morphism T preserves the grade. Then also we can get $t = s{\triangleleft}u$ and $v = u^{-1}\tau(b,b^R)$, which imply that $u = \tau(b,b^R)\,v^{-1}$ and $s = t{\triangleleft}u^{-1} = t{\triangleleft}\,v\tau(b,b^R)^{-1}$. Therefore

$$T^{-1}(\delta_v \otimes t) = \delta_s \otimes u\,,$$

with s and u as defined above. Note that it is automatic that if T is a morphism, with a linear map inverse T^{-1}, then T^{-1} is also a morphism. ■

Proposition 4.4 *Let A be the algebra in the category $\mathcal{C}$. Then the comultiplication Δ on A^* for any element $\alpha = (\delta_u \otimes s)$ in A^* with $u \in G$ and $s \in M$, can be given by*

$$\Delta(\delta_u \otimes s) = \sum_{v \in G} (\delta_v \otimes s) \otimes (\delta_{\tau(a^L, a) v^{-1} u \langle\alpha\rangle^{R^{-1}} a^{L^{-1}} a'} \otimes s \triangleleft v \tau(a^L, a)^{-1}),$$

where $a = \langle \delta_s \otimes v \rangle$ and $a' = \big((\langle\alpha\rangle \cdot a) \triangleleft \tau(a^L, a)^{-1}\big)^R$.

Proof. From Proposition (3.3), we know that

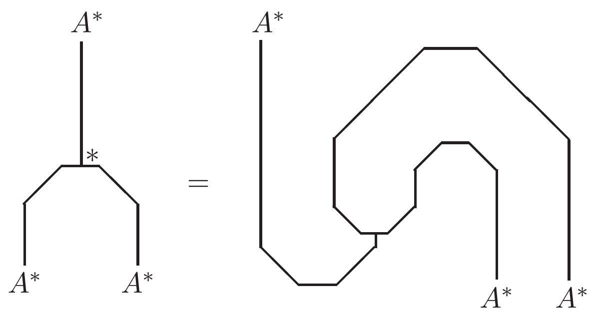

Figure 8

For $\alpha \in A^*$, we follow the above figure from top to bottom and calculate the following:

Put $\mathrm{coev_A}(1) = \beta \otimes \gamma = \beta' \otimes \gamma'$ (we suppress summations as usual) which implies $\langle\beta\rangle \cdot \langle\gamma\rangle = e$ and $\langle\beta'\rangle \cdot \langle\gamma'\rangle = e$. As all parts of the above diagram are morphisms in the category and preserve the grads we should have $\langle\alpha\rangle = \langle\gamma'\rangle \cdot \langle\gamma\rangle$. We start with

$$\alpha \otimes \mathrm{coev_A}(1) = \alpha \otimes (\beta \otimes \gamma).$$

According to the diagram we include $\mathrm{coev_A}(1)$ again to get

$$\alpha \otimes \big((\beta \otimes \mathrm{coev_A}(1)) \otimes \gamma\big) = \alpha \otimes \big((\beta \otimes (\beta' \otimes \gamma')) \otimes \gamma\big).$$

After that we apply the associator inverse Φ^{-1} and then the multiplication to get

$$\alpha \otimes \big(((\beta \bar{\triangleleft} \tau(\langle\beta'\rangle, \langle\gamma'\rangle)^{-1} \otimes \beta') \otimes \gamma') \otimes \gamma\big) \mapsto \alpha \otimes \big((((\beta \bar{\triangleleft} \tau(\langle\beta'\rangle, \langle\gamma'\rangle)^{-1}) \beta') \otimes \gamma') \otimes \gamma\big).$$

Now applying the associator Φ gives $\quad \alpha \otimes \big(((\beta\bar{\triangleleft}\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\beta^{'})\bar{\triangleleft}\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle) \otimes (\gamma^{'} \otimes \gamma)\big)$.

