The Characterization of $\mathrm{PGL}(2, p)$ for Some p by Their Element Orders

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Abstract

For any group G , $\pi_e(G)$ denotes the set of orders of its elements. If Ω is a subset of positive integers, $h(\Omega)$ stands for the number of distinct isomorphism classes of finite groups G such that $\pi_e(G) = \Omega$. Let $\Omega = \pi_e(\mathrm{PGL}(2, p))$, where p is a prime number and $5 \leq p < 100$. In this paper, we prove that $h(\Omega) \in \{1, \infty\}$.

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1 Introduction

For any group G , we denote by $\pi_e(G)$ the set of orders of elements in G . This set is *closed* and *partially* ordered by the divisibility relation, hence, it is uniquely determined by $\mu(G)$, a subset of its elements that is *maximal* under the divisibility relation. If Ω is a subset of positive integers, $h(\Omega)$ stands for the number of distinct isomorphism classes of finite groups G such that $\pi_e(G) =$

Ω . It is clear that $h(\pi_e(G)) \geq 1$ for any group G . A group G is called *k-distinguishable* if $h(\pi_e(G)) = k < \infty$; otherwise G is called *non-distinguishable*. Also a 1-distinguishable group is called a *characterizable* group.

In [7], V. D. Mazurov proved the following result: *Let $G = PGL(r, p^s)$. If p, r are odd primes, $p - 1$ is divisible by r but not by r^2 , and s is a natural number not divisible by r , then $h(\pi_e(G)) = \infty$.*

In [5] and [10], W. J. Shi proved the following result: *If $\Omega = \pi_e(PGL(2, p))$, where p is a prime number and $5 \leq p \leq 19$ then $h(\Omega) \in \{1, \infty\}$.*

In the present article, we prove the following:

Main Theorem *Let p be a prime number and $5 \leq p < 100$. Further, assume that $\Omega = \pi_e(PGL(2, p))$. Then $h(\Omega) \in \{1, \infty\}$.*

Throughout this paper, all groups considered are finite and simple groups are non-Abelian. Given a group G , denote by $\pi(G)$ the set of all prime divisors of the order of G . All further unexplained notations are standard and can be found for instance in [1] and [3].

2 Preliminary Results

Our arguments depend on the prime graph components of simple groups (see [4] and [12]). The *prime graph* $\Gamma(G)$ of a group G is a graph whose vertices are prime divisors of the order of G and two distinct primes p, q are adjacent, if G contains an element of order pq . Denote the *connected components* of the prime graph of G by $\pi_i = \pi_i(G)$, $i = 1, 2, \dots, t(G)$, where $t(G)$ is the number of connected components. When $|G|$ is even, let π_1 be the connected component containing 2. A group G is called *2-Frobenius* if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ of G such that H is the Frobenius kernel of K and K/H is the Frobenius kernel of G/H .

We shall also use the following unpublished result of K. W. Gruenberg and O. H. Kegel (see [12], Theorem A).

Lemma 2.1 *If G is a group such that $t(G) \geq 2$, then G has one of the following structures:*

- (a) *Frobenius or 2-Frobenius;*
- (b) *simple;*
- (c) *an extension of a π_1 -group by a simple group;*
- (d) *simple by π_1 ; or*
- (e) *π_1 by simple by π_1 .*

Remark. In fact, when G is neither Frobenius nor 2-Frobenius, by Lemma 2.1, G has a normal series $1 \trianglelefteq N \triangleleft G_1 \trianglelefteq G$ such that N is a nilpotent π_1 -group,

$\overline{G}_1 = G_1/N$ is a simple group, and G/G_1 is a solvable π_1 -group.

The following lemma is taken from [11].

