Self-recursions of the Hauptmodul $N(j_{1,3})$

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Abstract. We know by Borcherds [2] that Thompson series of odd level $N$ can be determined by the four coefficients $H_1, H_2, H_3$ and $H_5$. And, according to Kang et al [4], if such $N$ is relatively prime to 3 then it can be estimated just by $H_1, H_2$ and $H_3$. We show in this paper that even in the case $N = 3$ Kang’s result is true.

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1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} mod $N$ ($N$ is a positive integer.). Since the group $\Gamma_1(N)$ act on $\mathfrak{H}$ by linear fractional transformation, we get the modular curve $X_1(N) = \Gamma_1(N)\backslash \mathfrak{H}$ as the projective closure of a smooth affine curve $\Gamma_1(N)\backslash \mathfrak{H}$. Let $g_{1,N}$ be the genus of $X_1(N)$ and $K(X_1(N))$ the function field on the curve $X_1(N)$. Since the genus $g_{1,N} = 0$ for the eleven case $1 \leq N \leq 10$ and $N = 12$ [5], the function fields $K(X_1(N))$ in these cases are rational function fields $\mathbb{C}(j_{1,N})$ generated by some modular functions $j_{1,N}$ whose definitions are explicitly given in the appendix of [6].

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Since $\Gamma_1(N)$ contains $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, a modular function $f$ in $K(X_1(N))$ has the periodicity $f(z+1) = f(z)$. Therefore, $f$ has a Fourier expansion with respect to $q = e^{2\pi i z}$, which is called a $q$-expansion of $f$. We call $f$ normalized if its $q$-expansion begins with $q^{-1} + 0 + a_1 q + a_2 q^2 + \cdots$. If the genus of the function field $K(X_1(N))$ is zero, then there exists the unique normalized generator $N(j_{1,N})$ of $K(X_1(N))$ which we call the Hauptmodul and is written by
\[
N(j_{1,N}) = q^{-1} + 0 + \sum_{k \geq 1} H_k q^k
\]
as its $q$-expansion. In this paper we find recursion relations satisfied by the Fourier coefficients $H_k$'s when $N = 3$.

For a Fuchsian group $\Gamma$, we denote $\overline{\Gamma}$ the inhomogeneous group of $\Gamma$ (i.e. $\overline{\Gamma} = \Gamma / \pm I$). Let $\Gamma_0(N)$ be a congruence Hecke subgroup of $SL_2(\mathbb{Z})$. When $\Gamma_1(N) = \Gamma_0(N)$ (i.e. $N = 1, 2, 3$), $N(j_{1,N})$ becomes a monstrous function which is the Thompson series (of type 1A, 2B and 3B, respectively) $T_g(q) = \sum_{n \in \mathbb{Z}} Tr(g|_{V_n}) q^n$ for some element $g$ of the monster simple group $M$. Norton and Koike [9, 10] derived the 2-plicate recursion formulas for the coefficients of $q$-expansion with $t = N(j_{1,N})$ and its 2-plicate $t^{(2)} = N(j_{1,N/(2,N)}) = q^{-1} + 0 + \sum_{k \geq 1} H_k^{(2)} q^k$ as follows; for $k \geq 1$,
\[
H_{4k} = H_{2k+1} + \frac{H_k^2 - H_k^{(2)}}{2} + \sum_{1 \leq j \leq k-1} H_j H_{2k-j},
\]
\[
H_{4k+1} = H_{2k+3} - H_2 H_{2k} + \frac{H_{2k}^2 + H_{2k}^{(2)}}{2} + \frac{H_{2k+1}^2 - H_{k+1}^{(2)}}{2} + \sum_{1 \leq j \leq k} H_j H_{2k-j+2} + \sum_{1 \leq j \leq 2k-1} (-1)^j H_j H_{4k-j},
\]
\[
H_{4k+2} = H_{2k+2} + \sum_{1 \leq k \leq k} H_j H_{2k-j+1},
\]
\[
H_{4k+3} = H_{2k+4} - H_2 H_{2k+1} - \frac{H_{2k+1}^2 - H_{k+1}^{(2)}}{2} + \sum_{1 \leq j \leq k+1} H_j H_{2k-j+3} + \sum_{1 \leq j \leq k} H_j^{(2)} H_{4k-j+2} + \sum_{1 \leq j \leq 2k} (-1)^j H_j H_{4k-j+2}.
\]
We see from the above that if $m = 4$ or $m > 5$ then $H_m$ is determined by the coefficient $H_i$ and $H_i^{(2)}$ for $1 \leq i < m$. Moreover, if we know the coefficients $H_m^{(s)}$ for $m = 1, 2, 3, 5$ and $s = 2^l$, then we can obtain all the coefficients $H_m$ of
Self-recursions of the Hauptmodul $N(j_{1,N})$. This observation turns out to be one of the key steps in Borcherds’ proof \[2\].