Applying the associator inverse Φ^{-1} again will give

$$\begin{aligned}\big(\alpha\bar{\triangleleft}\tau(\langle\tilde{\beta}\rangle,\langle\gamma^{'}\otimes\gamma\rangle)^{-1}\otimes\tilde{\beta}\,\big)\otimes(\gamma^{'}\otimes\gamma) &= \big(\alpha\bar{\triangleleft}\tau(\langle\tilde{\beta}\rangle,\langle\gamma^{'}\rangle\cdot\langle\gamma\rangle)^{-1}\otimes\tilde{\beta}\,\big)\otimes(\gamma^{'}\otimes\gamma)\\ &= \big(\alpha\bar{\triangleleft}\tau(\langle\tilde{\beta}\rangle,\langle\alpha\rangle)^{-1}\otimes\tilde{\beta}\,\big)\otimes(\gamma^{'}\otimes\gamma)\end{aligned} \tag{6}$$

where

$$\tilde{\beta} = ((\beta\bar{\triangleleft}\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\beta^{'})\bar{\triangleleft}\,\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle), \tag{7}$$

which implies that

$$\langle\tilde{\beta}\rangle = (((\langle\beta\rangle\triangleleft\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\cdot\langle\beta^{'}\rangle)\triangleleft\,\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle). \tag{8}$$

Now we want to show that $\langle\tilde{\beta}\rangle = \langle\alpha\rangle^{L}$, which can be proved as follows

$$\begin{aligned}\langle\tilde{\beta}\rangle\cdot\langle\alpha\rangle = \langle\tilde{\beta}\rangle\cdot(\langle\gamma^{'}\rangle\cdot\langle\gamma\rangle) &= (((\langle\beta\rangle\triangleleft\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\cdot\langle\beta^{'}\rangle)\triangleleft\,\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle)\cdot(\langle\gamma^{'}\rangle\cdot\langle\gamma\rangle)\\ &= (((\langle\beta\rangle\triangleleft\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\cdot\langle\beta^{'}\rangle)\cdot\langle\gamma^{'}\rangle)\cdot\langle\gamma\rangle\\ &= (\langle\beta\rangle\cdot(\langle\beta^{'}\rangle\cdot\langle\gamma^{'}\rangle))\cdot\langle\gamma\rangle = (\langle\beta\rangle\cdot e)\cdot\langle\gamma\rangle = \langle\beta\rangle\cdot\langle\gamma\rangle = e\,.\end{aligned}$$

So if we apply the evaluation map to (6), it can be rewritten as

$$\begin{aligned}&\big(\alpha\bar{\triangleleft}\,\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)^{-1}\,\big)(\tilde{\beta})\,(\gamma^{'}\otimes\gamma) =\\ &\alpha\Big(\tilde{\beta}\bar{\triangleleft}\,\tau(\langle\tilde{\beta}\rangle^{L},\langle\tilde{\beta}\rangle)^{-1}\big(\langle\tilde{\beta}\rangle^{L}\triangleright\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)\big)\tau\big(\langle\tilde{\beta}\rangle^{L}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle),(\langle\tilde{\beta}\rangle^{L}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle))^{R}\big)\,(\gamma^{'}\otimes\gamma).\end{aligned} \tag{9}$$

To make this equation simpler we need to do the following calculations

$$\big(\langle\alpha\rangle^{LL}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)\big)\cdot(\langle\alpha\rangle^{L}\cdot\langle\alpha\rangle) = (\langle\alpha\rangle^{LL}\cdot\langle\alpha\rangle^{L})\cdot\langle\alpha\rangle,$$

which implies that $\quad \langle\alpha\rangle^{LL}\triangleleft\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle) = \langle\alpha\rangle$. Thus we can consider the following

$$\langle\alpha\rangle^{LL}\langle\alpha\rangle^{L}\langle\alpha\rangle = \langle\alpha\rangle^{LL}\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle) = \big(\langle\alpha\rangle^{LL}\triangleright\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)\big)\langle\alpha\rangle,$$

which implies that $\langle\alpha\rangle^{LL}\langle\alpha\rangle^{L} = \tau(\langle\alpha\rangle^{LL},\langle\alpha\rangle^{L}) = \langle\alpha\rangle^{LL}\triangleright\tau(\langle\alpha\rangle^{L},\langle\alpha\rangle)$. Now substituting in (9) gives $\quad \Delta_{A^{*}}(\alpha) = \alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle,\langle\alpha\rangle^{R})\big)\,(\gamma^{'}\otimes\gamma)$, where

$$\tilde{\beta} = ((\beta\bar{\triangleleft}\tau(\langle\beta^{'}\rangle,\langle\gamma^{'}\rangle)^{-1})\beta^{'})\bar{\triangleleft}\,\tau(\langle\gamma^{'}\rangle,\langle\gamma\rangle).$$