Lemma 2.2 *Let G be a group and N be a minimal normal subgroup of G . We further, suppose, N is an elementary Abelian p -group, then $h(\pi_e(G)) = \infty$. In particular, $h(\pi_e(G)) = \infty$ when G is a solvable group.*

We also need the following result:

Lemma 2.3 *Let G be a finite non-solvable group. Assume that there exists $n = 2^a \cdot 3^b \cdot 5^c$, where $a, b, c \geq 0$, and $n \neq 8, 12, 20$, such that $n \in \pi_e(G) \setminus \pi_e(\text{SL}(2, 5))$. Then G is neither Frobenius nor 2-Frobenius.*

Proof. Since G is non-solvable, G cannot be 2-Frobenius. If G is a Frobenius group with kernel K and complement H then H is non-solvable. Now by the structure of non-solvable Frobenius complements ([9], Theorem 18.6), we know that H has a normal subgroup H_0 of index ≤ 2 such that $H_0 \cong \text{SL}(2, 5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. Since $\pi_e(\text{SL}(2, 5)) = \{1, 2, 3, 4, 5, 6, 10\}$, the number of prime graph components of H is 1 and the corresponding number for G must be 2. In fact $T(G) = \{\pi_1(G) = \pi(H), \pi_2(G) = \pi(K)\}$. Now from $n = 2^a \cdot 3^b \cdot 5^c \in \pi_e(G) \setminus \pi_e(\text{SL}(2, 5))$ it follows that $n \in \pi_e(H)$, and so there exists $1 \neq x \in H$ such that $o(x) = n$. As $n \notin \pi_e(\text{SL}(2, 5))$, it follows that $x \notin H_0$ and hence $|H : H_0| = 2$. This implies $x^2 \in H_0$ and $o(x^2) = n/(2, n) \in \pi_e(\text{SL}(2, 5))$. Therefore, $(2, n) = 2$ and $n/2 \in \pi_e(\text{SL}(2, 5))$. Now we obtain $n \in \{8, 12, 20\}$, which contradicts the assumption. The lemma is proved. $\square$

Lemma 2.4 *Let G be a finite non-solvable group and $\pi_e(G) = \pi_e(\text{PGL}(2, p))$, where $p \geq 23$ is a prime number. Then G is neither Frobenius nor 2-Frobenius.*

Proof. We know $\mu(\text{PGL}(2, p)) = \{p-1, p, p+1\}$. Clearly G is not 2-Frobenius, since G is non-solvable. Assume G is a Frobenius group with kernel K and complement H , then as before H has a normal subgroup H_0 of index ≤ 2 and $H_0 \cong \text{SL}(2, 5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \pi(30) = \emptyset$. Moreover, we have

$$\pi_1(G) = \pi(H) = \pi(p^2 - 1) \quad \text{and} \quad \pi_2(G) = \pi(K) = \{p\}.$$

In particular $\{2, 3, 5\} \subset \pi(G)$. From Lemma 2.3, we get $15 \notin \pi_e(G)$, and so $5|p+1$ and $3|p-1$ or $5|p-1$ and $3|p+1$, by the structure consideration of $\pi_e(G)$. Therefore, $Z = 1$, which forces $\pi(H) = \{2, 3, 5\}$, whence $\pi_e(G) = \{2, 3, 5, p\}$. On the other hand, by Lemma 2.3 and the structure of $\pi_e(G)$, we must have

$p^2 - 1 = 2^\alpha \cdot 3 \cdot 5$ where $\alpha > 5$ as $p \geq 23$. But then $2^4 \in \pi_e(G)$, which is a contradiction by Lemma 2.3. The proof is complete now. $\square$

We also need the following Lemma.

Lemma 2.5 *Let G be a simple group such that $\{p\} \subset \pi(G) \subseteq \pi(p!)$, where p is a prime number and $23 \leq p < 100$. Then G is given in Table I. Moreover, if $\{p\} \subset \pi(G) \subseteq \pi(PSL(2, p))$ then:*

- (i) *For $p = 31$, $G \cong PSL(2, 31)$ or $PSL(3, 5)$,*
- (ii) *For $p \neq 31$, $G \cong PSL(2, p)$.*

Proof. Considering the fact that $\{p\} \subset \pi(G) \subseteq \pi(p!)$ and by referring to [2] and [8], for $p \neq 37$ we have the simple groups contained in Table I. When $p = 37$, the proof is similar as in ([2], Lemma 2.6). The rest of proof, when $\{p\} \subset \pi(G) \subseteq \pi(PSL(2, p))$, follows by checking the prime factors of $|G|$. $\square$

We also need the following result of V. D. Mazurov ([6], Lemma 1).

