A Note on Operators in Hilbert C*-Modules

M. Khanehgir and M. Hassani

Dept. of Math., Islamic Azad University of Mashhad
Mashhad P.O. Box 413-91735, Iran
khanehgir@mshdiau.ac.ir
hassani@mshdiau.ac.ir

Abstract

We explore commutativity up to a unitary factor for the pair of self-adjoint operators in Hilbert C*-modules.

Mathematics Subject Classification: 46H40, 46L57

Keywords: Hilbert C*-modules, partial isometry, unitary map

1. Introduction

Let $A$ be a $C^*$-algebra (not necessarily unital or commutative). An inner product $A$-module is a linear space $E$ which is right $A$-module (with compatible scalar multiplication: $\lambda(a) = \lambda(a)$ for $x \in E, a \in A, \lambda \in \mathcal{C}$), together with a map $(x, y) \rightarrow \langle x, y \rangle: E \times E \rightarrow A$ such that

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ (for $x \in E, a \in A$)
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ (for $x, y \in E$)
(iv) $\langle x, x \rangle \geq 0$ if $\langle x, x \rangle = 0$ then $x = 0$.

Given Hilbert $C^*$-modules $E$ and $F$, denoted by $L(E, F)$ the set of all (bounded) adjointable maps. Also $K(E, F)$ is the closed linear subspace
spanned by \( \{ \Theta_{x,y} : x \in F, y \in E \} \) in \( L(E, F) \), where \( \Theta_{x,y}(z) = x < y, z > \), for each \( z \in E \). Specially, if \( E = F \), we write \( L(E), K(E) \) respectively.

It is well known that \( L(E) \) is a \( C^* \)-algebra and \( K(E) \) is the closed two-sided ideal in \( L(E) \). The basic materials can see [1].

If \( E \) is a Hilbert \( C^* \)-module, \( x, y \) in \( E \) is said to be orthogonal if \( \langle x, y \rangle = 0 \). In this case, denote by \( x \perp y \).

Given a closed submodule \( F \) of \( E \), set \( F^\perp = \{ y \in E : \langle x, y \rangle = 0, \forall x \in F \} \), then \( F^\perp \) is called orthogonal complement of \( F \). Furthermore, if \( F \oplus F^\perp = E \) then \( F \) is said to be complemented submodule. We know that a closed submodule of a Hilbert \( C^* \)-module need not be complemented.

The following proposition gives an equivalent characterization of complemented closed submodule.

**Proposition 1.1:** Let \( A \) be a \( C^* \)-algebra, \( E \) is a Hilbert \( C^* \)-module, \( E_0 \) is a closed submodule in \( E \). The following are equivalent:

1. \( E_0 \) is a complemented submodule.
2. there is a projection \( p \) in \( L(E) \) such that \( K(E_0) \cong pK(E)p \), where \( R(p) \) is the range of \( p \).

**Proof:** See [2].

If \( F \) is complemented then for each \( z \) in \( E \) we can uniquely write \( z = x + y \) with \( x \) in \( F \) and \( y \) in \( F^\perp \). Just as in the case of Hilbert spaces, the equation \( x = Pz \) defines a projection \( P \) in \( L(E) \) whose range is \( F \). Conversely, if \( P \) is a projection in \( L(E) \) then the range of \( P \) is a complemented submodule of \( E \), since it is easy to check that \( ran(P)^\perp = ran(1 - P) = ker(P) \).

An operator \( u \in L(E, F) \) is said to be unitary if \( uu^* = 1_E \) and \( uu^* = 1_F \). Also if \( t \in L(E) \), then we define spectrum of \( t \) is the set \( sp(t) = \{ \lambda \in \mathbb{C} : \lambda I - t \text{ is not invertible} \} \) and we say \( t \) is positive and we denoted it by \( t \in L(E)^+ \) if \( sp(t) \subseteq \mathbb{R}^+ \).

The theorem of Miscenko which enables one to conclude that certain submodules are complemented.

**Theorem 1.2:** Let \( E, F \) be Hilbert \( A \)-modules and suppose that \( t \in L(E, F) \) has closed range. then

(i) \( \ker(t) \) is complemented submodule of \( E \) and \( \ker(t)^\perp = ran(t^*) \).
(ii) \( \text{ran}(t) \) is complemented submodule of \( F \) and \( \text{ran}(t) \perp = \ker(t^*) \).

(iii) the mapping \( t^* \in L(F, E) \) also has closed range.

**Proof:** See [1].

In the following we give some useful theorems which we need them.

**Theorem 1.3:** Let \( u \) be a linear map from \( E \) to \( F \). Then the following condition are equivalent:

(i) \( u \) is an isometric, surjective \( A \)-linear map.

(ii) \( u \) is a unitary element of \( L(E, F) \).

**Proof:** See [1].

We know that if \( u \) is isometric then \(| u(x) | = | x | \) for each \( x \in E \).

**Theorem 1.4:** For \( t \in L(E, F) \), \( t^*F \) and \( t^*tE \) have the same closures.

**Proof:** See [1].

**Theorem 1.5:** If \( t \in L(E)^+ \) has closed range then \( \text{ran}(t) = \text{ran}(t^{\frac{1}{2}}) \).

**Proof:** Trivially \( \text{ran}(t) \subseteq \text{ran}(t^{\frac{1}{2}}) \). Conversely consider \( C[z] \) denotes the algebra of all polynomials in an indeterminate \( z \) with complex coefficients and suppose that \( A \) is unital algebra. We know that the map from \( C[z] \) into \( A \) which takes \( P \) to \( P(a) \) which is \( a \in A \), is a unital homomorphism. By Stone-Weierstrass theorem (See [3]) there is a sequence of polynomials with zero constant coefficient which tends to continuous function \( \sqrt{x} \). Now we obtain \( \text{ran}(t^{\frac{1}{2}}) \subseteq \text{ran}(t) \).

