A Note on Operators in Hilbert C^* -Modules

M. Khanehgir and M. Hassani

Dept. of Math., Islamic Azad University of Mashhad

Mashhad P.O. Box 413-91735, Iran

khanehgir@mshdiau.ac.ir

hassani@mshdiau.ac.ir

Abstract

We explore commutativity up to a unitary factor for the pair of self-adjoint operators in Hilbert C^* -modules.

Mathematics Subject Classification: 46H40, 46L57

Keywords: Hilbert C^* -modules, partial isometry, unitary map

1. Introduction

Let A be a C^* -algebra (not necessarily unital or commutative). An inner product A -module is a linear space E which is right A -module (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E, a \in A, \lambda \in \mathbb{C}$), together with a map $(x, y) \rightarrow \langle x, y \rangle: E \times E \rightarrow A$ such that

- (i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
- (ii) $\langle x, ya \rangle = \langle x, y \rangle a \quad (x, y \in E, a \in A)$
- (iii) $\langle x, y \rangle^* = \langle y, x \rangle \quad (x, y \in E)$
- (iv) $\langle x, x \rangle \geq 0$ if $\langle x, x \rangle = 0$ then $x = 0$.

Given Hilbert C^* -modules E and F , denoted by $L(E, F)$ the set of all (bounded) adjointable maps. Also $K(E, F)$ is the closed linear subspace

spanned by $\{\Theta_{x,y} : x \in F, y \in E\}$ in $L(E, F)$, where $\Theta_{x,y}(z) = x \langle y, z \rangle$, for each $z \in E$. Specially, if $E = F$, we write $L(E)$, $K(E)$ respectively.

It is well known that $L(E)$ is a C^* -algebra and $K(E)$ is the closed two-sided ideal in $L(E)$. The basic materials can see [1].

If E is a Hilbert C^* -module, x, y in E is said to be orthogonal if $\langle x, y \rangle = 0$. In this case, denote by $x \perp y$. Given a closed submodule F of E , set $F^\perp = \{y \in E : \langle x, y \rangle = 0, \forall x \in F\}$, then $F^\perp$ is called orthogonal complement of F . Furthermore, if $F \oplus F^\perp = E$ then F is said to be complemented submodule. We know that a closed submodule of a Hilbert C^* -module need not be complemented.

The following proposition gives an equivalent characterization of complemented closed submodule.

Proposition 1.1: Let A be a C^* -algebra, E is a Hilbert C^* -module, E_0 is a closed submodule in E . The following are equivalent:

- (1) E_0 is a complemented submodule.
- (2) there is a projection p in $L(E)$ such that $K(E_0) \cong pK(E)p$, where $R(p)$ is the range of p .

Proof: See [2].

If F is complemented then for each z in E we can uniquely write $z = x + y$ with x in F and y in $F^\perp$. Just as in the case of Hilbert spaces, the equation $x = Pz$ defines a projection P in $L(E)$ whose range is F . Conversely, if P is a projection in $L(E)$ then the range of P is a complemented submodule of E , since it is easy to check that $\text{ran}(P)^\perp = \text{ran}(1 - P) = \ker(P)$.

An operator $u \in L(E, F)$ is said to be unitary if $u^*u = 1_E$, $uu^* = 1_F$. Also if $t \in L(E)$, then we define spectrum of t is the set $sp(t) = \{\lambda \in \mathbb{C} : \lambda I - t \text{ is not invertible}\}$ and we say t is positive and we denoted it by $t \in L(E)^+$ if $sp(t) \subseteq \mathbb{R}^+$.

The theorem of Miscenko which enables one to conclude that certain submodules are complemented.

Theorem 1.2: Let E, F be Hilbert A -modules and suppose that t in $L(E, F)$ has closed range. then

- (i) $\ker(t)$ is complemented submodule of E and $\ker(t)^\perp = \text{ran}(t^*)$.

(ii) $\text{ran}(t)$ is complemented submodule of F and $\text{ran}(t)^\perp = \ker(t^*)$.

(iii) the mapping $t^* \in L(F, E)$ also has closed range.

Proof: See [1].

In the following we give some useful theorems which we need them.

Theorem 1.3: Let u be a linear map from E to F . Then the following condition are equivalent:

(i) u is an isometric, surjective A -linear map.

(ii) u is a unitary element of $L(E, F)$.

Proof: See [1].

We know that if u is isometric then $\|u(x)\| = \|x\|$ for each $x \in E$.

Theorem 1.4: For t in $L(E, F)$, t^*F and t^*tE have the same closures.

Proof: See [1].

Theorem 1.5: If $t \in L(E)^+$ has closed range then $\text{ran}(t) = \text{ran}(t^{\frac{1}{2}})$.

Proof: Trivially $\text{ran}(t) \subseteq \text{ran}(t^{\frac{1}{2}})$. Conversely consider $C[z]$ denotes the algebra of all polynomials in an indeterminate z with complex coefficients and suppose that A is unital algebra. We know that the map from $C[z]$ into A which takes P to $P(a)$ which is $a \in A$, is a unital homomorphism. By Stone-Weierstrass theorem (See [3]) there is a sequence of polynomials with zero constant coefficient which tends to continuous function $\sqrt{x}$. Now we obtain $\text{ran}(t^{\frac{1}{2}}) \subseteq \text{ran}(t)$.

