International Mathematical Forum, 1, 2006, no. 9, 445-453

Intuitionistic Fuzzy Closed Ideals in BCI-algebras[1]

M. Akram

Punjab University College of Information Technology
University of the Punjab, Old Campus
Lahore-54000, Pakistan
m.akram@pucit.edu.pk

Abstract

In this paper, we apply the concept of an intuitionistic fuzzy set to closed ideals in BCI-algebras. The notion of an intuitionistic fuzzy closed ideal of a BCI-algebra is introduced, and some related properties are investigated. Characterizations of an intuitionistic fuzzy closed ideal are also given.

Keywords: IF closed ideals, upper(lower)-level cut, fuzzy characteristic

Mathematics Subject Classification: 06F35, 03G25

1 Introduction

The idea of *Intuitionistic Fuzzy Sets*(briefly, IFSs) was proposed by K. T. Atanassov in [1, 2], as a generalization of well-known notion of fuzzy sets [8]. Fuzzy sets give a degree of membership of an element in a given set, while intuitionistic fuzzy sets(IFSs) give both degrees of membership and of non-membership. Both degrees belong to the interval $[0, 1]$, and their sum should not exceed 1.

IFSs have also been defined by G. Takeuti and S. Titanti in [4]. G. Takeuti and S. Titanti however considered intuitionistic fuzzy logic in the narrow sense and derived a set theory from logic which they called intuitionistic fuzzy set theory.

Y. B. Jun [7]introduced the notion of closed fuzzy ideals in BCI-algebras and discussed their properties. In this paper, we apply the concept of an intuitionistic fuzzy set to closed ideals in BCI-algebras. The notion of an intuitionistic fuzzy closed ideal of a BCI-algebra is introduced, and some related properties

[1]This work was supported by PUCIT.

are investigated. Characterizations of an intuitionistic fuzzy closed ideal are also given.

2 Preliminaries

Definition 2.1. An algebra $(X; *, 0)$ of type (2, 0) is called a *BCI-algebra* if , for all x, y, $z \in X$ the following axioms hold:

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$.

In BCI-algebras, the following hold:

(5) $(x * 0) = x$.

(6) $(x * y) * z = (x * z) * y$.

(7) $0 * (y * x) = (0 * y) * (0 * x)$.

Definition 2.2. A subset I of a BCI-algebra $(X; *, 0)$ is called an *ideal* of X if for any $x, y \in X$:

(i) $0 \in I$,

(ii) $x * y$ and $y \in$ I$\Rightarrow x \in$ I.

Definition 2.3. An ideal I is called closed of X if $0 * x \in I$, $\forall\ x \in I$.

Definition 2.4. Let X be a non-empty set. A *fuzzy* (sub)set μ of the set X is a mapping $\mu : X \to [0, 1]$. The *complement* of a fuzzy set μ of a set X is denoted by $\overline{\mu}$ and defined by $\overline{\mu}(x) = 1 - \mu(x)$, $\forall x \in X$.

Definition 2.5. Let μ and λ be the fuzzy sets of a set X. For $s, t \in [0, 1]$, the set $U(\mu, s) = \{x \in X : \mu(x) \geq s\}$ is called *upper level* of μ. The set $L(\lambda, t) = \{x \in X : \lambda(x) \leq t\}$ is called *lower level* of λ.

Definition 2.6. An Intuitionistic Fuzzy Set (IFS) in a universe X is an object of the form $A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$, where the functions $\mu : X \to [0, 1]$ and $\lambda : X \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

3 Intuitionistic fuzzy closed ideals

Definition 3.1. An IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X is called an *intuitionistic fuzzy ideal* of X if it satisfies the following inequalities:

(IF1) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$,

(IF2) $\mu_A(x) \geq \min\{\mu_A(x * y), \mu_A(y)\}$,

(IF3) $\lambda_A(x) \leq \max\{\lambda_A(x * y), \lambda_A(y)\}$

for all x, $y \in X$.

Definition 3.2. An IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X is called an *intuitionistic fuzzy closed ideal(briefly, IFC-ideal)* of X if it satisfies (IF2), (IF3) and the following:

(IF4) $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$

for all $x \in X$.

