SOME NEW DIFFERENCE SEQUENCE SPACES OF INVARIANT MEANS

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Abstract

In this paper, we define the sequence spaces $w_\sigma(u, \Delta)$, $[w]_\sigma(u, \Delta)$ and $[w_\sigma](u, \Delta)$, where for any sequence $x = (x_k)$, the difference sequence Δx is given by $\Delta x = (\Delta x_k)_{k=1}^\infty = (x_k - x_{k-1})_{k=1}^\infty$. We also study some inclusion relations and properties of the above mentioned spaces. These are generalizations of those defined and studied by Mursaleen, Gaur and Chishti in 1997 and some others.

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1 Introduction

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ respectively normed by

$$\|x\| = \sup_k |x_k|.$$

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or σ -mean if and only if :

- (i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k ,

- (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
- (iii) $\varphi(x_{\sigma(k)}) = \varphi(x)$ for all $x \in l_\infty$.

In case σ is the translation mapping $k \rightarrow k + 1$, a σ -mean is often called a Banach limit (see [1]) and c_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergence sequences (see [4]). For $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$.

It can be shown that (see [9])

$$c_\sigma = \{x \in l_\infty : \lim_{m \rightarrow \infty} t_{km}(x) = L, \text{ uniformly in } m\}, \quad (1)$$

$L = \sigma\text{-}\lim x$, where

$$t_{km}(x) = (k+1)^{-1}[x_m + Tx_m + \dots + T^k x_m], \quad t_{-1,m} = 0 \quad (2)$$

A σ -mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if for all $n \geq 0, j \geq 1, \sigma^j(n) \neq n$ (see [6]).

A bounded sequence $x = (x_k)$ is said to be σ -convergent if and only if $x \in c_\sigma$ such that $\sigma^j(n) \neq n$ for all $n \geq 0, j \geq 1$. A bounded sequence $x = (x_k)$ is said to be strongly σ -convergent to a number L if and only if

$$(|x_k - l|) \in c_\sigma$$

with σ -limit zero (see [7]).

If $\sigma(n) = n + 1$, the space $[c_\sigma]$ of strongly σ -convergent sequences is the same as $[\widehat{c}]$, the space of strongly almost convergent sequences (see [7]). Note that $[c_\sigma] \subset c_\sigma \subset l_\infty$.

If p is any sublinear functional on l_∞ , then we write $\{l_\infty, p\}$ to denote the set of all linear functionals φ on l_∞ , such that $p(x) \geq \varphi(x)$, for all $x \in l_\infty$. A sublinear functional p generates σ -means if $\phi \in \{l_\infty, p\}$ implies ϕ is a σ -mean. We say that p dominates σ -means if ϕ is a σ -mean implies that $\phi \in \{l_\infty, p\}$. If p both generates and dominates σ -means, then $\{l_\infty, p\}$ is the set of all σ -means, and the σ -core of x is defined as $[-p(-x), p(x)]$ (see [5]).

Define a sublinear functional $V : l_\infty \rightarrow l_\infty$ by

$$V(x) = \sup_i \lim_k t_{ki}(x), \quad (3)$$

where $t_{ki}(x)$ is defined in (1.2). Then V generates and dominates σ -means and the σ -core of x is given by $[-V(-x), V(x)]$.

Using the concept of almost convergence, the sequence spaces w , $[w]$, and $[w_1]$ have been introduced and examined by Das and Sahoo (see [3]). After that, Mursaleen, Gaur and Chishti [8] generalized those spaces of Das and Sahoo [3] by introducing and examining the some new sublinear functionals and related sequence spaces involving σ -means as follows :

Let us define the following sublinear functionals on l_∞ :

$$q_\sigma(x) = \overline{\lim}_k \sup_i t_{ki}(x), \quad \phi_\sigma(x) = \overline{\lim}_n \sup_i \frac{1}{n+1} \sum_{k=0}^n t_{ki}(x),$$

$$\Psi_\sigma(x) = \overline{\lim}_n \sup_i \frac{1}{n+1} \sum_{k=0}^n |t_{ki}(x)|, \quad \theta_\sigma(x) = \overline{\lim}_n \sup_i \frac{1}{n+1} \sum_{k=0}^n t_{ki}(|x|).$$

By (1.1), it is trivial that

$$c_\sigma = \{x \in l_\infty : \varphi(x) = s \text{ for all } \varphi \in \{l_\infty, q_\sigma\}.$$

Let M_σ denote the set of all invariant means φ . Mursaleen, Gaur and Chishti [8] defined and studied the sequence spaces : w_σ , $[w]_\sigma$, and $[w_\sigma]$.

