International Mathematical Forum, 1, 2006, no. 8, 351-357

Locally Complete Spaces, Regularity, and the Banach Disk Closure Property

C. Bosch

Departamento de Matemáticas
Instituto Tecnológico Autónomo de México
Río Hondo No. 1, 01000 México, D.F.
bosch@itam.mx

C. L. García

Departamento de Matemáticas
Instituto Tecnológico Autónomo de México
Río Hondo No. 1, 01000 México, D.F.
clgarcia@itam.mx

F. Garybay-Bonales

Fac. Ing. Química
Universidad Michoacana, Ciudad Universitaria
58060 Morelia, Michoacán, México.
fgaribay@zeus.umich.mx

C. Gómez-Wulschner

Departamento de Matemáticas
Instituto Tecnológico Autónomo de México
Río Hondo No. 1, 01000 México, D.F.
claudiag@itam.mx

R. Vera

Esc. de Ciencias Fís.-Mat.
Universidad Michoacana, Ciudad Universitaria
58060 Morelia, Michoacán, México.
rvera@zeus.umich.mx

Abstract

We prove that for the inductive limit of sequentially complete spaces regularity or local completeness imply the Banach Disk Closure Property ($BDCP$) (an inductive limite enjoys the $BDCP$ if all Banach disks in the steps of the inductive limit are such that their closures, with respect to the inductive limit topology, are Banach discs as well). In particular we obtain that for an inductive limit of sequentially complete spaces, regularity is equivalent to the $BDCP$ plus an "almost regularity" condition.

Mathematics Subject Classification: 46A13, 46A30

Keywords: inductive limits of locally convex spaces, regularity, almost regularity, locally Baire and sequentially complete spaces

1 Introduction

In 1955 A. Grothendieck proved that a separated inductive limit of Fréchet spaces (an LF-space) is locally complete if and only if it is regular ([4]), that is, if $(E, \tau) = \text{ind}(E_n, \tau_n)$ then for every τ-bounded subset $A \subset E$ there exists an $n \in \mathbb{N}$ such that $A \subset E_n$ and A is τ_n-bounded. If the steps of the inductive limit are not necessarily Fréchet spaces, an old question attributed to Grothendieck, asks about the relationship between regularity and sequential completeness. In an attempt to answer Grothendieck's question, Bosch, Floret, Gómez-Wulschner, Kučera, Makarov, McKennon, and Qiu, among other researchers, (see, e.g., [1], [2], [3], [5], [7], [10], [9], [12], and the references therein) have obtained characterizations or sufficient conditions for the regularity of an inductive limit. New concepts and weaker properties than regularity and completeness have appeared to try to answer Grothendieck's question, some of them will be discussed in this article.

2 Definitions and notation

Throughout this paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex vector spaces with topologies τ_n and continuous identity maps $i_n : E_n \hookrightarrow E_{n+1}$, $n \in \mathbb{N}$. Their locally convex inductive limit will be denoted by $\text{ind}(E_n, \tau_n)$, and the inductive topology, $\text{ind}\tau_n$, will be denoted simply by τ. If $(E, \tau) = \text{ind}(E_n, \tau_n)$ we denote by σ_n and σ the weak topology in E_n and E respectively. An inductive limit $(E, \tau) = \text{ind}(E_n, \tau_n)$ is called *regular* if each bounded set in (E, τ) is contained and bounded in some (E_n, τ_n). This is an important and desirable property for an inductive limit and as we have said before it has been weakened in many different ways. If X is a locally convex vector space with topology α and $A \subset X$, we denote the closure of A in X by $\overline{A}^{\alpha}$.

A weaker notion than regularity that will be of interest to us is the notion of *G-almost regularity* (*GAR* in short) introduced in [5]. An inductive limit $(E,\tau) = \text{ind}(E_n,\tau_n)$ is said to be GAR if for any (E,τ)-bounded set B there exists an $n \in \mathbb{N}$ such that for any zero neighborhood, $U \in \tau_n$, the τ-closure of U, $\overline{U}^{\tau}$, absorbs B.