**Definition 1.6:** An element \( C \) in \( L(E, F) \) is called a partial isometry (from \( E_0 \) to \( F_0 \)) if \( F_0 = \text{ran}(C) \) is complemented in \( F \) and there exists a complemented submodule \( E_0 \) of \( E \) such that \( C \) is isometric from \( E_0 \) onto \( F_0 \) and \( C(E_0^\perp) = \{0\} \).

Suppose \( t \in L(E, F) \) and that the closures of the ranges of \( t \) and \( t^* \) are both complemented, then define \( u : \text{ran}(|t|) \to \text{ran}(t) \) by \( u(|t|x) = tx \). It is easily seen that \( u \) is isometric. We define \( u \) on \( \overline{\text{ran}(|t|)} \) as follows.

For \( x \in \overline{\text{ran}(|t|)} \) there is a sequence \( \{a_n\}_{n \in \mathbb{N}} \subseteq \text{ran}(|t|) \) in which \( a_n \to x \) as \( n \) tends to infinity. The sequence \( \{u(a_n)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( E \), so converges to an element of \( \overline{\text{ran}(t)} \) say \( a \). Now we define \( u(x) = a \). Also we define \( u \) is equal zero on \( \overline{\text{ran}(|t|)}^\perp \). Evidently \( t = u|t| \) and \( u \) is a partial
isometry, so $t$ has polar decomposition.

Just as for Hilbert space, one can easily that the following theorem holds for Hilbert $C^*$–modules.

**Theorem 1.7:** For an element $C$ of $L(E, F)$ the following condition are equivalent.

(i) $C$ is a partial isometry.

(ii) $C^*C$ is a projection in $L(E)$.

(iii) $CC^*$ is a projection in $L(F)$.

(iv) $C^*C = C$.

(v) $C^*CC^* = C^*$.

**Proof:** See [1].

**Lemma 1.8:** If $t \in L(E)$ is normal then $\ker(t) = \ker(t^*)$.

**Proof:** We have

$$\langle t^*tx - tt^*x, x \rangle = |tx|^2 - |t^*x|^2$$

So $t$ is normal iff $|tx| = |t^*x|$ and therefore $\ker(t) = \ker(t^*)$.

**2.Main Theorem**

**Theorem:** Let $t, s$ be bounded self-adjoint operators on Hilbert $C^*$–module $E$ and $ts$ and $|ts|$ have closed range. The following are equivalent:

(i) $ts^2t = st^2s$.

(ii) $ts = ust$ for some unitary $u$.

**Proof:** (ii) $\rightarrow$ (i) : Observe that $st = tsu^*$, and so $ts = ustu^*$. Hence $ts$ commutes with $u$ and $u^*$, and similarly for $st$. Thus we get

$$ts^2t = ustu^*ts = stuu^*ts = st^2s.$$  

(ii) $\rightarrow$ (i) : The condition (i) is equivalent to $|ts| = |st|$, where we have $|c| = (c^*c)^{\frac{1}{2}}$. Thus $ts$ is normal. It follows that

$$\ker(ts) = \ker(st) = \ker(|ts|) = \ker(|st|).$$

Now let $Q$ denotes the associated projection. By theorem 1.4 and 1.5 we have the ranges of these four operators are closed and all coincide. Since we have

$$st(E) = \overline{st(E)} = \overline{(ts)^*(E)} = \overline{(ts)(ts)(E)} = \overline{ts^2tE} = \overline{(st)^*E} = ts(E).$$

Let $P$ denotes the associated projection to $\text{ran}(ts) = \text{ran}(st) = \text{ran}(|ts|) =$
ran(\mid st \mid).

Hence $P + Q = I$. By polar decomposition theorem there are partial isometries $v, w$ such that $ts = v \mid ts \mid$, $st = w \mid st \mid$. Note that $vv^* = v^*v = ww^* = w^*w = P$. Since $P$ is a projection in $L(E)$ associated to $\text{ran}(ts) = \text{ran}(v) = \ker(v)^\perp$ hence if $x \in \ker(v)^\perp$ then $\langle v^*v(x), x \rangle = \| v(x) \|^2 = \| x \|^2 = \langle x, x \rangle = \langle P(x), x \rangle$ and if $x \in \ker(v)$ then $\langle v^*v(x), x \rangle = 0 = \langle P(x), x \rangle$. Hence by polarization identity $v^*v = P$.

Similarly $vv^* = w^*w = ww^* = P$. It is easily verified that $\mid ts \mid = \mid ts \mid$. From this it obtains that $v, v^*, w$ and $w^*$ commute with $\mid ts \mid$. By assumption it follows that $\mid ts \mid = \mid st \mid$. It yields that

$$ts = v \mid ts \mid = st \mid w^* = w^* \mid ts \mid$$

and since the range of $\mid ts \mid$ is dense in $P(E)$ it follows that $v = w^*$ and $v^* = w$.

Now we have

$$ts = v \mid ts \mid = v \mid st \mid = vw^*st = v^2st.$$  

We can extend $v^2 \mid_{P(E)}$ to a unitary map $u = v^2 \mid_{P(E)} \oplus Q$.

References:


Received: May 2, 2006