A Fractional Order of Convergence Rate for Successive Methods: Examples on Integral Equations

D. Rostamy V. F.¹ and M. Jabbari

Department of Mathematics, Faculty of Science
Imam Khomeini International University and,
Department of Mathematics, Statistics and Computer Science
University of Tehran
P. O. Box 14155-6455, Tehran, Iran

Abstract

This paper is devoted to investigating the fractional order of convergence of operator iteration schemes and, as a means of obtaining rapid convergence, rediscovering fixed-point methods by fractional calculus. In addition, a new definition for fractional Gateaux derivative is used which seems a key idea for accelerating the convergence of iterative methods. At the end, the relevance of the theory and applications of integral equations are given.

Mathematics Subject Classification: 65M12, 26A33

Keywords: Theoretical numerical analysis, fixed point theorems, convergence of successive approximations, fractional calculus, integral equations.

1. Introduction

Finding the roots of a given operator $F : S \mapsto S$ is one of the most commonly occurring problems in applied mathematics. Assume that α is a unique root for $F(x)$. Usually, it is not possible to determine a root of the operator explicitly within a finite number of steps. Thus we have to appeal to approximation methods. These methods are usually iterative and have the following three steps:

(1) Successive approximations, x_i , $i = 1, 2, \dots$ to the root are computed with

¹rostamy@khayam.ut.ac.ir rostamyd@hotmail.com

the aid of an iteration operator such as $G : S \mapsto S$ with $x_{i+1} = G(x_i)$.

(2) α is a unique fixed point for G (that is $\alpha = G(\alpha)$).

(3) Beginning with a starting value x_0 .

If fixed points of G are also roots of F , and if G is continuous in a neighborhood of each of its fixed points, then each accumulation point of the sequence x_i , $i = 1, 2, \dots$ is a fixed point of G , and hence it is a root of F . The following questions arise in this connection:

(a) Under which conditions the sequence $\{x_i\}$ will be converge?

(b) What is the convergence rate?

(c) How can the convergence rate be improved?

Remark 1. Successive approximation goes under several names, including fixed point iteration, Picard iteration, geometric series iteration, and the Numann series iteration.

There is an enormous amount of literature on the fixed point theorem (see [2, 4, 5, 6, 8]) our point of view is to observe classical answers to the above equations based on fixed point theorem. But in this paper we will emphasize on a new idea based on the fractional calculus and the fixed point theorem. Of course, we know that the fractional calculus is used in various sciences such as physics, mechanics, chemistry and biological processes (see [6]). The relevance of the subject in large is due to the fact that fractional calculus provides novel mathematical tools for modelling physical phenomena (in fluid mechanics, bio-heat equation, nonlinear damped vibrations suspension brigs and viscous-elasticity (see [9])).

The outline of this paper is as follows: In Section for preliminaries, we implement the concept of the fractional calculus on fixed point theorem in Theorem 2. The Theorem 3 is a generalization of Theorem 2 and theorem 4 is a new version of the fixed point theorem of Banach. On the other hand, the paper has relevance for theory and application of inverse problem such as finding roots and integral equations, in Section 3. Finally, in Section 4, concluding remarks are given.

2. Preliminaries

Definition 1. Let the sequence $x_0, x_1, x_2, \dots$ converge to α . If there exist constants C and p such that

$$\|x_{n+1} - \alpha\| \leq C\|x_n - \alpha\|^p,$$

then the sequence is said to converge to α with order p . A method which produces such a sequence is said to have an order of convergence p .

Based on the fixed point theorem, we can prove the following result (see [6]).

Theorem 1. Let α be a fixed point of G and assume that G is p times continuously differentiable in some neighborhood

$$\overline{b(\alpha, r)} = \{x | \|x - \alpha\| \leq r\}$$

of α . If $D^1G(\alpha) = D^2G(\alpha) = \dots = D^{p-1}G(\alpha) = 0$, $D^pG(\alpha) \neq 0$ where $D^k = \frac{d^k}{dx^k}$, then the order of convergence for this method is p , such that x_0 as initial point is sufficiently close to α . More specifically, the following inequality is hold:

$$\|x_n - \alpha\| \leq k_n \|x_0 - \alpha\|^{p^n},$$

with $k_n = k^{\frac{p^n - 1}{p - 1}}$ where k is an upper bound for $\frac{\|D^pG\|}{p!}$ in $b(\alpha, r)$.

