A note on the Dittert conjecture for permanents

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Abstract

Let K_n denote the convex set consisting of all real nonnegative $n \times n$ matrices whose entries have sum n . For $A \in K_n$ with row sums $r_1, \dots, r_n$ and column sums $c_1, \dots, c_n$, let ψ be defined by

$$\psi(A) = \prod_{i=1}^n r_i + \prod_{j=1}^n c_j - \text{per}(A),$$

where per stands for the permanent function. E. Dittert conjectures that the maximum of ψ on K_n occurs uniquely at $J_n = [1/n]_{n \times n}$. The conjecture is still unsolved for $n \geq 4$. In this paper, we obtain some sufficient conditions for which the Dittert conjecture holds.

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1. Introduction

A square real nonnegative matrix is called *row (resp. column) stochastic* if all its row (resp. column) sums are equal to 1. A matrix which is both row

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stochastic and column stochastic is called *doubly stochastic*. As usual, the set of all $n \times n$ doubly stochastic matrices is denoted by Ω_n , and the $n \times n$ matrix all of whose entries equal $\frac{1}{n}$ is denoted by J_n .

For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of A , $\text{per}(A)$, is defined by

$$\text{per}(A) = \sum_{\pi \in S_n} a_{1\pi(1)} \cdots a_{n\pi(n)},$$

where S_n denotes the permutations of $1, \dots, n$ onto itself.

In 1926, van der Waerden [8] posed the problem of determining the minimum of the permanent function on Ω_n .

It was conjectured that for any $A \in \Omega_n$,

$$\text{per}(A) \geq \frac{n!}{n^n},$$

with equality holds if and only if $A = J_n$.

The conjecture remained unsolved for over half a century until Egoryčev [2] and Falikman [3] proved it independently. We will call this problem as the *van der Waerden-Egoryčev-Falikman Theorem*.

In this paper, we consider a conjecture generalizing the van der Waerden-Egoryčev-Falikman Theorem, which has known as the *Dittert conjecture* [6, Conjecture 28].

Throughout this paper, let K_n denote the set of all real nonnegative $n \times n$ matrices whose entries have sum n , and let ψ denote a real valued function on K_n defined by

$$\psi(A) = \prod_{i=1}^n \sum_{j=1}^n a_{ij} + \prod_{j=1}^n \sum_{i=1}^n a_{ij} - \text{per}(A) \quad (1)$$

for $A = [a_{ij}] \in K_n$. Since K_n is a compact subset of a finite dimensional euclidean space, it contains a matrix A such that $\psi(A) \geq \psi(X)$ for all $X \in K_n$. Such a matrix A will be called a *ψ -maximizing matrix* on K_n .

The following conjecture due to E. Dittert is still open for $n \geq 4$.

DITTERT CONJECTURE For any $A \in K_n$,

$$\psi(A) \leq 2 - \frac{n!}{n^n}$$

with equality holds if and only if $A = J_n$.

The Dittert Conjecture asserts that J_n is the unique ψ -maximizing matrix on K_n . It is proved for $n = 2$ [7] and for $n = 3$ [4], and the other partial results

for it are found in [1,4,5,7]. In particular, as noted in [5] (also see [7]), we see that if A is a ψ -maximizing matrix on K_n then

$$0 < \text{per}(A) \leq \frac{n!}{n^n}. \quad (2)$$

Clearly, from the van der Waerden-Egoryčev-Falikman Theorem, the Dittert Conjecture holds for a special subset Ω_n of K_n .

In this note, it is obtained a sufficient condition on K_n for which the Dittert conjecture holds. As a consequence, we see that a ψ -maximizing matrix on K_n depends on only its permanent.

2. A sufficient condition for which the Dittert conjecture holds

Throughout this section, let A be a matrix on K_n with row sums $r_1, \dots, r_n$ and column sums $c_1, \dots, c_n$ such that $r_1 + \dots + r_n = c_1 + \dots + c_n = n$. Then $\psi(A)$ in (1) can be written as

$$\psi(A) = \prod_{i=1}^n r_i + \prod_{j=1}^n c_j - \text{per}(A).$$

Since $\text{per}(A) = \text{per}(A^T)$, it will enable us to replace “row” by “column” in all results of $\psi(A)$. Moreover, noticing for any $n \times n$ permutation matrices P and Q , $\text{per}(PAQ) = \text{per}(A)$ and thus $\psi(PAQ) = \psi(A)$, without loss of generality we may assume that

$$r_1 \leq \dots \leq r_n \text{ and } c_1 \leq \dots \leq c_n. \quad (3)$$

Clearly, $r_1 \leq 1$ and $c_1 \leq 1$, and if $r_1 = c_1 = 1$ then A is a doubly stochastic matrix.

