The Characterization of $\text{PGL}(2, p)$ for Some $p$ by Their Element Orders

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Abstract

For any group $G$, $\pi_e(G)$ denotes the set of orders of its elements. If $\Omega$ is a subset of positive integers, $h(\Omega)$ stands for the number of distinct isomorphism classes of finite groups $G$ such that $\pi_e(G) = \Omega$. Let $\Omega = \pi_e(\text{PGL}(2, p))$, where $p$ is a prime number and $5 \leq p < 100$. In this paper, we prove that $h(\Omega) \in \{1, \infty\}$.

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1 Introduction

For any group $G$, we denote by $\pi_e(G)$ the set of orders of elements in $G$. This set is closed and partially ordered by the divisibility relation, hence, it is uniquely determined by $\mu(G)$, a subset of its elements that is maximal under the divisibility relation. If $\Omega$ is a subset of positive integers, $h(\Omega)$ stands for the number of distinct isomorphism classes of finite groups $G$ such that $\pi_e(G) = \Omega$. 

Ω. It is clear that $h(\pi_e(G)) \geq 1$ for any group $G$. A group $G$ is called $k$-distinguishable if $h(\pi_e(G)) = k < \infty$; otherwise $G$ is called non-distinguishable. Also a 1-distinguishable group is called a characterizable group.

In [7], V. D. Mazurov proved the following result: Let $G = \text{PGL}(r, p^s)$. If $p, r$ are odd primes, $p - 1$ is divisible by $r$ but not by $r^2$, and $s$ is a natural number not divisible by $r$, then $h(\pi_e(G)) = \infty$.

In [5] and [10], W. J. Shi proved the following result: If $\Omega = \pi_e(\text{PGL}(2, p))$, where $p$ is a prime number and $5 \leq p \leq 19$ then $h(\Omega) \in \{1, \infty\}$.

In the present article, we prove the following:

**Main Theorem** Let $p$ be a prime number and $5 \leq p < 100$. Further, assume that $\Omega = \pi_e(\text{PGL}(2, p))$. Then $h(\Omega) \in \{1, \infty\}$.

Throughout this paper, all groups considered are finite and simple groups are non-Abelian. Given a group $G$, denote by $\pi(G)$ the set of all prime divisors of the order of $G$. All further unexplained notations are standard and can be found for instance in [1] and [3].

## 2 Preliminary Results

Our arguments depend on the prime graph components of simple groups (see [4] and [12]). The prime graph $\Gamma(G)$ of a group $G$ is a graph whose vertices are prime divisors of the order of $G$ and two distinct primes $p, q$ are adjacent, if $G$ contains an element of order $pq$. Denote the connected components of the prime graph of $G$ by $\pi_i = \pi_i(G), i = 1, 2, \ldots, t(G)$, where $t(G)$ is the number of connected components. When $|G|$ is even, let $\pi_1$ be the connected component containing 2. A group $G$ is called 2-Frobenius if there exists a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ of $G$ such that $H$ is the Frobenius kernel of $K$ and $K/H$ is the Frobenius kernel of $G/H$.

We shall also use the following unpublished result of K. W Gruenberg and O. H. Kegel (see [12], Theorem A).

**Lemma 2.1** If $G$ is a group such that $t(G) \geq 2$, then $G$ has one of the following structures:

(a) Frobenius or 2-Frobenius;
(b) simple;
(c) an extension of a $\pi_1$-group by a simple group;
(d) simple by $\pi_1$; or
(e) $\pi_1$ by simple by $\pi_1$.

**Remark.** In fact, when $G$ is neither Frobenius nor 2-Frobenius, by Lemma 2.1, $G$ has a normal series $1 \triangleleft N \triangleleft G_1 \triangleleft G$ such that $N$ is a nilpotent $\pi_1$-group,
$G_1 = G_1/N$ is a simple group, and $G/G_1$ is a solvable $\pi_1$-group.

The following lemma is taken from [11].

**Lemma 2.2** Let $G$ be a group and $N$ be a minimal normal subgroup of $G$. We further, suppose, $N$ is an elementary Abelian $p$-group, then $h(\pi_e(G)) = \infty$. In particular, $h(\pi_e(G)) = \infty$ when $G$ is a solvable group.

We also need the following result:

**Lemma 2.3** Let $G$ be a finite non-solvable group. Assume that there exists $n = 2^a3^b5^c$, where $a, b, c \geq 0$, and $n \neq 8, 12, 20$, such that $n \in \pi_e(G)\setminus\pi_e(SL(2, 5))$. Then $G$ is neither Frobenius nor 2-Frobenius.