Our aim in this paper is to modify the above theorem with fractional calculus [2]. Hence we need the following definition:

Definition 2. Suppose that G is continuous on $[0, X]$. Then the Riemann-Liouville fractional integral of order v is

$$D^{-v}G(x) = \frac{1}{\Gamma(v)} \int_0^x (x - \xi)^{v-1} G(\xi) d\xi = \frac{1}{\Gamma(v)} \int_0^x z^{v-1} G(x - z) dz, \quad (1)$$

for $v > 0$ and $0 < x \leq X$.

Furthermore, we assume that $G(x)$ be analytic at $x = a$ for all $a \in [0, X]$ and the power series

$$G(x - z) = G(x) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k G(x)}{k!} z^k, \quad (2)$$

converges for all z in an interval that property contains $[0, X]$. Thus it converges uniformly on $[0, X]$. Now substitute (2) in (1)

$$D^{-v}G(x) = \frac{1}{\Gamma(v)} G(x) \int_0^x z^{v-1} dz + \frac{1}{\Gamma(v)} \int_0^x z^{v-1} \left[\sum_{k=1}^{\infty} (-1)^k \frac{D^k G(x)}{k!} z^k \right] dz.$$

By the uniform convergence we may interchange the order of summation and integration to obtain

$$D^{-v}G(x) = \frac{1}{\Gamma(v)} \sum_{p=0}^{\infty} \frac{(-1)^p D^p G(x)}{p!(v+p)} x^{v+p}, \quad 0 \leq x \leq X. \quad (3)$$

Thus we have expressed the fractional integral of an analytic function in terms of ordinary derivatives of the function.

Remark 2. We can develop the idea of fractional derivative of an operator by generalizing the of Gateaux derivative (see [1]). Hence, we need to the following definitions.

Definition 3. Let $G : S \longrightarrow S$ be an operator where S is a normed linear space. If for $x_0 \in S$ there exists a linear operator $DG_w(x_0)$ such that for $h \in S$

$$\lim_{s \rightarrow 0} \frac{G(x_0 + sh)[\] - G(x_0)[\]}{s} = DG_w(x_0)[\]h,$$

then we say that G is weakly differentiable at x_0 . The operator $DG_w(x_0)[\]$ is called the weak or Gateaux derivative of G at x_0 .

Hence, we can rewrite (3) based on Definition 3:

$$\begin{aligned} D^{-v}G(x_0)[\] &= \frac{1}{\Gamma(v)} \sum_{p=0}^{\infty} \frac{(-1)^p x_0^{v+p}}{p!(p+v)} \lim_{s \rightarrow 0} \frac{D^{p-1}G(x_0 + sh)[\] - D^{p-1}G(x_0)[\]}{s} = \\ &= \frac{1}{\Gamma(v)} \sum_{p=0}^{\infty} \frac{(-1)^p x_0^{v+p}}{p!(p+v)} D^p G_w(x_0)[\]h. \end{aligned} \quad (4)$$

We call (4) fractional Gateaux derivative of G at x_0 .

Of course in (4), we assume that S is a Banach space such that G is infinity differentiable in neighborhood $b(x_0, r)$, $r > 0$ of x_0 . Therefore, we can state Theorem 2 based on Definition 3.

Now, based on the above definition and Theorem 1 we can have the following interesting result:

Theorem 2. Assume α is a root of $x = G(x)$, where G is an operator, infinitely differentiable continuous on S . Then the following hold:

- (i) $G(\overline{\alpha_n}) = 0$
 - (ii) $DG(\overline{\alpha_n}) = 0$
 - (iii) $D^{-(v)}G(\overline{\alpha_n}) = \alpha - x_{n+1}$ for $v \in \mathbb{R}^+ \setminus \{0\}$
 - (iv) for all of $x \in S$, $\|D^n G(x)\|$ is sufficiently close to zero for $n = 3, 4, \dots$
- where $\overline{\alpha_n} = \alpha - x_n$ and

$$x_{n+1} = G(x_n), \quad n = 0, 1, 2, \dots \quad (5)$$

Also, if the initial guess x_0 is chosen sufficiently close to α (or $x_0 \in b(\alpha, r)$), then (5) have order of convergence $v + 2$.