If $N = 1$, the Hauptmodul is none other that the modular invariant $j$ and 2-plicate of $j$ is itself, so the above 2-plicate recursion formulas become self-recursions. Thus only four coefficients $H_1$, $H_2$, $H_3$ and $H_5$ are necessary to determine all the coefficients of $j$. However, Koo and Oh \[8\] showed that, in fact, it is enough to have the first three ones $H_1$, $H_2$ and $H_3$ to do the same work by using the Peterson formula for generalized Kac-Moody superalgebras.

On the other hand, Koike \[9\] showed that monstrous functions are completely replicable and Norton \[10\] proved that any completely replicable function $q^{-1} + \sum_{k \geq 1} H_k q^k$ is determined by its 12-coefficients $H_m$ for $m = 1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 18, 23$.

Unlike the monstrous cases, that is $\Gamma_1(N) \neq \Gamma_0(N)$ (if $4 \leq N \leq 10$ or $N = 12$), $N(j_{1,N})$ may not be completely replicable and hence the coefficients $H_k$ do not satisfy (1.1) any more. For instances, Kim and Koo \[6, 7\] showed that for some even levels $N$ the Hauptmoduln $N(j_{1,N})$ satisfy some kind of self-recursion formulas by using Hecke operator and investigating the poles of certain functions, which reads that all such coefficients of $N(j_{1,N})$ can be determined by either $H_1$, $H_2$, $H_3$, $H_4$ or $H_1$, $H_3$, $H_5$. However, for odd $N$, Kang, Kim, Koo and Oh \[4\] showed that if $(N, 3) = 1$, then we are able to determine all the coefficients of $N(j_{1,N})$ just by the first three ones $H_1$, $H_2$ and $H_3$.

In this article we concentrate on the case $N = 3$, which is not covered in the work of Kang et al \[4\]. Since $\Gamma_1(3)/\{\pm 1\} = \Gamma_0(3)/\{\pm 1\}$, the Hauptmodul $N(j_{1,3})$ of $\Gamma_1(3)$ is a monstrous function and so the coefficients of $N(j_{1,3})$ satisfy (1.1). But we can derive the following self-recursion formulas without 2-plicate:

**Theorem 1.1.** Let $t = N(j_{1,3})$ be the Hauptmodul of $\Gamma_1(3)$ and write $t = q^{-1} + \sum_{k \geq 1} H_k q^k$. Let $t(1)$ be the value of $t$ at the cusp 1. Then we have self-recursion formulas as follows: for $k \geq 1$,

\[
H_{9k-3} = t(1)H_{9k-6} - \sum_{1 \leq j \leq 3k-3} H_j H_{9k-3j-6},
\]

\[
H_{9k} = t(1)H_{9k-3} - \sum_{1 \leq j \leq 3k-2} H_j H_{9k-3j-3},
\]

\[
H_{9k+3} = t(1)H_{9k} - \sum_{1 \leq j \leq 3k-1} H_j H_{9k-3j}.
\]
\[ H_{3k+1} = -\frac{1}{2} \sum_{i+j=3k} H_i H_j - \frac{3}{2} \sum_{i+j=k \geq 2} H_{3i} H_{3j} - t(1)H_{3k}, \]

\[ H_{3k+2} = -\sum_{i+j=3k+1 \atop i,j \geq 1} H_i H_j - \frac{1}{3} \sum_{i+j+l=3k \atop i,j,l \geq 1} H_i H_j H_l + \frac{1}{3} H_k - 6 \sum_{i+j+l=k \atop i,j,l \geq 1} H_{3i} H_{3j} H_{3l} + 6t(1) \sum_{i+j=k \atop i,j \geq 1} H_{3i} H_{3j} - 2t(1)^2 H_{3k} + 3H_1 H_{3k}. \]