**Proof.** Since $G$ is non-solvable, $G$ cannot be 2-Frobenius. If $G$ is a Frobenius group with kernel $K$ and complement $H$ then $H$ is non-solvable. Now by the structure of non-solvable Frobenius complements ([9], Theorem 18.6), we know that $H$ has a normal subgroup $H_0$ of index $\leq 2$ such that $H_0 \cong SL(2, 5) \times Z$, where every Sylow subgroup of $Z$ is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. Since $\pi_e(SL(2, 5)) = \{1, 2, 3, 4, 5, 6, 10\}$, the number of prime graph components of $H$ is 1 and the corresponding number for $G$ must be 2. In fact $T(G) = \{\pi_1(G) = \pi(H), \pi_2(G) = \pi(K)\}$. Now from $n = 2^a3^b5^c \in \pi_e(G)\setminus\pi_e(SL(2, 5))$ it follows that $n \in \pi_e(H)$, and so there exists $1 \neq x \in H$ such that $o(x) = n$. As $n \notin \pi_e(SL(2, 5))$, it follows that $x \notin H_0$ and hence $|H : H_0| = 2$. This implies $x^2 \in H_0$ and $o(x^2) = n/(2, n) \in \pi_e(SL(2, 5))$. Therefore, $(2, n) = 2$ and $n/2 \in \pi_e(SL(2, 5))$. Now we obtain $n \in \{8, 12, 20\}$, which contradicts the assumption. The lemma is proved. $\square$

**Lemma 2.4** Let $G$ be a finite non-solvable group and $\pi_e(G) = \pi_e(PGL(2, p))$, where $p \geq 23$ is a prime number. Then $G$ is neither Frobenius nor 2-Frobenius.

**Proof.** We know $\mu(PGL(2, p)) = \{p−1, p, p+1\}$. Clearly $G$ is not 2-Frobenius, since $G$ in non-solvable. Assume $G$ is a Frobenius group with kernel $K$ and complement $H$, then as before $H$ has a normal subgroup $H_0$ of index $\leq 2$ and $H_0 \cong SL(2, 5) \times Z$, where every Sylow subgroup of $Z$ is cyclic and $\pi(Z)\cap\pi(30) = \emptyset$. Moreover, we have

$\pi_1(G) = \pi(H) = \pi(p^2 − 1)$ and $\pi_2(G) = \pi(K) = \{p\}.$

In particular $\{2, 3, 5\} \subset \pi(G)$. From Lemma 2.3, we get $15 \notin \pi_e(G)$, and so $5|p+1$ and $3|p−1$ or $5|p−1$ and $3|p+1$, by the structure consideration of $\pi_e(G)$. Therefore, $Z = 1$, which forces $\pi(H) = \{2, 3, 5\}$, whence $\pi_e(G) = \{2, 3, 5, p\}$. On the other hand, by Lemma 2.3 and the structure of $\pi_e(G)$, we must have
$p^2 - 1 = 2^\alpha \cdot 3.5$ where $\alpha > 5$ as $p \geq 23$. But then $2^4 \in \pi_e(G)$, which is a contradiction by Lemma 2.3. The proof is complete now. □

We also need the following Lemma.

**Lemma 2.5** Let $G$ be a simple group such that $\{p\} \subset \pi(G) \subseteq \pi(p!)$, where $p$ is a prime number and $23 \leq p < 100$. Then $G$ is given in Table I. Moreover, if $\{p\} \subset \pi(G) \subseteq \pi(\text{PSL}(2,p))$ then:

(i) For $p = 31$, $G \cong \text{PSL}(2,31)$ or $\text{PSL}(3,5)$,
(ii) For $p \neq 31$, $G \cong \text{PSL}(2,p)$.

**Proof.** Considering the fact that $\{p\} \subset \pi(G) \subseteq \pi(p!)$ and by referring to [2] and [8], for $p \neq 37$ we have the simple groups contained in Table I. When $p = 37$, the proof is similar as in ([2], Lemma 2.6). The rest of proof, when $\{p\} \subset \pi(G) \subseteq \pi(\text{PSL}(2,p))$, follows by checking the prime factors of $|G|$. □

We also need the following result of V. D. Mazurov ([6], Lemma 1).

**Lemma 2.6** Let $N$ be a normal subgroup of $H$. Assume that $H/N$ is a Frobenius group with kernel $A$ and cyclic complement $B$. If $(|A|, |N|) = 1$ and $A$ is not contained in $NC_G(N)/N$, then $p|B| \in \pi_e(H)$, where $p$ is a prime factor of $|N|$.