Proof. According to (4) and using the above assumptions we can write:

$$\begin{aligned}\alpha - x_{n+1} &= D^{-v}G(x)|_{x=\overline{\alpha_n}} = \frac{1}{\Gamma(v)} \sum_{p=0}^{\infty} \frac{(-1)^p D^p G(\overline{\alpha_n})}{p!(v+p)} x^{v+p}|_{x=\overline{\alpha_n}} = \\ &= \frac{1}{\Gamma(v)} (\overline{\alpha_n}^{(v+2)}) \left[\frac{D^2 G(\overline{\alpha_n})}{2!(v+2)} + \frac{D^3 G(\overline{\alpha_n})}{3!(v+3)} \overline{\alpha_n} + \dots \right] \simeq \\ &\simeq \frac{1}{\Gamma(v)} (\overline{\alpha_n}^{(v+2)}) \left[\frac{D^2 G(\overline{\alpha_n})}{2!(v+2)} \right].\end{aligned}$$

Therefore according to (iv) we have:

$$\|\alpha - x_{n+1}\| \leq \frac{1}{\Gamma(v)} \|\alpha - x_n\|^{v+2} \left[\frac{\|D^2 G(\overline{\alpha_n})\|}{2!(v+2)} \right],$$

hence, the proof is complete. $\square$

Remark 3. In practical hypothesis (iii) in Theorem 2 appears to be a natural condition because we can transfer it to the condition $D^2 G(\overline{\alpha_n}) = 2!(v+2)\Gamma(v)$ in a special case (see (4), (i) and (ii)). Therefore, we can obtain (iii). In the following we give an example that deals with finding root.

Example 1. Assume $G(x) = \Gamma(v)2!(v+2)x^2 + c_1x + c_2$ in Theorem 2 such that $c_1 = -4\Gamma(v)(v+2)$ and $c_2 = -c_1\overline{\alpha_n} - \Gamma(v)2!(v+2)\overline{\alpha_n}^2$. It is clear that the conditions (i), (ii) and (iii) in Theorem 2 are fulfilled. Therefore, the sequence of $x_{n+1} = G(x_n)$ has order of convergence $v+2$ for $v > 0$ and this is more than the convergence rate in Newton method.

Based on the above theorems, we can give the further results in the next Section.

3. Main Theorems

Theorem 3. Assume α is a root of $x = G(x)$ and G is an operator which infinitely differentiable continuous on S . Then, the following hold:

- (i) $G(\overline{\alpha_n}) = 0$
- (ii) $DG(\overline{\alpha_n}) = D^2G(\overline{\alpha_n}) = \dots = D^lG(\overline{\alpha_n}) = 0, l \in \mathbf{N}$
- (iii) $D^{-v}G(\overline{\alpha_n}) = \alpha - x_{n+1} \quad v \in \mathbb{R}^+ \setminus \{0\}$
- (iv) $\|D^n G\|$ is sufficiently close to zero for $n = l+2, l+3, \dots$

where $\overline{\alpha_n} = \alpha - x_n$ and $x_{n+1} = G(x_n)$, $n = 0, 1, 2, \dots$. Also, if the initial guess x_0 is chosen sufficiently close to α , then the iteration (4) will have order of

convergence $v + l + 1$.

Proof. The proof is completed by induction and observing Theorem 2. $\square$

Theorem 4. (Fractional Convergence of Fixed Point Theorem)

Let α be a fixed point of an operator $G : S \mapsto S$ such that it is infinitely differentiable continuous on S . Assume that, $x_0 \in S$ is a string point and $x_{i+1} = G(x_i)$, $i = 0, 1, \dots$. Further, let a neighborhood $b(x_0, r) = \{x \mid \|x - x_0\| < r\}$ be a neighborhood of x_0 and there exist a constant k , $0 \leq k < 1$, exist such that

- (i) $\|G(x) - G(y)\| \leq k\|x - y\|$ for all $x, y \in \overline{b(x_0, r)} = \{x \mid \|x - x_0\| \leq r\}$,
 - (ii) $\|x_1 - x_0\| \leq (1 - k)r < r$,
 - (iii) $G(\overline{\alpha_n}) = 0$ where $\overline{\alpha_n} = \alpha - x_n$,
 - (iv) $DG(\overline{\alpha_n}) = D^2G(\overline{\alpha_n}) = \dots = D^lG(\overline{\alpha_n}) = 0$, $l \in \mathbf{N}$
 - (v) $D^{-v}G(\overline{\alpha_n}) = \alpha - x_{n+1}$ $v \in \mathbb{R}^+ \setminus \{0\}$
 - (vi) for all $x \in [0, X]$, $\|D^n G\|$ is sufficiently close to zero for $n = l + 2, l + 3, \dots$
- Then we have:

- (1) $x_i \in b(x_0, r)$ for all $i = 0, 1, \dots$
- (2) G has exactly one fixed point α , $G(\alpha) = \alpha$, in $\overline{b(x_0, r)}$, and $\{x_i\}$ has order of convergence $v + l + 1$.