Definition 1.6: An element C in $L(E, F)$ is called a partial isometry (from E_0 to F_0) if $F_0 = \text{ran}(C)$ is complemented in F and there exists a complemented submodule E_0 of E such that C is isometric from E_0 onto F_0 and $C(E_0^\perp) = \{0\}$.

Suppose $t \in L(E, F)$ and that the closures of the ranges of t and t^* are both complemented, then define $u : \text{ran}(|t|) \rightarrow \text{ran}(t)$ by $u(|t|x) = tx$. It is easily seen that u is isometric. We define u on $\overline{\text{ran}(|t|)}$ as follows.

For $x \in \overline{\text{ran}(|t|)}$ there is a sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq \text{ran}(|t|)$ in which $a_n \rightarrow x$ as n tends to infinity. The sequence $\{u(a_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in E , so converges to an element of $\overline{\text{ran}(t)}$ say a . Now we define $u(x) = a$. Also we define u is equal zero on $\overline{\text{ran}(|t|)}^\perp$. Evidently $t = u|t|$ and u is a partial

isometry, so t has polar decomposition.

Just as for Hilbert space, one can easily that the following theorem holds for Hilbert C^* -modules.

Theorem 1.7: For an element C of $L(E, F)$ the following condition are equivalent.

- (i) C is a partial isometry.
- (ii) C^*C is a projection in $L(E)$.
- (iii) CC^* is a projection in $L(F)$.
- (iv) $CC^*C = C$.
- (v) $C^*CC^* = C^*$.

Proof: See [1].

Lemma 1.8: If $t \in L(E)$ is normal then $\ker(t) = \ker(t^*)$.

Proof: We have

$$\langle t^*tx - tt^*x, x \rangle = |tx|^2 - |t^*x|^2$$

So t is normal iff $|tx| = |t^*x|$ and therefore $\ker(t) = \ker(t^*)$.

2. Main Theorem

Theorem: Let t, s be bounded self-adjoint operators on Hilbert C^* -module E and ts and $|ts|$ have closed range. The following are equivalent:

- (i) $ts^2t = st^2s$.
- (ii) $ts = ust$ for some unitary u .

Proof: (ii) $\rightarrow$ (i) : Observe that $st = tsu^*$, and so $ts = utsu^*$. Hence ts commutes with u and u^* , and similarly for st . Thus we get

$$ts^2t = ust u^* ts = st u u^* ts = st^2s.$$

(ii) $\rightarrow$ (i) : The condition (i) is equivalent to $|ts| = |st|$, where we have $|c| = (c^*c)^{\frac{1}{2}}$. Thus ts is normal. It follows that

$$\ker(ts) = \ker(st) = \ker(|ts|) = \ker(|st|).$$

Now let Q denotes the associated projection. By theorem 1.4 and 1.5 we have the ranges of these four operators are closed and all coincide. Since we have

$$st(E) = \overline{st(E)} = \overline{(ts)^*(E)} = \overline{(ts)^*(ts)(E)} = \overline{ts^2tE} = \overline{(st)^*E} = ts(E).$$

Let P denotes the associated projection to $\text{ran}(ts) = \text{ran}(st) = \text{ran}(|ts|) =$

$\text{ran}(|st|)$.

Hence $P+Q=I$. By polar decomposition theorem there are partial isometries v, w such that $ts = v|ts|$, $st = w|st|$. Note that $vv^* = v^*v = ww^* = w^*w = P$. Since P is a projection in $L(E)$ associated to $\text{ran}(ts) = \text{ran}(v) = \ker(v)^\perp$ hence if $x \in \ker(v)^\perp$ then $\langle v^*v(x), x \rangle = \|v(x)\|^2 = \|x\|^2 = \langle x, x \rangle = \langle P(x), x \rangle$ and if $x \in \ker(v)$ then $\langle v^*v(x), x \rangle = 0 = \langle P(x), x \rangle$. Hence by polarization identity $v^*v = P$.

Similarly $vv^* = w^*w = ww^* = P$. It is easily verified that $|ts|P = |ts|$. From this it obtains that v, v^*, w and w^* commute with $|ts|$. By assumption it follows that $|ts| = |st|$. It yields that

$$ts = v|ts| = |st|w^* = w^*|ts|$$

and since the range of $|ts|$ is dense in $P(E)$ it follows that $v = w^*$ and $v^* = w$.

Now we have

$$ts = v|ts| = v|st| = vw^*st = v^2st.$$

We can extend $v^2|_{P(E)}$ to a unitary map $u = v^2|_{P(E)} \oplus Q$.

References:

- [1] E. C. Lance, Hilbert C^* -Module, Cambridge University Press, London, 1995.
- [2] Zhang Lunchuan, The Characterization of Bounded Generalized Inverse Module Maps and Application to C^* -Algebras Factor Decomposition. preprint.
- [3] W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, New York, 1976.

Received: May 2, 2006