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

We define an IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X by $\mu_A(0) = 0.8$, $\mu_A(a) = 0.5$, $\mu_A(b) = \mu_A(c) = 0.3$, $\lambda_A(0) = 0.1$, $\lambda_A(a) = 0.3$, and $\lambda_A(b) = \lambda_A(c) = 0.4$. By routine calculations, we know that $A = \langle X, \mu_A, \lambda_A \rangle$ is an IFC-ideal of X.

Proposition 3.4. *Every IFC-ideal is an intuitionistic fuzzy ideal.*

Proof. Routine. □

The converse of Proposition 3.4 is not true in general as seen in the following example.

Example 3.5. Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

We define an IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X by $\mu_A(0) = 0.8$, $\mu_A(a) = 0.5$, $\mu_A(b) = \mu_A(c) = 0.3$, $\lambda_A(0) = 0.1$, $\lambda_A(a) = 0.3$, and $\lambda_A(b) = \lambda_A(c) = 0.4$. we know that $A = \langle X, \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy ideal of X. But it is not an IFC-ideal of X since $\mu_A(0 * a) \ngeq \mu_A(a)$ and $\lambda_A(0 * a) \nleq \lambda_A(a)$.

Proposition 3.6. *If an IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X is an intuitionistic fuzzy closed ideal, then $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$, for all $x \in X$.*

Proof. Straightforward. □

Proposition 3.7. *If an IFS $A = \langle X, \mu_A, \lambda_A \rangle$ in X is an intuitionistic fuzzy subalgebra, then $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$, for all $x \in X$.*

Proof. For any $x \in X$, we have
$\mu_A(0 * x) \geq \min\{\mu_A(0), \mu_A(x)\} = \min\{\mu_A(x * x), \mu_A(x)\}$
$\geq \min\{min\{\mu_A(x), \mu_A(x)\}, \mu_A(x)\} = \mu_A(x)$ and $\lambda_A(0*x) \leq \max\{\lambda_A(0), \lambda_A(x)\} =$ $\max\{\lambda_A(x * x), \lambda_A(x)\} \leq \max\{\max\{\lambda_A(x), \lambda_A(x)\}, \lambda_A(x)\} = \lambda_A(x)$.This completes the proof. □

Corollary 3.8. *Every intuitionistic fuzzy subalgebra satisfying (IF2)and (IF3) is an intuitionistic fuzzy closed ideal.*

Theorem 3.9. *An intuitionistic fuzzy set $A = \langle X, \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy closed ideal of X if and only if the non-empty upper s-level cut $U(\mu_A, s)$ and the non-empty lower t-level cut $L(\lambda_A, t)$ are closed ideals of X for every s, $t \in [0, 1]$.*

Proof. Suppose that $A = \langle X, \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy closed ideal of X. For $s \in [0, 1]$, obviously, $0 * x \in U(\mu_A, s)$. Let x, $y \in X$ be such that $x*y \in U(\mu_A, s)$ and $y \in U(\mu_A, s)$, then $\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\} \geq s$. For $t \in [0, 1]$, obviously, $0*x \in L(\lambda_A, t)$. Let x, $y \in X$ be such that $x * y \in L(\lambda_A, t)$ and $y \in L(\lambda_A, t)$, then $\lambda_A(x) \leq \max\{\lambda_A(x * y)), \lambda_A(y)\} \leq t$. Hence $U(\mu_A, s)$ and $L(\lambda_A, t)$ are closed ideals of X.
Conversely, Assume that each non-empty level subsets $U(\mu_A, s)$ and $L(\lambda_A, t)$ are closed ideals of X. For any $x \in X$, let $\mu_A(x) = s$ and $\lambda_A(x) = t$. Then $x \in U(\mu_A, s)$ and $L(\lambda_A, t)$. Since 0*x $\in U(\mu_A, s) \cap L(\lambda_A, t)$, it follows that $\mu_A(0 * x) \geq s = \mu_A(x)$ and $\lambda_A(0 * x) \leq t = \lambda_A(x)$, for all $x \in X$. If there exist x_1, $y_1 \in X$ such that $\mu_A(x_1) \leq \min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}$, then by taking $s_0 := \frac{1}{2}[\mu_A(x_1) + \min\{\mu_A(x_1 * y_1), \mu_A(y_1)\}]$. It follows that $x_1 * y_1 \in U(\mu_A, s_0)$ and $y_1 \in U(\mu_A, s_0)$, but $x_1 \notin U(\mu_A, s_0)$, which is contradiction. Hence $U(\mu_A, s_0)$ is not closed ideal of X. Finally, suppose that there exist a, $b \in X$ such that $\lambda_A(a) > \max\{\lambda_A(a * b), \lambda_A(b)\}$. Taking $t_0 := \frac{1}{2}[\mu_A(a) + \max\{\mu_A(a * b)), \mu_A(b)\}]$. It follows that $a * b \in L(\lambda_A, t_0)$ and $b \in L(\lambda_A, t_0)$, but $a \notin L(\lambda_A, t_0)$, which is contradiction. Hence $L(\lambda_A, t_0)$ is not closed ideal of X. □