Proof. The proof follows immediately from the contraction property, the fixed point theorem [7] and the above theorems that is 2 and 3. $\square$

Remark 4. The above theorems are valid for G as an operator that has infinity differentiable in the sense of Gateaux derivative and S is a Banach space.

In the following we give some examples based on Gateaux derivative that confirm the above theory:

Example 2. Let κ be the integral operator such that

$$\kappa : S = C[0, 1] \rightarrow S = C[0, 1].$$

Defined $\kappa x(s) = \int_0^1 k(s, t)x^2(t)dt$. Then

$$\kappa'(x_0)[\](s) = 2 \int_0^1 k(s, t)x_0(t)[\]dt,$$

$$\kappa''(x_0)[\][\](s) = 2 \int_0^1 k(s, t)[\][\]dt$$

and $\kappa'''(x_0) = 0$. If we consider $x(s) = G(x(s)) = \sum_{i=0}^2 \alpha_i (\kappa[x(s)])^i$ then it is clear that we can determine coefficients of $\{\alpha_i\}_{i=0}^2$ such that conditions of Theorems 3, 4 are fulfilled.

Example 3. Let G be an operator form $X = Y_1 \times Y_2$ into the product space $X = Y_1 \times Y_2$ defined by

$$x = Gx = (\lambda_1 y_1, \lambda_2 y_2) = \begin{cases} y_1 = \alpha_1 \kappa_1 y_1 + \alpha_2 \kappa_2 y_2 \\ y_2 = \beta_1 \kappa_1 y_1 + \alpha_2 \kappa_2 y_2 \end{cases}.$$

If λ_0 and λ_1 are both differentiable at $x_0 = (y_{0,1}, y_{0,2})^T$ then

$$G'(x_0)[\] = (\lambda'_1(y_{0,1})[\], \lambda'_2(y_{0,2})[\]).$$

Also, we assume that $\kappa_1 y(s) = \int_0^1 k_1(s, t) y^2(t) dt$, $\kappa_2 y(s) = \int_0^1 k_2(s, t) y(t)^2 dt$. It is clear that we have the following results:

$$G''(x_0)[\] = (\lambda''_1 y_{0,1}[\], \lambda''_2 y_{0,2}[\])$$

and

$$G'''(x_0) = (0, 0).$$

Therefore, we can determine coefficients of $\{\alpha_i, \beta_i\}_{i=1}^2$ such that conditions of Theorems 3, 4 are fulfilled.

Example 4. If in Example 3 we put $\kappa_1 y(s) = \frac{dy(s)}{ds}$ and $\kappa_2 y(s) = \int_0^1 k_2(s, t) y(t)^2 dt$ then our problem is as follows:

$$G(x) = \begin{cases} y_1 = \alpha_1 \kappa_1 y_1 + \alpha_2 \kappa_2 y_2 \\ y_2 = \beta_1 \kappa_1 y_1 + \alpha_2 \kappa_2 y_2 \end{cases}.$$

On the other hand, we know that κ_1 is a linear operator thus its second derivative is zero. Therefore, we can determine coefficients of $\{\alpha_i, \beta_i\}_{i=1}^2$ such that the conditions of Theorems 3, 4 are fulfilled.

4. Conclusion Remarks

In this paper, we remark the following alternatives:

- Implementing the above theorems for solving integral equations and systems of integral equations such that it converges under slightly weaker assumptions for slowly the rate of convergence and strongly assumptions for fast the rate of convergence.
- Extension the above theorems for solving convergence of solutions of hyper-singular integral equations.
- Construction of suitable successive approximations for some inverse problems.

Acknowledgement

The authors would like to thank Professor R. Zaare-Nahandi for suggestions improving the revised version of this paper. The first author also acknowledges the partial support of Institute for Studies in Theoretical Physics and Mathematics.

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Received: October 2, 2005