Lemma 2.6 *Let N be a normal subgroup of H . Assume that H/N is a Frobenius group with kernel A and cyclic complement B . If $(|A|, |N|) = 1$ and A is not contained in $NC_G(N)/N$, then $p|B| \in \pi_e(H)$, where p is a prime factor of $|N|$.*

3 Proof of the Main Theorem

Proof of the Main Theorem. It is well known that

$$\text{Aut}(PSL(2, p^n)) = \text{PGL}(2, p^n) : Z_n.$$

When $n = 1$, we have $\text{Aut}(PSL(2, p)) = \text{PGL}(2, p)$, and hence

$$\mu(\text{PGL}(2, p)) = \mu(\text{Aut}(PSL(2, p))) = \{p - 1, p, p + 1\}.$$

Let G be a group such that $\pi_e(G) = \pi_e(\text{PGL}(2, p)) = \{p, \text{all factors of } p - 1 \text{ and } p + 1\}$. Then the connected components of the prime graph of G are:

$$\pi_1 = \pi(p^2 - 1) \quad \text{and} \quad \pi_2 = \{p\}.$$

When $p < 23$, it has been proved that $h(\pi_e(G)) \in \{1, \infty\}$ (see [8] and [9]). Thus, we assume $p \geq 23$. If G is solvable, then $h(\pi_e(G)) = \infty$, by Lemma 2.2. Hence, we suppose G is non-solvable. Now by Lemma 2.4, G is neither Frobenius nor 2-Frobenius. On the other hand, by Lemma 2.1, there exists a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1 -group, $\overline{G}_1 = G_1/N$ is a simple group, and G/G_1 is a solvable π_1 -group. Since $\overline{G}_1 \trianglelefteq G/N$ and $C_{G/N}(\overline{G}_1) = 1$, we have

$$\frac{G}{N} = \frac{N_{G/N}(\overline{G}_1)}{C_{G/N}(\overline{G}_1)} \cong \text{a subgroup of } \text{Aut}(\overline{G}_1),$$

thus we may assume that $G/N \leq \text{Aut}(\overline{G}_1)$.

Table I. Simple groups with condition $\{p\} \subset \pi(G) \subseteq \pi(p!)$.

p	Finite simple groups
23	$A_n, n = 23, 24, 25, 26, 27, 28; M_{23}, M_{24}, Co_3, Co_2, Co_1, Fi_{23}; A_1(23), {}^2A_2(23).$
29	$A_n, n = 29, 30; Ru, Fi'_{24}; A_1(17^2), S_4(17), {}^2A_3(17), A_1(29).$
31	$A_n, n = 31, 32, 33, 34, 35, 36; O'N, Th; A_4(2), A_5(2), A_4(2^2), A_5(2^2), A_1(2^5), S_{10}(2), S_{12}(2), O_{10}^+(2), O_{12}^+(2), O_{12}^-(2), A_2(5), A_3(5), A_2(5^2), A_1(5^3), O_7(5), S_6(5), O_8^+(5), G_2(5), A_1(31), {}^2A_2(31).$
37	$A_n, n = 37, 38, 39, 40; {}^2A_2(3^3), {}^2A_3(3^3), {}^2G_2(3^3), A_1(11^3), G_2(11), {}^2A_2(11), A_1(31^2), S_4(31), {}^2A_3(31), A_1(37).$
41	$A_n, n = 41, 42; A_1(2^{10}), A_3(3^2), S_4(2^5), {}^2A_4(2^2), {}^2A_5(2^2), Sz(2^5), A_1(3^4), O_9(3), S_4(3^2), S_8(3), O_{10}^+(3), O_8^-(3), A_1(41), A_1(41^2), S_4(41).$
43	$A_n, n = 43, 44, 45, 46; J_4; {}^2A_6(2), {}^2A_7(2), {}^2A_8(2), {}^2A_9(2), O_{14}^-(2), A_2(7^2), A_1(7^3), O_7(7), S_6(7), O_8^+(7), G_2(7), {}^2A_2(7), {}^2A_3(7), {}^2A_2(37), A_1(43), A_1(43^2), S_4(43).$
47	$A_n, n = 47, 48, 49, 50, 51, 52; B = F_{2+}; A_1(47), A_1(47^2), S_4(47).$
53	$A_n, n = 53, 54, 55, 56, 57, 58; A_1(23^2), B_2(23), {}^2A_3(23), A_1(53).$
59	$A_n, n = 59, 60; A_1(59).$
61	$A_n, n = 61, 62, 63, 64, 65, 66; A_4(3^2), A_1(3^5), B_5(3), C_5(3), A_3(11), A_1(11^2), A_2(11^2), B_2(11), B_3(11), C_3(11), D_4(11), {}^2A_3(11), A_2(13), A_3(13), A_2(47), A_3(47), A_1(61).$
67	$A_n, n = 67, 68, 69, 70; Ly; A_2(29), A_2(37), A_1(37^3), G_2(37), A_1(67).$
71	$A_n, n = 71, 72; F_1; A_4(5), A_5(5), A_1(71).$
73	$A_n, n = 73, 74, 75, 76, 77, 78; A_2(2^3), A_1(2^9), G_2(2^3), C_4(2^3), E_6(2), B_3(2^2), C_3(2^3), A_1(3^6), A_5(3^2), B_6(3), B_2(3^3), C_6(3), D_4(3^2), G_2(3^2), F_4(3), {}^2A_2(3^2), {}^2A_3(3^3), {}^2D_6(3), {}^2E_6(3), A_1(73), A_1(73^2), B_2(73).$
79	$A_n, n = 79, 80, 81, 82; A_2(23), A_3(23), A_2(23^2), A_1(23^3), B_3(23), C_3(23), D_4(23), G_2(23), A_1(79), A_2(79).$
83	$A_n, n = 83, 84, 85, 86, 87, 88; A_1(83), A_1(83^2).$
89	$A_n, n = 89, 90, 91, 92, 93, 94, 95, 96; A_1(89).$
97	$A_n, n = 97, 98, 99, 100; A_2(61), A_1(97).$