Theorem 3.10. *Let B be a non-empty subset of X and $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in X defined by*

$$\mu_A(x) := \begin{cases} s_0 & \text{if } x \in B, \\ s_1 & \text{otherwise.} \end{cases} \text{ and } \lambda_A(x) := \begin{cases} t_0 & \text{if } x \in B, \\ t_1 & \text{otherwise.} \end{cases}$$

for all $x \in X$ and $s_i, t_i \in [0,1]$ such that $s_0 > s_1$, $t_0 < t_1$ and $s_i + t_i \leq 1$ for $i = 0, 1$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy closed ideal of X if and only if $B = (\mu_B, \lambda_B)$ is a closed ideal of X. Moreover, $U(\mu_A; s_0) = B = L(\lambda_A; t_0)$.

Proof. Suppose that B is a closed ideal of X. Note that $U(\mu_A; s_1) = X = L(\lambda_A; t_1)$ and $U(\mu_A; s_0) = X = L(\lambda_A; t_0)$, which are closed ideals of X. Using Theorem 3.9, we know that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy closed ideals of X.
The converse part is straightforward. □

Corollary 3.11. *Let χ_H be a characteristic function of X. Then the intuitionistic fuzzy set $\chi_H = (\mu_H, \neg\mu_H)$ is an intuitionistic fuzzy closed ideals of X.*

Proof. Straightforward. □

Definition 3.12. An intuitionistic fuzzy closed ideal $A = (\mu_A, \lambda_A)$ of a BCI-algebra X is called an *intuitionistic fuzzy characteristic* if $\mu_A^f(x) = \mu_A(x)$ and $\lambda_A^f(x) = \lambda_A(x)$, for all $x \in X$ and $f \in \mathrm{Aut}(X)$.

Lemma 3.13. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy closed ideal of a BCI-algebra X and let $x \in X$. Then $\mu_A(x) = t$, $\lambda_A(x) = s$ if and only if $x \in U(\mu_A; t)$, $x \notin U(\mu_A; s)$ and $x \in L(\lambda_A, s)$, $x \notin L(\lambda_A, t)$, for all $s > t$.*

Proof. Straightforward. □

Theorem 3.14. *For an intuitionistic fuzzy closed ideal $A = (\mu_A, \lambda_A)$ of a BCI-algebra X, the following are equivalent:*

(i) *A is fuzzy characteristic.*

(ii) *each level closed ideal of A is characteristic.*

Proof. Suppose that A is fuzzy characteristic and Let $t \in Im(\mu_A)$, $f \in \mathrm{Aut}(X)$ and $x \in U(\mu_A; t)$. Then $\mu_A^f(x) = \mu_A(x) \geq t$
$\Rightarrow \mu_A(f(x)) \geq t$
$\Rightarrow f(x) \in U(\mu_A; t)$.
Likewise, $f(x) \in L(\lambda_A; s)$.
Thus $f(U(\mu_A; t)) \subseteq U(\mu_A; t)$ and $f(L(\lambda_A; s)) \subseteq L(\lambda_A; s)$.