From the definition of the coevaluation map we know

$$\mathrm{coev}_A(1) = \sum_{\xi \in \text{ basis of } V} \xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \otimes \hat{\xi},$$

so we put $\beta = \xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1}$ and $\gamma = \hat{\xi}$, and also in the same way we put $\beta' = \eta \,\bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}$ and $\gamma' = \hat{\eta}$. These imply that

$$\langle \beta \rangle = \langle \xi \rangle \triangleleft \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \qquad \text{and} \qquad \langle \gamma \rangle = \langle \hat{\xi} \rangle = \langle \xi \rangle^L,$$

$$\langle \beta' \rangle = \langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \qquad \text{and} \qquad \langle \gamma' \rangle = \langle \hat{\eta} \rangle = \langle \eta \rangle^L.$$

Now we substitute these in $\tilde{\beta}$ and try to simplify it as follows

$$\tilde{\beta} = \Big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \, \tau(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}, \langle \eta \rangle^L)^{-1} (\eta \,\bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}) \Big) \bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \xi \rangle^L).$$

Put $\pi = \big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \, \tau\big(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}, \langle \eta \rangle^L\big)^{-1}\big) \big(\eta \,\bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \big)$, so

$$\pi \bar{\triangleleft} \tau(\langle \eta \rangle^L, \langle \eta \rangle) =$$

$$\Big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \, \tau\big(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}, \langle \eta \rangle^L\big)^{-1} \big((\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}) \triangleright \tau(\langle \eta \rangle^L, \langle \eta \rangle) \big) \Big) \, \eta$$

$$= \Big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \, \tau\big(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}, \langle \eta \rangle^L\big)^{-1} \big(\langle \eta \rangle \triangleright \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \big)^{-1} \Big) \, \eta$$

$$= \big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \big) \, \eta.$$

The last equivalence is due to:

$$\tau\big(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1}, \langle \eta \rangle^L\big) = \big(\langle \eta \rangle \triangleleft \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \big) \, \langle \eta \rangle^L =$$

$$\big(\langle \eta \rangle \triangleright \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \big)^{-1} \, \langle \eta \rangle (\langle \eta \rangle^L \, \langle \eta \rangle)^{-1} \, \langle \eta \rangle^L = \big(\langle \eta \rangle \triangleright \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \big)^{-1}. \quad \text{So}$$

$$\tilde{\beta} = \big(\pi \bar{\triangleleft} \tau(\langle \eta \rangle^L, \langle \eta \rangle) \big) \bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \, \tau(\langle \eta \rangle^L, \langle \xi \rangle^L)$$

$$= \Big(\big(\xi \,\bar{\triangleleft}\, \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1} \big) \, \eta \Big) \bar{\triangleleft}\, \tau(\langle \eta \rangle^L, \langle \eta \rangle)^{-1} \, \tau(\langle \eta \rangle^L, \langle \xi \rangle^L).$$

Now put $\xi = \delta_t \otimes v$, $a = \langle \xi \rangle = \langle \delta_t \otimes v \rangle$, $\eta = \delta_{t'} \otimes v'$, $a' = \langle \eta \rangle = \langle \delta_{t'} \otimes v' \rangle$ and $w = \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1}$, then $\xi \bar{\triangleleft} w = (\delta_t \otimes v) \bar{\triangleleft} w = \delta_{t \triangleleft (a \triangleright w)} \otimes (a \triangleright w)^{-1} v w$. Hence

$$(\delta_{t \triangleleft (a \triangleright w)} \otimes (a \triangleright w)^{-1} v w)(\delta_{t'} \otimes v') = \delta_{t', t \triangleleft v w} \, \delta_{t \triangleleft (a \triangleright w) \tau(a \triangleleft w, a')} \otimes \tau(a \triangleleft w, a')^{-1} (a \triangleright w)^{-1} v w v'.$$