Combining the 2-plicate recursion formulas (1.1) and the above theorem, we have:

**Corollary 1.2.** The coefficients of \( \mathcal{N}(j_{1,3}) \) can be determined only by the first three coefficients \( H_1, H_2, \) and \( H_3. \)

Considering this result and the work of Kang et al [4], one can expect to improve Norton’s result [10] further to be \( m = 1, 2, 3, \) and hence questions in this theme are worthy of studying.

2. **Proof of theorem and corollary**

Let \( f(z) \) be a modular function with respect to a congruence group \( \Gamma. \) From now on, for brevity, we refer it by saying “\( f(z) \) is on \( \Gamma\)”. For a prime \( p \) dividing \( N, \) we define a Hecke operator \( U_p \) by

\[ f|_{U_p} = p^{-1} \sum_{i=0}^{p-1} f|_{(1, i \atop 1, 1)}. \]

We first recall the following elementary lemmas. (cf. [1, theorem 4.5], [6] §2)

**Lemma 2.1.**
1. If \( f \) is on \( \Gamma_1(N), \) then so is \( f|_{U_p}. \)
2. If \( f(z) = \sum_{l \geq 0} a_l q^l, \) then \( f(z)|_{U_p} = \sum_{l \geq -[\frac{l_0}{p}]} a_p q^l. \)
3. If \( f(z)|_{U_3} = g(z), \) then \( f(z) = \text{terms of degree } 3k \pm 1 + g(3z). \)

For two cusps \( a, b \in \mathbb{Q} \cup \{\infty\}, \) by \( a \sim b \) we mean that \( a \) is equivalent to \( b \) under \( \Gamma_1(3). \)

**Lemma 2.2.** \( \Gamma_1(3) \) has two cusps at \( \infty (\sim 2/3) \) and at \( 0 (\sim 1). \) Moreover, the cusp \( \infty \) has width 1 and the cusp \( 0 \) has width 3.

**Proof.** It is immediate from [6, lemma 3]. \( \square \)
Let $t$ be the normalized Hauptmodul $N(j_{1,3})$ of $\Gamma_1(3)$. For each $n \geq 1$, there exist a unique Faber polynomial $X_n(t)$ in $t$ such that

$$X_n(t) \equiv \frac{1}{n}q^{-n} \text{ mod } q\mathbb{C}[q].$$

For instance, $X_1(t) = t$, $X_2(t) = \frac{1}{2}t^2 - H_1$, $X_3(t) = \frac{1}{3}t^3 - H_1t - H_2$.

**Proposition 2.3.** 1. $t|_{U_3}$ has only a simple pole at the cusp 1, and no other poles.

2. $X_2(t)|_{U_3}$ has a double pole at the cusp 1, and no other poles.

3. $X_3(t)|_{U_3}$ has poles at the cusps $\infty$ and 1 which are simple and triple, respectively, and no more.

**Proof.** We first compute explicitly $q_h$-expansions of $X_1(t)$, $X_2(t)$, and $X_3(t)$ at each cusp because we just need some coefficients of $q_h$-expansions to prove theorem 1.1.

Note that $t|_{U_3}$ is holomorphic at the cusp $\infty$ by lemma 2.1. Since

$$t|_{U_3}|_{(1 \ 0)} = \frac{1}{3}t|_{(\frac{1}{3} \ 0)} + \frac{1}{3}t|_{(2 \ \frac{1}{3})} + \frac{1}{3}t|_{(3 \ \frac{2}{3})} = \frac{1}{3}t|_{(\frac{1}{3} \ 0)} + \frac{1}{3}t|_{(\frac{2}{3} \ \frac{1}{3})} + \frac{1}{3}t|_{(\frac{3}{3} \ 2)}$$