Let $x \in U(\mu_A; t)$ and $y \in X$ such that $f(y) = x$. Then
$\mu_A(y) = \mu_A^f(y) = \mu_A(f(y)) = \mu_A(x) \geq t$
$\Rightarrow y \in U(\mu_A; t)$ so that $x = f(y) \in U(\mu_A; t)$.
Similarly, $y \in L(\lambda_A; s)$ so that $x = f(y) \in L(\lambda_A; s)$.
Consequently, $U(\mu_A; t) \subseteq f(U(\mu_A; t))$ and $L(\lambda_A; s) \subseteq f(L(\lambda_A; s))$.
Hence $f(U(\mu_A; t)) = U(\mu_A; t)$ and $f(L(\lambda_A; s)) = L(\lambda_A; s)$,
i.e., $U(\mu_A; t)$ and $L(\lambda_A; s)$ are characteristic.
Conversely, suppose that each level closed ideal of A is characteristic and let $x \in X$,
$f \in Aut(X)$ and $\mu_A(x) = t$, $\lambda_A(x) = s$. Then, by virtue of Lemma 3.13, $x \in U(\mu_A; t)$, $x \notin U(\mu_A; s)$ and $x \in L(\lambda_A; s)$, $x \notin L(\lambda_A; t)$, for all $s > t$. It follows from the assumption that $f(x) \in f(U(\mu_A; t)) = U(\mu_A; t)$ and $f(x) \in f(L(\lambda_A; s)) = L(\lambda_A; s)$, so that $\mu_A^f(x) = \mu_A(f(x)) \geq t$ and $\lambda_A^f(x) = \lambda_A(f(x)) \leq s$. Let $s_1 = \mu_A^f(x)$, $s_2 = \lambda_A^f(x)$ and assume that $s_1 > t$, $s_2 > s$. Then $f(x) \in U(\mu_A; s_1) = f(U(\mu_A; s_1))$, $f(x) \in L(\lambda_A, s_2) = f(L(\lambda_A; s_2))$ which implies from the injectivity of f that $x \in U(\mu_A; s_1)$, $x \in L(\mu_A; s_2)$, a contradiction. Hence $\mu_A^f(x) = \mu_A(x)$ and $\lambda_A^f(x) = \lambda_A(x)$ showing that A is fuzzy characteristic. □

Theorem 3.15. *Let $A = (\mu_A, \lambda_A)$ be a closed ideal of X with the finite image. Then every descending chain of closed ideals of X terminates at finite step.*

Proof. Suppose that there exists a strictly descending chain $G_0 \supset G_1 \supset G_2 \supset \cdots$ of closed ideals of X which does not terminate at finite step. Define an IFS $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) := \begin{cases} \frac{n}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

$$\lambda_A(x) := \begin{cases} \frac{1}{n+1} & \text{if } x \in G_n \setminus G_{n+1}, n = 0, 1, 2, \cdots, \\ 0 & \text{if } x \in \bigcap_{n=0}^{\infty} G_n, \end{cases}$$

where $G_0 = X$. We prove that $A = (\mu_A, \lambda_A)$ is a closed ideal of X. It is easy to show that $A = (\mu_A, \lambda_A)$ satisfies (IF4). Let $x, y \in X$. Assume that $x * y \in G_n \setminus G_{n+1}$ and $y \in G_k \setminus G_{k+1}$ for $n = 0, 1, 2, \cdots$; $k = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $x * y$ and $y \in G_n$, so $x \in G_n$ because G_n is a closed ideal of X. Hence

$$\mu_A(x) \geq \tfrac{n}{n+1} = \min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq \tfrac{1}{n+1} = \max\{\lambda_A(x * y), \lambda_A(y)\}.$$

If $x * y, y \in \bigcap_{n=0}^{\infty} G_n$, then $x \in \bigcap_{n=0}^{\infty} G_n$. Thus

$$\mu_A(x) = 1 = \min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) = 0 = \max\{\lambda_A(x * y), \lambda_A(y)\}.$$

If $x * y \notin \bigcap_{n=0}^{\infty} G_n$ and $y \in \bigcap_{n=0}^{\infty} G_n$, then there exists $k \in \mathbb{N}$ such that $x * y \in G_k \setminus G_{k+1}$. It follows that $x \in G_k$ so that

$$\mu_A(x) \geq \tfrac{k}{k+1} = \min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq \tfrac{1}{k+1} = \max\{\lambda_A(x * y), \lambda_A(y)\}.$$

Finally suppose that $x * y \in \bigcap_{n=0}^{\infty} G_n$ and $y \notin \bigcap_{n=0}^{\infty} G_n$. Then $y \in G_r \setminus G_{r+1}$ for some $r \in \mathbb{N}$. Hence $x \in G_r$, and so

$$\mu_A(x) \geq \tfrac{r}{r+1} = \min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq \tfrac{1}{r+1} = \max\{\lambda_A(x * y), \lambda_A(y)\}.$$