Now put $p = \tau(\langle\eta\rangle^L, \langle\eta\rangle)^{-1}\,\tau(\langle\eta\rangle^L, \langle\xi\rangle^L)$, then we get

$$\tilde{\beta} = \Big(\big(\xi\,\bar{\triangleleft}\,\tau(\langle\xi\rangle^L, \langle\xi\rangle)^{-1}\big)\,\eta\Big)\bar{\triangleleft}\,p =$$

$$\delta_{t',t\triangleleft vw}\,\delta_{t\triangleleft(a\triangleright w)\tau(a\triangleleft w,a')\big(((a\triangleleft w)\cdot a')\triangleright p\big)} \otimes \big(((a\triangleleft w)\cdot a')\triangleright p\big)^{-1}\tau(a\triangleleft w, a')^{-1}(a\triangleright w)^{-1}vwv'p.$$

If we put $q = \tau(\langle\alpha\rangle, \langle\alpha\rangle^R)$, then we get

$$\alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle, \langle\alpha\rangle^R)\big) = \delta_{t',t\triangleleft vw}\,\alpha\Big(\delta_{t\triangleleft(a\triangleright w)\tau(a\triangleleft w,a')\big(((a\triangleleft w)\cdot a')\triangleright p\big)\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big)}$$
$$\otimes\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big)^{-1}\big(((a\triangleleft w)\cdot a')\triangleright p\big)^{-1}\tau(a\triangleleft w, a')^{-1}(a\triangleright w)^{-1}vwv'pq\Big). \tag{10}$$

To make this simpler we do the following:

$$\big(((a\triangleleft w)\cdot a')\triangleright p\big)\big(\big(((a\triangleleft w)\cdot a')\triangleleft p\big)\triangleright q\big) = ((a\triangleleft w)\cdot a')\triangleright pq,$$

and also we have $pq = {a'}^{-1}{a'}^{L^{-1}}{a'}^{L}a^L\langle\alpha\rangle^{-1}\langle\alpha\rangle\langle\alpha\rangle^R = {a'}^{-1}\,a^L\,\langle\alpha\rangle^R$. If we put $F = (a\triangleright w)\tau(a\triangleleft w, a')\big(((a\triangleleft w)\cdot a')\triangleright pq\big)$, then F will be equal to the G-part of the following unique factorization:

$$(a\triangleright w)(a\triangleleft w)a'pq = (a\triangleright w)\tau(a\triangleleft w, a')((a\triangleleft w)\cdot a')pq$$
$$= (a\triangleright w)\tau(a\triangleleft w, a')\big(((a\triangleleft w)\cdot a')\triangleright pq\big)\big(((a\triangleleft w)\cdot a')\triangleleft pq\big),$$

but we also have $(a\triangleright w)(a\triangleleft w)a'pq = awa'pq = aa^{-1}a^{L^{-1}}a'{a'}^{-1}a^L\langle\alpha\rangle^R = \langle\alpha\rangle^R$. So $F = e$, which means that equation (10) can be rewritten as

$$\alpha\big(\tilde{\beta}\bar{\triangleleft}\tau(\langle\alpha\rangle, \langle\alpha\rangle^R)\big) = \delta_{t',t\triangleleft vw}\,\alpha\big(\delta_t\otimes vwv'{a'}^{-1}\,a^L\,\langle\alpha\rangle^R\big).$$