Let (E,τ) be a Hausdorff locally convex vector space and let $D \subset E$ be a bounded, closed, and absolutely convex set. Let $E_D = \cup_{k=1}^{\infty} kD$, and for each $x \in E_D$, let $\rho_D(x) := \inf\{r > 0 : x \in rD\}$ be the Minkowski seminorm associated to D. The boundedness of D implies that the inclusion map $i : (E_D,\rho_D) \to (E,\tau)$ is continuous and that ρ_D is a norm. A bounded, closed, and absolutely convex set $D \subset E$ is called a *τ-disk*. D is a *Banach (Baire) disk* if (E_D,ρ_D) is a Banach (Baire) space. If every bounded set B is contained in a Banach (Baire) disk we say that E is *locally complete (Baire)*. It is easy to see that sequentially complete spaces are locally complete and that locally complete spaces are locally Baire.

3 Locally complete spaces, regularity, and reflexivity

In this section we study the relationship between GAR and the reflexivity of an inductive limit. J. Kučera in [8] proved that if the spaces (E_n,τ_n) are all reflexive then the inductive limit $(E,\tau) = \text{ind}(E_n,\tau_n)$ is reflexive if and only if for any bounded set B in (E,τ) there exists a set A, bounded in some (E_n,τ_n), such that $B \subset \overline{A}^{\tau}$. This last regularity property, that we call KAR, clearly implies GAR. Here we will prove a theorem similar to Kučera's theorem (Proposition 3.5 below) by using an essentially different technique (we use locally Baire spaces and GAR). For the sake of completeness we write down most of the proofs although some of them are standard arguments in the field.

First we prove the following technical Lemma:

Lemma 3.1 *Let* $(E,\tau) = \text{ind}(E_n,\tau_n)$. *If* V *is a neighborhood of zero in* (E_n,τ_n) *then* $\overline{(E_n)_V}^{\tau} = (\overline{E_n}^{\tau})_{\overline{V}^{\tau}}$ *where, recall,* $(E_n)_V$ *denotes* $\cup_{k=1}^{\infty} kV$.

Proof. Clearly $\overline{(E_n)_V}^{\tau} \supset (\overline{E_n}^{\tau})_{\overline{V}^{\tau}} = \cup_{k=1}^{\infty} k\overline{V}^{\tau}$. Now suppose that $x_0 \in \overline{(E_n)_V}^{\tau} \setminus (\overline{E_n}^{\tau})_{\overline{V}^{\tau}}$. For each $k \geq 1$, there exists $y_k \in (k\overline{V}^{\tau})^{\circ} \subset (\overline{E_n}^{\tau}, \overline{\tau_n}^{\tau})'$ such that $|y_k(x_0)| > 1$.

Since $\overline{V}^{\tau} \subset 2\overline{V}^{\tau} \subset \cdots$ we have that $(\overline{V}^{\tau})^{\circ} \supset (2\overline{V}^{\tau})^{\circ} \supset \cdots$. Now, by Alaoglu's theorem $(\overline{V}^{\tau})^{\circ}$ is σ_n^*-compact in $(\overline{E_n}^{\tau})'$ and since $(y_k) \subset (\overline{V}^{\tau})^{\circ}$ then there exists a σ_n^* convergent subsequence (y_{k_m}) of (y_k), say $y_{k_m} \overset{\sigma_n^*}{\to} y$. Thus $y \in (\overline{V}^{\tau})^{\circ}$ and $|y(x_0)| \geq 1$.

Now we prove that $y \in \cap_{k=1}^{\infty}(k\overline{V}^{\tau})^{\circ}$. Let $k_0 \geq 1$, then $(y_k)_{k\geq k_0} \subset (k_0\overline{V}^{\tau})^{\circ}$. Then, there exists $m_0 \in \mathbb{N}$ such that $(y_{k_m})_{m\geq m_0} \subset (k_0\overline{V}^{\tau})^{\circ}$. Thus $y \in (k_0\overline{V}^{\tau})^{\circ}$. On the other hand, since $(\overline{V}^{\tau})^{\circ} \supset (2\overline{V}^{\tau})^{\circ} \supset \cdots$ we have that $\cap_{k=1}^{\infty}(k\overline{V}^{\tau})^{\circ} = \{0\}$ hence $y \equiv 0$ which contradicts that $|y(x_0)| \geq 1$.