2 Some new sequence spaces and sublinear functionals

If $u = (u_k)$ is any sequence such that $u_k \neq 0 (k = 1, 2, 3, \dots)$, then we further generalize the last mentioned spaces and define the following sequence spaces :

$$w_\sigma(u, \Delta) = \{x : \limsup_n \sup_i \frac{1}{n+1} \sum_{k=0}^n t_{km}(u\Delta x - se) = 0, \text{ for some } s \in \mathbb{C}\},$$

$$[w]_\sigma(u, \Delta) = \{x : \lim_n \sup_i \frac{1}{n+1} \sum_{k=0}^n |t_{km}(u\Delta x - se)| = 0, \text{ for some } s \in \mathbb{C}\},$$

and

$$[w_\sigma](u, \Delta) = \{x : \limsup_n \sup_i \frac{1}{n+1} \sum_{k=0}^n t_{km}(|u\Delta x - se|) = 0, \text{ for some } s \in \mathbb{C}\},$$

and we call these spaces as the spaces of $w_\sigma(u, \Delta)$ -convergent, $[w]_\sigma(u, \Delta)$ -convergent, and $[w_\sigma](u, \Delta)$ -convergent sequences respectively. If $u = e$, and $\Delta x_k = x_k$, for all k , then these spaces reduce to those defined and studied by Mursaleen, Gaur and Chishti [8]. We prove the following theorems :

Theorem 2.1. The sublinear functional ϕ_σ on l_∞ both dominates and generates σ -means, that is

$$\phi_\sigma(u\Delta x) = q_\sigma(u\Delta x) \text{ for all } x \in l_\infty.$$

Proof. It is easy to see that $\phi_\sigma(u\Delta x) \leq q_\sigma(u\Delta x)$ for $x \in l_\infty$. Hence ϕ_σ generates σ -means. Now, for $\varphi \in M_\sigma$, we have

$$\varphi(u\Delta x) = \frac{1}{n+1} \varphi\left(\sum_{k=0}^n t_{ki}(u\Delta x)\right) \leq \sup_i \frac{1}{n+1} \sum_{k=0}^n t_{ki}(u\Delta x)$$

which implies that $\varphi(u\Delta x) \leq \phi_\sigma(u\Delta x)$ for all $x \in l_\infty$, and so ϕ_σ dominates σ -means. Therefore ϕ_σ dominates and generates σ -means and $\phi_\sigma(u\Delta x) = q_\sigma(u\Delta x)$ for all $x \in l_\infty$. Hence $\{l_\infty, \phi_\sigma\} = M_\sigma$.

If $f(u\Delta x - se) = 0$ for all $f \in \{l_\infty, \Psi_\sigma\}$, then we say that x is $\Psi_\sigma(u\Delta x)$ -convergent to s . Similarly, we define the $\theta_\sigma(u\Delta x)$ -convergent sequences.

Theorem 2.2. We have

- (i) $[w_\sigma](u, \Delta) \cap l_\infty = \{x : \Psi_\sigma(u\Delta x - se) = 0, \text{ for some } s \in \mathbb{C}\},$
 $= \{x : f(u\Delta x - se) = 0, \text{ for all } f \in \{l_\infty, \Psi_\sigma\}\}.$
- (ii) $[w]_\sigma(u, \Delta) \cap l_\infty = \{x : \theta_\sigma(u\Delta x - se) = 0, \text{ for some } s \in \mathbb{C}\},$
 $= \{x : f(u\Delta x - se) = 0, \text{ for all } f \in \{l_\infty, \theta_\sigma\}\}.$

Proof. By Hahn-Banach Theorem, $\{l_\infty, \Psi_\sigma\}$ is non-empty. If $f \in \{l_\infty, \Psi_\sigma\}$, then we have

$$-\Psi_\sigma(-u\Delta x) \leq f(u\Delta x) \leq \Psi_\sigma(u\Delta x), \quad x \in l_\infty$$

or equivalently

$$\Psi_\sigma(-u\Delta x + se) \leq f(u\Delta x - se) \leq \Psi_\sigma(u\Delta x - se).$$