Note that one of the components of the prime graph of $\overline{G}_1$ must be $\{p\}$ and so $\{p\} \subseteq \pi(\overline{G}_1) \subseteq \pi(G) = \pi(\text{PSL}(2, p))$. Now according to Lemma 2.5, if $p = 31$ we have $\overline{G}_1 \cong \text{PSL}(2, 31)$ or $\text{PSL}(3, 5)$, and if $p \neq 31$ then $\overline{G}_1 \cong \text{PSL}(2, p)$. We claim that in all cases $\overline{G}_1 \cong \text{PSL}(2, p)$. In fact, if $p = 31$

and $\overline{G}_1 \cong \text{PSL}(3, 5)$, then $24 \in \pi_e(\overline{G}_1) = \pi_e(\text{PSL}(5, 3))$ but $24 \notin \pi_e(G) = \pi_e(\text{PGL}(2, 31))$, and this is a contradiction. Therefore $\overline{G}_1 \cong \text{PSL}(2, p)$.

Let P/N be a Sylow p -subgroup of $\overline{G}_1$ and X/N be the normalizer in $\overline{G}_1$ of P/N . Then X/N is a Frobenius group of order $p(p-1)/2$, with cyclic complement of order $(p-1)/2$. Now, by lemma 2.6, we deduce that N is a 2-group. First, suppose that $N = 1$. Then $G_1 = \text{PSL}(2, p)$. Moreover we may assume that $G_1 \leq G \leq \text{Aut}(G_1)$. Since $|\text{Out}(\text{PSL}(2, p))| = 2$, it follows that $G \cong \text{PSL}(2, p)$ or $G \cong \text{Aut}(\text{PSL}(2, p)) \cong \text{PGL}(2, p)$. But since $p+1 \in \pi_e(\text{PGL}(2, p)) \setminus \pi_e(\text{PSL}(2, p))$, we have $G \cong \text{PGL}(2, p)$. Next, suppose that $N \neq 1$. Without loss of generality, we may assume that N is an elementary Abelian 2-subgroup of G , by considering G/M , where M is the preimage of $\Phi(N/O_2(G))$ in G . Now from Lemma 2.2, we see that $h(\pi_e(G)) = \infty$. Therefore, the proof of Main Theorem is complete. $\square$

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