Proposition 3.2 *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$. If E locally Baire then E is GAR.*

Proof. Let $B \subset E$ a τ-disk. Since E is locally Baire we have that (E_B,ρ_B) is a Baire space. Now, $E_B = \cup_{n=1}^{\infty}(E_B \cap E_n)$ so by Baire's Category Theorem there exists $N \in \mathbb{N}$ such that $\operatorname{int}(\overline{E_B \cap E_N}^{\rho_B}) \neq \emptyset$. Let $x_0 \in \operatorname{int}(\overline{E_B \cap E_N}^{\rho_B})$ and $r > 0$ such that $x_0 + rB \subset \overline{E_B \cap E_N}^{\rho_B}$. Since $\overline{E_B \cap E_N}^{\rho_B}$ is a subspace we have that $rB \subset -x_0 + \overline{E_B \cap E_N}^{\rho_B} \subset \overline{E_B \cap E_N}^{\rho_B}$. Then $B \subset \overline{E_B \cap E_N}^{\rho_B}$ and $E_B \subset \overline{E_B \cap E_N}^{\rho_B} \subset E_B$. Thus $E_B = \overline{E_B \cap E_N}^{\rho_B} \subset \overline{E_B \cap E_N}^{\tau} \subset \overline{E_N}^{\tau}$.

Now, let V be a zero neighborhood in (E_N,τ_N). By Lemma 3.1 we have $\overline{(E_N)_V}^{\tau} = (\overline{E_N}^{\tau})_{\overline{V}^{\tau}} = \overline{E_N}^{\tau}$ so $E_B = \cup_{n=1}^{\infty}(E_B \cap k\overline{V}^{\tau})$. Since (E_B,ρ_B) is a Baire space, we have, by Baire's Category Theorem, that there exists $k_0 \in \mathbb{N}$ such that $\operatorname{int}(\overline{E_B \cap k_0\overline{V}^{\tau}}^{\rho_B}) \neq \emptyset$. Let $z_0 \in \operatorname{int}(\overline{E_B \cap k_0\overline{V}^{\tau}}^{\rho_B}) = \operatorname{int}(E_B \cap k_0\overline{V}^{\tau})$ and $r_0 > 0$ such that $z_0 + r_0B \subset E_B \cap k_0\overline{V}^{\tau} \subset k_0\overline{V}^{\tau}$. Then $r_0B \subset -z_0 + k_0\overline{V}^{\tau} \subset 2k_0\overline{V}^{\tau}$ and $B \subset \frac{2k_0}{r_0}\overline{V}^{\tau}$. Thus E is *GAR*.

Proposition 3.3 *If (X,τ) is reflexive then it is locally complete and hence locally Baire.*

Proof. Let $B \subset X$ a disk. Let (x_n) be a ρ_B-Cauchy sequence. Clearly (x_n) is weakly-Cauchy, then there is a $k_0 > 0$ such that $(x_n) \subset k_0B$. Since X is reflexive k_0B is weakly compact. Then there exists a subsequence (x_{n_k}) of (x_n) which weakly-converges to some $x \in k_0B$. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $x_n - x_m \in \varepsilon B$ if $m,n \geq N$, hence $x_n - x \in \overline{\varepsilon B}^{weak} = \varepsilon B$ for all $n \geq N$. Thus (x_n) ρ_B-converges to x and (X_B,ρ_B) is Banach.

Corollary 3.4 *Let $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$. If (E,τ) is locally complete or reflexive then it is GAR.*

Proof. Note that either one local completness or reflexivity implies that (E,τ) is locally Baire. Hence by Proposition 3.2 (E,τ) is *GAR*.