### 3 Proof of the Main Theorem

**Proof of the Main Theorem.** It is well known that

$\text{Aut}(\text{PSL}(2,p^n)) = \text{PGL}(2,p^n) : Z_n$.

When $n = 1$, we have $\text{Aut}(\text{PSL}(2,p)) = \text{PGL}(2,p)$, and hence

$\mu(\text{PGL}(2,p)) = \mu(\text{Aut}(\text{PSL}(2,p))) = \{p - 1, p, p + 1\}$.

Let $G$ be a group such that $\pi_e(G) = \pi_e(\text{PSL}(2,p)) = \{p, \text{all factors of } p - 1 \text{ and } p + 1\}$. Then the connected components of the prime graph of $G$ are:

$\pi_1 = \pi(p^2 - 1)$ and $\pi_2 = \{p\}$.

When $p < 23$, it has been proved that $h(\pi_e(G)) \in \{1, \infty\}$ (see [8] and [9]). Thus, we assume $p \geq 23$. If $G$ is solvable, then $h(\pi_e(G)) = \infty$, by Lemma 2.2. Hence, we suppose $G$ is non-solvable. Now by Lemma 2.4, $G$ is neither Frobenius nor 2-Frobenius. On the other hand, by Lemma 2.1, there exists a normal series $1 \leq N < G_1 \leq G$ such that $N$ is a nilpotent $\pi_1$-group, $G_1/N$ is a simple group, and $G/G_1$ is a solvable $\pi_1$-group. Since $G_1/N(G_1) = 1$, we have
\[ G = \frac{N_{G/N}(G_1)}{C_{G/N}(G_1)} \cong \text{a subgroup of } \text{Aut}(G_1), \]

thus we may assume that \( G/N \leq \text{Aut}(G_1) \).

### Table I. Simple groups with condition \( \{p\} \subset \pi(G) \subseteq \pi(p!) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>Finite simple groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>( A_n, n = 23, 24, 25, 26, 27, 28; M_{23}, M_{24}, Co_3, Co_2, Co_1, Fi_{23}; A_1(23), 2A_2(23). )</td>
</tr>
<tr>
<td>29</td>
<td>( A_n, n = 29, 30; Ru, F_{24}'; A_1(17^2), S_4(17), 2A_3(17), A_1(29). )</td>
</tr>
<tr>
<td>31</td>
<td>( A_n, n = 31, 32, 33, 34, 35, 36; O'N, Th; A_4(2), A_5(2), A_4(2^2), A_5(2^2), A_1(2^5), S_12(2), O_{10}^+(2), O_{12}^+(2), O_{12}(2), A_2(5), A_3(5), A_2(5^2), A_1(5^3), O_7(5), S_5(5), O_8^+(5), G_2(5), A_1(31), 2A_2(31). )</td>
</tr>
<tr>
<td>37</td>
<td>( A_n, n = 37, 38, 39, 40; 2A_2(3^4), 2A_3(3^3), 2G_2(3^3), A_1(11^3), G_2(11), 2A_2(11), A_1(31^2), S_4(31), 2A_3(31), A_1(37). )</td>
</tr>
<tr>
<td>41</td>
<td>( A_n, n = 41, 42; A_2(1)^4), A_3(3^2), S_4(2^2), A_4(2^2), A_5(2^2), S_5(2^2), A_1(3^4), O_9(3), S_1(3^2), S_8(3), O_{10}^+(3), O_{10}^-(3), O_{11}(3), A_1(41), A_1(41^2), S_4(41). )</td>
</tr>
<tr>
<td>44</td>
<td>( A_n, n = 43, 44, 45, 46; A_4; 2A_6(2), 2A_7(2), 2A_8(2), 2A_9(2), O_{14}^+(2), A_2(7^2), A_1(7^3), O_7(7), S_7(7), O_8^+(7), G_2(7), A_2(7), 2A_3(7), 2A_2(37), A_1(43), A_1(43^2), S_4(43). )</td>
</tr>
<tr>
<td>47</td>
<td>( A_n, n = 47, 48, 49, 50, 51, 52; B = F_{24}'; A_1(47), A_1(47^2), S_4(47). )</td>
</tr>
<tr>
<td>53</td>
<td>( A_n, n = 53, 54, 55, 56, 57, 58; A_1(23^2), B_2(23), 2A_3(23), A_1(53). )</td>
</tr>
<tr>
<td>59</td>
<td>( A_n, n = 59, 60; A_1(59). )</td>
</tr>
<tr>
<td>61</td>
<td>( A_n, n = 61, 62, 63, 64, 65, 66; A_4(3^2), A_1(3^3), A_2(5), B_5(3), C_5(3), A_1(11^2), A_2(11^2), B_2(11), B_3(11), C_3(11), D_4(11), 2A_3(11), A_2(13), A_3(13), A_2(47), A_3(47), A_1(61). )</td>
</tr>
<tr>
<td>67</td>
<td>( A_n, n = 67, 68, 69, 70; Ly; A_2(29), A_2(37), A_1(37^2), G_2(37), A_1(67). )</td>
</tr>
<tr>
<td>71</td>
<td>( A_n, n = 71, 72; F_1; A_4(5), A_5(5), A_1(71). )</td>
</tr>
<tr>
<td>73</td>
<td>( A_n, n = 73, 74, 75, 76, 77, 78; A_2(2^4), A_1(2^4), G_2(2^4), C_4(2^4), E_6(2), B_3(2^4), C_3(2^4), A_1(3^6), A_5(3^2), B_6(3), B_2(3^3), C_6(3), D_4(3^2), G_2(3^2), F_4(3), 2A_2(3^2), 2A_3(3^3), 2D_6(3), 2E_6(3), A_1(73), A_1(73^2), B_2(73). )</td>
</tr>
<tr>
<td>79</td>
<td>( A_n, n = 79, 80, 81, 82; A_2(23), A_3(23), A_2(23^2), A_1(23^4), B_3(23), C_3(23), D_4(23), G_2(23), A_1(79), A_2(79). )</td>
</tr>
<tr>
<td>83</td>
<td>( A_n, n = 83, 84, 85, 86, 87, 88; A_1(83), A_1(83^2). )</td>
</tr>
<tr>
<td>89</td>
<td>( A_n, n = 89, 90, 91, 92, 93, 94, 95, 96; A_1(89). )</td>
</tr>
<tr>
<td>97</td>
<td>( A_n, n = 97, 98, 99, 100; A_2(61), A_1(97). )</td>
</tr>
</tbody>
</table>