$$= \frac{1}{3}t \left( \frac{z}{3} \right) + \frac{1}{3}t \left( \frac{z-1}{3} \right) + \frac{1}{3}t \left( \frac{3z+2}{3z+3} \right),$$

it has $q_3$-expansion at the cusp 1 such as

$$(2.1) \quad t|_{U_3}|_{(1 \ 0)} = -\frac{\omega^2}{3}q_3^{-1} + \frac{t(1)}{3} + \frac{-\omega H_1}{3}q_3 + O(q_3^2)$$

with $q_3 = q^{1/3} = e^{2\pi i/3}$ and $\omega = e^{2\pi i/3}$. Hence $t|_{U_3}$ has a pole only at the cusp 1, which is simple.

Since $X_2(t) \equiv \frac{1}{2}q^{-2}$ mod $q\mathbb{C}[q]$, we see from lemma 2.1 that $X_2(t)|_{U_3}$ is holomorphic at the cusp $\infty$. In a similar way as in the case of $t$,

$$X_2(t)|_{U_3}|_{(1 \ 0)} = \frac{1}{3}X_2(t)|_{(\frac{1}{3} \ 0)} + \frac{1}{3}X_2(t)|_{(\frac{2}{3} \ \frac{1}{3})} + \frac{1}{3}X_2(t)|_{(\frac{3}{3} \ 2)}$$

has the following $q_3$-expansion at the cusp 1:

$$(2.2) \quad X_2(t)|_{U_3}|_{(1 \ 0)} = \frac{1}{6} + \frac{\omega^2}{q_3}q_3^{-2} + \frac{X_2(t)(1)}{3} + O(q_3) = \frac{-\omega}{6}q_3^{-2} + \frac{t(1)^2}{6} - \frac{H_1}{3} + O(q_3).$$

Thus $X_2(t)|_{U_3}$ has a pole only at the cusp 1, which is double. In like manner, we are able to show that $X_3(t)$ has $q$-expansion at the cusp $\infty$ and $q_3$-expansion at the cusp 1 as follows:

$$X_3(t)|_{U_3} = \frac{1}{3}q^{-1} + O(q),$$
Comparing the coefficients of \(q(1.2)\).

Therefore, \(X_3(t)|_{U_3}\) has a simple pole at the cusp \(\infty\) and a triple pole at the cusp \(1\).

Observe that \(t^2|_{U_3}\) and \(t^3|_{U_3}\) have \(q_3\)-expansions at the cusp \(1\) as follows:

\[
(2.4) \quad \left( t|_{U_3}|(1, 0, 1) \right)^2 = \frac{\omega}{9} q_3^{-2} + \frac{-2\omega^2 t(1)}{9} q_3^{-1} + O(1)
\]

\[
(2.5) \quad \left( t|_{U_3}|(1, 0, 1) \right)^3 = -\frac{1}{27} q_3^{-3} + \frac{\omega t(1)}{9} q_3^{-2} + \left( -\frac{\omega^2 t(1)^2}{9} - \frac{\omega^2 H_1}{9} \right) q_3^{-1} + O(1).
\]

**Proof of Theorem 1.1.** By [6, Cor 9 (ii)] we obtain an identity \((t - t(1)) \times t|_{U_3} = H_3\), from which we get

\[
(q^{-1} + \sum H_k q^k - t(1)) \times (\sum H_3k q^k) = H_3.
\]

By comparing the coefficients of \(q^{3k-2}, q^{3k-1}\) and \(q^{3k}\) on both sides, we have (1.2).

Since \(X_2(t)|_{U_3}\) and \(\frac{3}{2}(t|_{U_3})^2 - t(1)t|_{U_3}\) have a double pole at the cusp \(1\) and are holomorphic at the cusp \(\infty\) by (2.1), (2.2) and (2.4), \(X_2(t)|_{U_3} + \frac{3}{2}(t|_{U_3})^2 - t(1)t|_{U_3}\) is holomorphic on the modular curve \(\Gamma_1(3) \setminus \mathcal{S}^*\). Hence, it is a constant. Replacing \(X_2(t)\) by \(\frac{1}{2}t^2 - H_1\), we obtain