For a matrix $A \in K_n$, $A(i|j)$ denotes the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i -th row and the j -th column, and let

$$\psi_{ij}(A) = \prod_{k \neq i} r_k + \prod_{l \neq j} c_l - \text{per}(A(i|j)).$$

The following lemma is due to S. G. Hwang [5].

LEMMA 2.1 *Let $A = [a_{ij}]$ be a ψ -maximizing matrix on K_n . Then*

$$\psi(A) \geq \psi_{ij}(A),$$

with equality holds if $a_{ij} > 0$.

For $r_i > 0$ and $c_j > 0$, let

$$r = \prod_{i=1}^n r_i, \quad c = \prod_{j=1}^n c_j, \quad \bar{r}_i = \frac{r}{r_i} \text{ and } \bar{c}_j = \frac{c}{c_j}.$$

THEOREM 2.2 *Let $A = [a_{ij}]$ be a ψ -maximizing matrix on K_n . Then*

$$\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A) \quad \text{iff} \quad A = J_n.$$

Proof. Clearly, if $A = J_n$ then $\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A)$ holds. Suppose that $\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A)$. Lemma 2.1 implies that $\psi(A) = \psi_{i,j}(A)$ for all $i, j \in \{1, 2, \dots, n\}$. Noticing for any i, s, t ($s \neq t$) in $\{1, 2, \dots, n\}$

$$0 = \psi_{is} - \psi_{it} = \bar{c}_s - \bar{c}_t + \text{per}A(i|s) - \text{per}A(i|t)$$

we obtain

$$\text{per}A(i|s) - \text{per}A(i|t) = \left(\frac{1}{c_s} - \frac{1}{c_t} \right) \prod_{j=1}^n c_j.$$

By the (average) Lemma 2 [5], if B is the matrix obtained from A by averaging columns s and t then $\psi(A) = \psi(B)$. The similar result holds for rows. Consequently, after a sequence of averaging on columns and rows of A , we obtain $A = J_n$.

LEMMA 2.3 *Let A be a ψ -maximizing matrix on K_n with (3). Then*

$$0 < \text{per}(A) \leq \text{per}(A(1|1)),$$

with equality holds if $r_s = r_t$.

Proof. The $0 < \text{per}(A)$ follows from (2). By Lemma 2.1, we get

$$\begin{aligned} \psi(A) - \psi_{11}(A) &= r + c - \text{per}(A) - (\bar{r}_1 + \bar{c}_1 - \text{per}(A(1|1))) \\ &= \bar{r}_1(r_1 - 1) + \bar{c}_1(c_1 - 1) + \text{per}(A(1|1)) - \text{per}(A) \geq 0, \end{aligned}$$

which implies that

$$\bar{r}_1(r_1 - 1) + \bar{c}_1(c_1 - 1) \geq \text{per}(A) - \text{per}(A(1|1)).$$

Since $r_1 \leq 1$ and $c_1 \leq 1$ we have $\text{per}(A) \leq \text{per}(A(1|1))$, which completes the proof.

LEMMA 2.4 *Let $A = [a_{ij}]$ be a ψ -maximizing matrix on K_n with (3) and let k be an integer such that $1 \leq k \leq n$. If $a_{sk} > 0$ and $a_{tk} > 0$ for any integers s and t such that $1 \leq s < t \leq n$, then*

$$\text{per}(A(s|k)) \geq \text{per}(A(t|k)),$$

with equality holds if $r_s = r_t$.