Now $f(u\Delta x - se) = 0$ if and only if $\Psi_\sigma(u\Delta x - se) = -\Psi_\sigma(-u\Delta x + se) = 0$ (see [2], Theorem 6). But since by definition $\Psi_\sigma(u\Delta x) = \Psi_\sigma(-u\Delta x)$, it follows that $f(u\Delta x) = s$ for $f \in \{l_\infty, \Psi_\sigma\}$ if and only if $\Psi_\sigma(u\Delta x - se) = 0$. It is easy to verify that $\Psi_\sigma(u\Delta x - se) = 0$ is equivalent to the fact that

$$\frac{1}{n+1} \sum_{k=0}^n |t_{ki}(u\Delta x - se)| \rightarrow 0, (n \rightarrow \infty, \text{ uniformly in } i).$$

This completes the proof of (i). The proof of (ii) follows on the same lines.

It is evident from Theorem 2.2 that $[w_\sigma](u, \Delta) \cap l_\infty$ and $[w]_\sigma(u, \Delta) \cap l_\infty$ are the set of all $\Psi_\sigma(u\Delta x)$ -convergent sequences and all $\theta_\sigma(u\Delta x)$ -convergent sequences respectively.

3 Inclusion relations

In this section, we prove some inclusion relations for these newly introduced sequence spaces.

Theorem 3.1. Let $\sigma^j(n) \neq n$ for all $n \geq 0, j \geq 1$. Then

$$[w_\sigma](u, \Delta) \subset [w]_\sigma(u, \Delta) \subset w_\sigma(u, \Delta).$$

Proof. Let $x \in [w_\sigma](u, \Delta)$ and $x \rightarrow s[w_\sigma](u, \Delta)$. Then $t_{ni}(|u\Delta x - se|) \rightarrow 0$, as $n \rightarrow \infty$, uniformly in i , hence its arithmetic mean also converges to s as $n \rightarrow \infty$, uniformly in i . This proves that $x \in [w]_\sigma(u, \Delta)$ and $x \rightarrow s[w_\sigma](u, \Delta)$. Also since

$$\frac{1}{m+1} \left| \sum_{n=0}^m t_{ni}(u\Delta x - se) \right| \leq \frac{1}{m+1} \left| \sum_{n=0}^m t_{ni}(u\Delta x - se) \right| \leq \frac{1}{m+1} \left| \sum_{n=0}^m t_{ni}(u\Delta x - se) \right|.$$

It follows that $[w_\sigma](u, \Delta) \subset [w]_\sigma(u, \Delta) \subset w_\sigma(u, \Delta)$.

Theorem 3.2. $X \in s[w]_\sigma(u, \Delta)$ if and only if

- (i) $x \rightarrow s[w_\sigma](u, \Delta)$,
- (ii) $\frac{1}{m+1} \sum_{n=0}^m |t_{ni}(u\Delta x - se) - D_{ni}(u\Delta x - se)| \rightarrow 0$, as $m \rightarrow \infty$, uniformly in i , where

$$D_{ni} = D_{ni}(x) = \frac{1}{m+1} \sum_{n=0}^m t_{ni}(x), \quad m \geq 1.$$

Proof. Suppose $x \in s[w_\sigma](u, \Delta)$. Now

$$\begin{aligned}
\frac{1}{m+1} \sum_{n=0}^m |t_{ni}(u\Delta x - se) - D_{ni}(u\Delta x - se)| \\
\leq \frac{1}{m+1} \sum_{n=0}^m |t_{ni}(u\Delta x - se)| + \frac{1}{m+1} \sum_{n=0}^m |D_{ni}(u\Delta x - se)| \\
\leq \sum_1 + \sum_2,
\end{aligned}$$

, say. But by the hypothesis, $\sum_1 = o(1)$ as $m \rightarrow \infty$, uniformly in i . The conclusion (i) implies that $D_{ni} \rightarrow 0$, as $n \rightarrow \infty$, uniformly in i , and hence $\sum_2 = o(1)$ as $m \rightarrow \infty$, uniformly in i .

The converse part immediately follows from

$$\begin{aligned}
\frac{1}{m+1} \sum_{n=0}^m |t_{ni}(u\Delta x - se)| \leq \frac{1}{m+1} \sum_{n=0}^m |t_{ni}(u\Delta x - se) - D_{ni}(u\Delta x - se)| \\
+ \frac{1}{m+1} \sum_{n=0}^m |D_{ni}(u\Delta x - se)|.
\end{aligned}$$

This completes the proof.

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