Note that one of the components of the prime graph of \( \overline{G}_1 \) must be \( \{p\} \) and so \( \{p\} \subseteq \pi(\overline{G}_1) \subseteq \pi(G) = \pi(\text{PSL}(2, p)) \). Now according to Lemma 2.5, if \( p = 31 \) we have \( \overline{G}_1 \cong \text{PSL}(2, 31) \) or \( \text{PSL}(3, 5) \), and if \( p \neq 31 \) then \( \overline{G}_1 \cong \text{PSL}(2, p) \). We claim that in all cases \( \overline{G}_1 \cong \text{PSL}(2, p) \). In fact, if \( p = 31 \)
and $G_1 \cong \text{PSL}(3, 5)$, then $24 \in \pi_e(G_1) = \pi_e(\text{PSL}(5, 3))$ but $24 \notin \pi_e(G) = \pi_e(\text{PSL}(2, 31))$, and this is a contradiction. Therefore $G_1 \cong \text{PSL}(2, p)$.

Let $P/N$ be a Sylow $p$-subgroup of $G_1$ and $X/N$ be the normalizer in $G_1$ of $P/N$. Then $X/N$ is a Frobenius group of order $p(p - 1)/2$, with cyclic complement of order $(p - 1)/2$. Now, by lemma 2.6, we deduce that $N$ is a 2-group. First, suppose that $N = 1$. Then $G_1 = \text{PSL}(2, p)$. Moreover we may assume that $G_1 \leq G \leq \text{Aut}(G_1)$. Since $|\text{Out}(\text{PSL}(2, p))| = 2$, it follows that $G \cong \text{PSL}(2, p)$ or $G \cong \text{Aut}(\text{PSL}(2, p)) \cong \text{PGL}(2, p)$. But since $p + 1 \in \pi_e(\text{PGL}(2, p)) \setminus \pi_e(\text{PSL}(2, p))$, we have $G \cong \text{PGL}(2, p)$. Next, suppose that $N \neq 1$. Without loss of generality, we may assume that $N$ is an elementary Abelian 2-subgroup of $G$, by considering $G/M$, where $M$ is the preimage of $\Phi(N/O_2(G))$ in $G$. Now from Lemma 2.2, we see that $h(\pi_e(G)) = \infty$. Therefore, the proof of Main Theorem is complete. \qed

References


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