Proposition 3.5 *Let (E_n,τ_n) be reflexive for all n. If $(E,\tau) = \operatorname{ind}(E_n,\tau_n)$ is KAR then (E,τ) is reflexive and regular.*

Proof. Let $B \subset E$ a disk. Since (E,τ) is *AR* there is an $N \in \mathbb{N}$ and a disk $A \subset E_N$ such that $B \subset \overline{A}^{\tau}$. (E_N,τ_N) is reflexive so A is σ_N-compact hence A is τ-closed so $B \subset \overline{A}^{\tau} = A \subset E_N$. So (E,τ) is regular. Now B is τ-closed so B is τ_N-closed since $i : (E_N,\tau_N) \to (E,\tau)$ is continuous. We have that B is a τ_N-disk and since (E_N,τ_N) is reflexive then B is σ_N-compact. Thus, B is σ-compact, i.e., (E,τ) is reflexive.

4 Regularity and the BDCP

It is well known that the closure of a Banach disk is not necessarily a Banach disk. For instance, in [11, Obs. 8.3.31], an example of an (LB) space is given, where a Banach disk in some step of the inductive limit is such that its closure with respect to the inductive limit topology is not a Banach disk. When Banach disks in the steps of an inductive limit are such that their closures (again with respect to the inductive limit topology) are Banach discs too, we have what we call the *Banach Disk Closure Property.* For further reference we write this property as,

Definition 4.1 *We say that* $(E, \tau) = \text{ind}(E_n, \tau_n)$ *satisfies the* Banach Disk Closure Property *(BDCP) if for every* τ_n*-Banach disk* $A \subset E_n$*,* $\overline{A}^{\tau}$ *is a Banach disk.*

The following theorem shows that the $BDCP$ is enjoyed by sequentially complete inductive limits which in addition are either regular, or locally complete. Moreover, the $BDCP$ characterizes locally complete inductive limits which are KAR.

Theorem 4.2 *Let* $(E, \tau) = \text{ind}(E_n, \tau_n)$ *be such that* (E_n, τ_n) *is sequentially complete for all* $n \in \mathbb{N}$*. Then,*

(a) if (E, τ) *is regular then it satisfies* $BDCP$

(b) if (E, τ) *is locally complete it satisfies* $BDCP$

(c) if (E, τ) *is* KAR *then* E *satisfies* $BDCP$ *if and only if* E *is locally complete.*

Proof. To prove (a) let $A \subset E_n$ be a τ_n-Banach Disk. Since $\tau|_{E_n} < \tau_n$, A is τ-bounded in E, hence $\overline{A}^{\tau}$ is also τ-bounded in E. By regularity, there exists $m \in \mathbb{N}$ such that $\overline{A}^{\tau} \subset E_m$ and it is τ_m-bounded there. In this case $E_{\overline{A}^{\tau}} = (E_m)_{\overline{A}^{\tau}} \subset E_m$ On the other hand, since (E_m, τ_m) is sequentially complete it is locally complete, that is, $(E_{\overline{A}^{\tau}}, \rho_{\overline{A}^{\tau}}) = ((E_m)_{\overline{A}^{\tau}}, \rho_{\overline{A}^{\tau}})$ is a Banach space.

For (b) let $A \subset E_n$ be a τ_n-disk. Since $\tau|_{E_n} < \tau_n$, A and consequently $\overline{A}^{\tau}$ are τ-bounded, hence $\overline{A}^{\tau}$ is a τ-disk. Since E is locally complete, there exists $D \subset E$ τ-disk such that $\overline{A}^{\tau} \subset D$ and (E_D, ρ_D) is a Banach space. More over, $\rho_{\overline{A}^{\tau}} \geq \rho_D|_{E_{\overline{A}^{\tau}}} \geq \tau|_{E_{\overline{A}^{\tau}}}$.