Consequently, we conclude that $A = (\mu_A, \lambda_A)$ is a closed ideal of X and $A = (\mu_A, \lambda_A)$ has infinite number of different values, which is a contradiction. This completes the proof. □

Theorem 3.16. *The following are equivalent:*

(i) *every ascending chain of closed ideals of X terminates at finite step.*

(ii) *the set of values of any intuitionistic fuzzy closed ideals is a well-ordered subset of [0, 1].*

Proof. Let A be an intuitionistic fuzzy closed ideals of X. Suppose that the set of values of A is not a well-ordered subset of $[0, 1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\mu_A(x) = \alpha_n$. Let $B_n := \{x \in X | \mu_A(x) \geq \alpha_n\}$. Then

$$B_1 \subset B_2 \subset B_3 \subset \cdots$$

is a strictly ascending chain of closed ideals of X which is not terminating. This is a contradiction. If there exists a strictly increasing sequence $\{\beta_n\}$ such that $\lambda_A(x) = \beta_n$. Let $G_n := \{x \in X | \lambda_A(x) \leq \beta_n\}$. Then

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

is a strictly ascending chain of closed ideals of X which is not terminating. This is a contradiction.
Conversely, suppose that there exists a strictly ascending chain

$$G_1 \subset G_2 \subset G_3 \subset \cdots - - - - - (*)$$

of closed ideals of X which does not terminate at finite step. Define an IFS $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) := \begin{cases} \frac{1}{k} & \text{where } k = min\{n \in \mathbb{N} | x \in G_n\}, \\ 0 & \text{if } x \notin G_n. \end{cases}$$

$$\lambda_A(x) := \begin{cases} \frac{1}{k} & \text{where } k = max\{n \in \mathbb{N} | x \in G_n\}, \\ 1 & \text{if } x \in G_n. \end{cases}$$

where $G = \bigcup_{n \in \mathbb{N}} G_n$. We prove that $A = (\mu_A, \lambda_A)$ is a closed ideals of X. It is easy to show that A satisfies (IF4). Let x, $y \in X$. If $x * y$, $y \in G_n \backslash G_{n-1}$ for $n = 2, 3, \cdots$, then $x \in G_n$. It follows that

$$\mu_A(x) \geq \frac{1}{n} = min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq \frac{1}{n} = max\{\lambda_A(x * y), \lambda_A(y)\}.$$

Assume that $x * y \in G_n$ and $y \in G_n \backslash G_m$ for all $m < n$. Since A is a closed ideals of X, therefore, $x \in G_n$. Thus

$$\mu_A(x) \geq \frac{1}{n} \geq \frac{1}{m+1} \geq \mu_A(y),$$

$$\lambda_A(x) \leq \frac{1}{n} \leq \frac{1}{m+1} \leq \lambda_A(y).$$

and hence

$$\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq max\{\lambda_A(x * y), \lambda_A(y)\}.$$

Similarly, for the case $x * y \in G_n \backslash G_m$ and $y \in G_n$, we have

$$\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\},$$

$$\lambda_A(x) \leq max\{\lambda_A(x * y), \lambda_A(y)\}.$$

Hence A is a closed ideals of X. Since the chain (*)is not terminating, A has strictly descending sequence of values. This contradicts that the value set of any intuitionistic fuzzy closed ideal is well-ordered. □

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1)(1983), 87-96.

[2] K. T. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **61**(2)(1994), 137-142.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets. Theory and applications*, Studies in Fuzziness and Soft Computing, **35**. Heidelberg; Physica-Verl ag 1999.

[4] G. Takeuti S. Titants , *Intuitionistic fuzzy logic and Intuitionistic fuzzy set theory* , Journal of Symbolic Logic, **49**(1984), 851-866.

[5] Y. B. Jun and K. H. Kim, *Intuitionistic fuzzy ideals of BCK-algebras*, Internat. J. Math. and Math. Sci. **24**(12) (2000), 839-849.

[6] Y. B. Jun and W. A. Dudek, *Intuitionistic fuzzy closed ideals of BCH-algebras*, J. of Fuzzy Math., **9**(3) (2001), 535-544.

[7] Y. B. Jun, *Closed fuzzy ideals in BCI-algebras*, Math. Japonica, **3**(1993), 199-202.

[8] L. A. Zadeh, *Fuzzy sets*, Information Control, **8**(1965) 338-353.

[9] K. Imai, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42**, 26-29(1966).

Received: June 27, 2005