\[
(\frac{1}{2}t^2 - H_1)|_{U_3} = -\frac{3}{2}(t|_{U_3})^2 + t(1)t|_{U_3} + c
\]

for some constant \(c\). By applying lemma 2.1 we derive that

\[
\frac{1}{2}t^2 - H_1 = (\text{terms of degree } 3k \pm 1) - \left( \frac{3}{2}(t|_{U_3})^2 - t(1)t|_{U_3} \right) (3z) + c
\]

and

\[
\frac{1}{2}(q^{-1} + \sum H_k q^k)^2 - H_1 = (\text{terms of degree } 3k \pm 1) - \left( \sum H_3k q^{3k} \right)^2 + t(1)(\sum H_3k q^{3k}) + c.
\]

Comparing the coefficients of \(q^{3k}\), we obtain (1.3).

Similarly, \(X_3(t)|_{U_3} - \frac{1}{3}t\) can have a triple pole only at the cusp \(1\) and the function

\[
\left( X_3(t)|_{U_3} - \frac{1}{3}t \right) + 6(t|_{U_3})^3 - 6t(1)(t|_{U_3})^2 - (2H_1 - 2t(1)^2)t|_{U_3}
\]
Self-recursions of the Hauptmodul $N(j_{1,3})$ is holomorphic on $\Gamma_1(3)\backslash S^*$ (use (2.1) $\sim$ (2.5)), and hence it is a constant. Substituting $X_3(t)$ for $\frac{1}{3}t^3 - H_1t - H_2$, we get

$$\left(\frac{1}{3}t^3 - H_1t - H_2\right)\big|_{U_3} = \frac{1}{3}t - 6(t|_{U_3})^3 + 6t(1)(t|_{U_3})^2 + (2H_1 - 2t(1)^2)t|_{U_3} + c$$

for a constant $c$. So we have

$$\frac{1}{3}t^3 - H_1t - H_2 = \text{(terms of degree } 3k \pm 1) + \left(\frac{1}{3}t - 6(t|_{U_3})^3 + 6t(1)(t|_{U_3})^2 + (2H_1 - 2t(1)^2)t|_{U_3}\right)(3z) + c,$$

and

$$\frac{1}{3}(q^{-1} + \sum H_k q^k)^3 - H_1(q^{-1} + \sum H_k q^k) - H_2 = \text{(terms of degree } 3k \pm 1) + \frac{1}{3}q^{-3} + \frac{1}{3}\sum H_k q^{3k} - 6(\sum H_k q^{3k})^3 + 6t(1)(\sum H_k q^{3k})^2 + (2H_1 - 2t(1)^2)(\sum H_k q^{3k}) + c.$$

Comparing the coefficients of $q^{3k}$, we have (1.4). Therefore, we complete the proof.

**Proof of Corollary 1.2.** First, by the 2-plicate formulas (1.1) we know that that four coefficients $H_1$, $H_2$, $H_3$ and $H_5$ are necessary to determine all the coefficients. From the appendix of [6] we see that the Hauptmodul $t$ is given by

$$t = \frac{240}{(j_{1,3} - 1)} + 9 = q^{-1} + 54q - 76q^2 - 234q^3 + 1188q^4 - 1384q^5 - 2916q^6 + \cdots,$$

where $E_4(z)$ is the normalized Eisenstein series of weight 4 and $j_{1,3} = E_4(z)/E_4(3z)$. By the self-recursion formula (1.3) and the 2-plicate formulas, we obtain the following identities:

$$H_4 = H_3 + \frac{H_1^2 - H_1}{2} = -H_1H_2 - t(1)H_3.$$

Thus $H_1$, $H_2$ and $H_3$ determine $H_4 = 1188$ and $t(1) = -12$. Since the self-recursion formula (1.4) yields

$$H_5 = -2(H_1H_3 + H_2^2) - \frac{1}{3}H_1^3 + \frac{1}{3}H_1 - 2t(1)^2H - 3 + 3H_1H_3,$$
we conclude that the first three coefficients $H_1$, $H_2$ and $H_3$ are needed to determine all the coefficients of the Hauptmodul of $\Gamma_1(3)$. This proves the corollary.

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