If we put $\alpha = \delta_u\otimes s$ the R.H.S. of the above equation becomes

$$\delta_{t',t\triangleleft vw}\,\mathrm{ev}\big((\delta_\mathrm{u}\otimes \mathrm{s})\otimes(\delta_\mathrm{t}\otimes \mathrm{vwv'}\mathrm{a}'^{-1}\,\mathrm{a^L}\,\langle\alpha\rangle^\mathrm{R})\big) = \delta_{\mathrm{t}',\,\mathrm{t}\triangleleft\mathrm{vw}}\,\delta_{\mathrm{s},\,\mathrm{t}}\,\delta_{\mathrm{u},\,\mathrm{vwv'}\mathrm{a}'^{-1}\,\mathrm{a^L}\,\langle\alpha\rangle^\mathrm{R}},$$

which implies that $t = s$, $t' = s\triangleleft v\,\tau(a^L, a)^{-1}$ and $u = v\tau(a^L, a)^{-1}v'{a'}^{-1}a^L\,\langle\alpha\rangle^R$, or $v' = \tau(a^L, a)v^{-1}u\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a'$. We know that $\langle\alpha\rangle = \langle\delta_s\otimes u\rangle^L$, or $\langle\alpha\rangle^R = \langle\delta_s\otimes u\rangle$, so

$$s\cdot\langle\alpha\rangle^R = s\cdot\langle\delta_s\otimes u\rangle = s\triangleleft u, \quad t\cdot a = t\cdot\langle\delta_t\otimes v\rangle = t\triangleleft v \;\text{ and }\; \mathrm{t}'\cdot\mathrm{a}' = \mathrm{t}'\cdot\langle\delta_{\mathrm{t}'}\otimes\mathrm{v}'\rangle = \mathrm{t}'\triangleleft\mathrm{v}'.$$

To confirm our calculations we prove the last equation substituting by its values as follows:

$$t^{'}\triangleleft v^{'}=s\triangleleft u\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a^{'}=(s\cdot\langle\alpha\rangle^{R})\triangleleft\,\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a^{'}=\big(s\triangleleft(\langle\alpha\rangle^{R}\triangleright z)\big)\cdot(\langle\alpha\rangle^{R}\triangleleft z),\qquad(11)$$

where $z=\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a^{'}$. But we know that

$$\langle\alpha\rangle^{R}z=(\langle\alpha\rangle^{R}\triangleright z)(\langle\alpha\rangle^{R}\triangleleft z)=\langle\alpha\rangle^{R}\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a^{'}=a^{L^{-1}}a^{'}=\tau(a^{L^{-1}},a^{'})(a^{L^{-1}}\cdot a^{'}).$$

So by the uniqueness of the factorization we get $(\langle\alpha\rangle^{R}\triangleright z)=\tau(a^{L^{-1}},a^{'})$ and $(\langle\alpha\rangle^{R}\triangleleft z)=(a^{L^{-1}}\cdot a^{'})$, then substituting in (11) gives

$$\begin{aligned}t^{'}\triangleleft v^{'}&=\big(s\triangleleft\tau(a^{L^{-1}},a^{'})\big)\cdot(a^{L^{-1}}\cdot a^{'})=(s\cdot a^{L^{-1}})\cdot a^{'}\\&=(s\cdot a\tau(a^{L},a)^{-1})\cdot a^{'}=(s\triangleleft v\tau(a^{L},a)^{-1})\cdot a^{'}=t^{'}\cdot a^{'}.\end{aligned}$$

Finally, we calculate $a^{'}$ which we do as the following:

$$\langle\alpha\rangle\cdot a=(a^{'L}\cdot a^{L})\cdot a=(a^{'L}\triangleleft\tau(a^{L},a))\cdot(a^{L}\cdot a)=a^{'L}\triangleleft\tau(a^{L},a),$$

so $a^{'L}=(\langle\alpha\rangle\cdot a)\triangleleft\tau(a^{L},a)^{-1}$, or $a^{'}=\big((\langle\alpha\rangle\cdot a)\triangleleft\tau(a^{L},a)^{-1}\big)^{R}$. Therefore,

$$\Delta(\delta_u\otimes s)=\sum_{v}(\delta_v\otimes s)\otimes(\delta_{\tau(a^{L},a)v^{-1}u\,\langle\alpha\rangle^{R^{-1}}a^{L^{-1}}a^{'}}\otimes s\triangleleft v\tau(a^{L},a)^{-1}).\qquad\blacksquare$$

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