Proof. By Lemma 2.1, we get

$$\begin{aligned}\psi_{sk}(A) - \psi_{tk}(A) &= (\bar{r}_s + \bar{c}_k - \text{per}(A(s|k)) - (\bar{r}_t + \bar{c}_k - \text{per}(A(t|k))) \\ &= \left(\frac{r}{r_s} - \frac{r}{r_t}\right) - (\text{per}(A(s|k)) - \text{per}(A(t|k))) \\ &= \frac{r(r_t - r_s)}{r_s r_t} - (\text{per}(A(s|k)) - \text{per}(A(t|k))) = 0.\end{aligned}$$

Since $r_t \geq r_s$, we have $\text{per}(A(s|k)) \geq \text{per}(A(t|k))$, and if $r_s = r_t$ then the equality holds. Thus the proof is completed.

The following is an immediate consequence of Lemma 2.4.

COROLLARY 2.5 *Let $A = [a_{ij}]$ be a ψ -maximizing matrix on K_n with (3) and let k be an integer such that $1 \leq k \leq n$. If $a_{ik} > 0$ for all $i = 1, 2, \dots, n$, then*

$$\text{per}(A(1|k)) \geq \text{per}(A(2|k)) \geq \dots \geq \text{per}(A(n|k)),$$

with all equalities hold if A is a row stochastic matrix.

THEOREM 2.6 *Let A be a ψ -maximizing matrix on K_n with (3). If*

$$\text{per}(A) = \text{per}(A(1|1)), \quad (4)$$

then the Dittert conjecture holds.

Proof. Note that $r_1 \leq 1$ and $c_1 \leq 1$. Suppose that $r_1 < 1$ or $c_1 < 1$. Then we get

$$\begin{aligned}\psi(A) &= r + c - \text{per}(A) \\ &= r_1 \cdot \bar{r}_1 + c_1 \cdot \bar{c}_1 - \text{per}(A(1|1)) + \text{per}(A(1|1)) - \text{per}(A) \\ &< \bar{r}_1 + \bar{c}_1 - \text{per}(A(1|1)) + \text{per}(A(1|1)) - \text{per}(A) \\ &= \psi_{11}(A) + \text{per}(A(1|1)) - \text{per}(A).\end{aligned}$$

If $\text{per}(A) = \text{per}(A(1|1))$ then $\psi(A) < \psi_{11}(A)$, which contradicts to Lemma 2.1. Thus $r_1 = 1$ and $c_1 = 1$. It follows that $r_1 = \dots = r_n = 1$ and $c_1 = \dots = c_n = 1$. From the van der Waerden-Egoryčev-Falikman Theorem we have $A = J_n$, and the proof is completed.

Now, we obtain a sufficient condition for which (4) holds. To prove our next theorem, we shall use Alexandrov's inequality for permanents:

$$(\text{per}(A))^2 \geq \text{per}[a_1, a_1, a_2, \dots, a_{n-1}] \text{per}[a_2, \dots, a_{n-1}, a_n, a_n]$$

for any nonnegative $n \times n$ matrix $A = [a_1, a_2, \dots, a_n]$, which is a reformulation of the original one due to Egoryčev [2], where a_i is the i -th column vector of A .

THEOREM 2.7 *Let $A = [a_{ij}]$ be a ψ -maximizing matrix on K_n with (3). If $\text{per}(A(1|1)) \leq \text{per}(A(i|j))$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, n\}$, where $c_1 c_n \geq 1$ then*

$$\text{per}(A) = \text{per}(A(1|1)).$$

Proof. From Lemma 2.3, suppose that $\text{per}(A) < \text{per}(A(1|1))$. Now Alexandrov's inequality (also see [2]) implies that

$$\begin{aligned} (\text{per}(A))^2 &\geq \left[\sum_{i=1}^n a_{i1} \text{per}(A(i|n)) \right] \left[\sum_{i=1}^n a_{in} \text{per}(A(i|1)) \right] \\ &\geq \left(\sum_{i=1}^n a_{i1} \right) \text{per}(A(1|1)) \left(\sum_{i=1}^n a_{in} \right) \text{per}(A(1|1)) \\ &= c_1 c_n (\text{per}(A(1|1)))^2 \\ &> c_1 c_n (\text{per}(A))^2. \end{aligned}$$

Since $c_1 c_n \geq 1$ we have $(\text{per}(A))^2 > (\text{per}(A))^2$, which is a contradiction. Hence $\text{per}(A) = \text{per}(A(1|1))$. This completes the proof.

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