Let $\{x_n\} \subset E_{\overline{A}^{\tau}}$ be a $\rho_{\overline{A}^{\tau}}$-Cauchy sequence, then, it is ρ_D-Cauchy, hence, there is $x \in E_D$ such that $\{x_n\} \overset{\rho_D}{\to} x$. We claim that $\{x_n\} \overset{\rho_{\overline{A}^{\tau}}}{\to} x \in E_{\overline{A}^{\tau}}$. Indeed, let $r > 0$, then, there is $N \in \mathbb{N}$ such that $x_n - x_m \in rA$ for all n, $m \geq N$. By taking the ρ_D-limit on m and since $\overline{A}^{\tau}$ is ρ_D-closed we have, $x_n - x \in rA$

for all $n \geq N$. Since we know that $(E_{\overline{A}^\tau}, \rho_{\overline{A}^\tau})$ is Banach, $(E, \tau) = \mathrm{ind}(E_n, \tau_n)$ satisfies $BDCP$.

Finally, to prove (c), note that the sufficiency follows immediately from (b). On the other hand, let $A \subset E$ be a τ-disk. Since E is KAR there exists $N \in \mathbb{N}$ and $B \subset E_N$ a τ_N-disk such that $A \subset \overline{B}^\tau = D$. Now, since (E_N, τ_N) is sequentially complete then B is a τ_N-Banach disk and the $BDCP$ implies that D is a τ-Banach disk. This proves that $(E, \tau) = \mathrm{ind}(E_n, \tau_n)$ is locally complete.

Note that since the $BDCP$ and KAR imply local completeness of the inductive limit then we obtain from Theorem 4.2 and Grothendieck's theorem the following,

Corollary 4.3 *Let (E_n, τ_n) be a sequentially complete space for every $n \in \mathbb{N}$. Then $(E, \tau) = \mathrm{ind}(E_n, \tau_n)$ is KAR and satisfies the $BDCP$ if and only if it is regular.*

ACKNOWLEDGEMENTS. The authors C. Bosch, C.L. García and C. Gómez-Wulschner would like to thank the support from the Asociación Mexicana de Cultura, A.C. The authors F. Garybay-Bonales and R. Vera would like to thank the support from the Coordinación de la Investigación Científica de la Universidad Michoacana, México.

References

[1] C. Bosch, J. Kučera, On regularity of inductive limits, *Czechoslovak Math. J.* **45(120)** (1995), no. 1, 171–173.

[2] C. Bosch, J. Kučera, Sequential completeness and regularity of inductive limits of webbed spaces, *Czechoslovak Math. J.* **52(127)** (2002), no. 2, 329–332.

[3] K. Floret, Folgenretraktive Sequenzen lokalkonvexer Räume, *J. Reine Angew. Math.* **259** (1973), 65–85.

[4] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* 1955 (1955), no. 16, 140 pp.

[5] C. Gómez-Wulschner, J. Kučera, Sequentially complete inductive limits and regularity, *Czechoslovak Math. J.* **54(129)** (2004), no. 3, 697–699.

[6] G. Köthe, *Topological Vector Spaces I,* Die Grundlehren der mathematischen Wissenschaften Band 159, Springer-Verlag, New York, 1969.

[7] J. Kučera, Sequential completeness of LF spaces, *Czechoslovak Math. J.* **51(126)** (2001), no. 1, 181–183.

[8] J. Kučera, Reflexivity of inductive limits, *Czechoslovak Math. J.* **54(129)** (2004), no. 1, 103–106. 4.

[9] J. Kučera, K. McKennon, Bounded sets in inductive limits, *Proc. Amer. Math. Soc.* **69** (1978), no. 1, 62–64.

[10] B. M. Makarov, Pathological examples of inductive limits of Banach spaces, *Uspehi Mat. Nauk.* **18** (1963), no. 3, 171–178.

[11] P. Pérez-Carreras, J. Bonet, *Barreled Locally Convex Spaces,* North-Holland Mathematics Studies 131, North-Holland Publishing Co., Amsterdam, 1987.

[12] J-H. Qiu, Sequential retractivities and regularity on inductive limits, *Czechoslovak Math. J.* **50(125)** (2000), no. 4, 847–851.

Received: September 7, 2005