A note on the Dittert conjecture
for permanents

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Abstract

Let \(K_n\) denote the convex set consisting of all real nonnegative \(n \times n\) matrices whose entries have sum \(n\). For \(A \in K_n\) with row sums \(r_1, \ldots, r_n\) and column sums \(c_1, \ldots, c_n\), let \(\psi\) be defined by

\[
\psi(A) = \prod_{i=1}^{n} r_i + \prod_{j=1}^{n} c_j - \text{per}(A),
\]

where per stands for the permanent function. E. Dittert conjectures that the maximum of \(\psi\) on \(K_n\) occurs uniquely at \(J_n = [1/n]_{n \times n}\). The conjecture is still unsolved for \(n \geq 4\). In this paper, we obtain some sufficient conditions for which the Dittert conjecture holds.

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1. Introduction

A square real nonnegative matrix is called row (resp. column) stochastic if all its row (resp. column) sums are equal to 1. A matrix which is both row

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Stochastic and column stochastic is called **doubly stochastic**. As usual, the set of all $n \times n$ doubly stochastic matrices is denoted by $\Omega_n$, and the $n \times n$ matrix all of whose entries equal $\frac{1}{n}$ is denoted by $J_n$.

For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of $A$, $\text{per}(A)$, is defined by

$$\text{per}(A) = \sum_{\pi \in S_n} a_{1\pi(1)} \cdots a_{n\pi(n)},$$

where $S_n$ denotes the permutations of $1, \ldots, n$ onto itself.

In 1926, van der Waerden [8] posed the problem of determining the minimum of the permanent function on $\Omega_n$.

It was conjectured that for any $A \in \Omega_n$,

$$\text{per}(A) \geq \frac{n!}{n^n},$$

with equality holds if and only if $A = J_n$.

The conjecture remained unsolved for over half a century until Egorychev [2] and Falikman [3] proved it independently. We will call this problem as the *van der Waerden-Egorychev-Falikman Theorem*.

In this paper, we consider a conjecture generalizing the van der Waerden-Egorychev-Falikman Theorem, which has known as the *Dittert conjecture* [6, Conjecture 28].

Throughout this paper, let $K_n$ denote the set of all real nonnegative $n \times n$ matrices whose entries have sum $n$, and let $\psi$ denote a real valued function on $K_n$ defined by

$$\psi(A) = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} a_{ij} - \text{per}(A)$$

for $A = [a_{ij}] \in K_n$. Since $K_n$ is a compact subset of a finite dimensional euclidean space, it contains a matrix $A$ such that $\psi(A) \geq \psi(X)$ for all $X \in K_n$. Such a matrix $A$ will be called a $\psi$-maximizing matrix on $K_n$.

The following conjecture due to E. Dittert is still open for $n \geq 4$.

**DITTERT CONJECTURE** For any $A \in K_n$,

$$\psi(A) \leq 2 - \frac{n!}{n^n}$$

with equality holds if and only if $A = J_n$.

The Dittert Conjecture asserts that $J_n$ is the unique $\psi$-maximizing matrix on $K_n$. It is proved for $n = 2$ [7] and for $n = 3$ [4], and the other partial results
for it are found in [1,4,5,7]. In particular, as noted in [5] (also see [7]), we see that if $A$ is a $\psi$-maximizing matrix on $K_n$ then

$$0 < \text{per}(A) \leq \frac{n!}{n^n}. \quad (2)$$

Clearly, from the van der Waerden-Egoryčev-Falikman Theorem, the Dittert Conjecture holds for a special subset $\Omega_n$ of $K_n$.

In this note, it is obtained a sufficient condition on $K_n$ for which the Dittert conjecture holds. As a consequence, we see that a $\psi$-maximizing matrix on $K_n$ depends on only its permanent.

2. A sufficient condition for which the Dittert conjecture holds

Throughout this section, let $A$ be a matrix on $K_n$ with row sums $r_1, \ldots, r_n$ and column sums $c_1, \ldots, c_n$ such that $r_1 + \cdots + r_n = c_1 + \cdots + c_n = n$. Then $\psi(A)$ in (1) can be written as

$$\psi(A) = \prod_{i=1}^{n} r_i + \prod_{j=1}^{n} c_j - \text{per}(A).$$

Since $\text{per}(A) = \text{per}(A^T)$, it will enable us to replace “row” by “column” in all results of $\psi(A)$. Moreover, noticing for any $n \times n$ permutation matrices $P$ and $Q$, $\text{per}(PAQ) = \text{per}(A)$ and thus $\psi(PAQ) = \psi(A)$, without loss of generality we may assume that

$$r_1 \leq \cdots \leq r_n \text{ and } c_1 \leq \cdots \leq c_n. \quad (3)$$

Clearly, $r_1 \leq 1$ and $c_1 \leq 1$, and if $r_1 = c_1 = 1$ then $A$ is a doubly stochastic matrix.

For a matrix $A \in K_n$, $A(i|j)$ denotes the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i$-th row and the $j$-th column, and let

$$\psi_{ij}(A) = \prod_{k \neq i} r_k + \prod_{l \neq j} c_l - \text{per}(A(i|j)).$$

The following lemma is due to S. G. Hwang [5].

**LEMMA 2.1** Let $A = [a_{ij}]$ be a $\psi$-maximizing matrix on $K_n$. Then

$$\psi(A) \geq \psi_{ij}(A),$$

with equality holds if $a_{ij} > 0$. 
For $r_i > 0$ and $c_j > 0$, let

$$r = \prod_{i=1}^{n} r_i, \quad c = \prod_{j=1}^{n} c_j, \quad \bar{r}_i = \frac{r}{r_i} \quad \text{and} \quad \bar{c}_j = \frac{c}{c_j}.$$

**THEOREM 2.2** Let $A = [a_{ij}]$ be a ψ-maximizing matrix on $K_n$. Then

$$\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A) \quad \text{iff} \quad A = J_n.$$

**Proof.** Clearly, if $A = J_n$ then $\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A)$ holds. Suppose that $\psi(A) = \frac{1}{n^2} \sum_{i,j} \psi_{ij}(A)$. Lemma 2.1 implies that $\psi(A) = \psi_{i,j}(A)$ for all $i, j \in \{1, 2, \ldots, n\}$. Noticing for any $i, s, t \neq t$ in $\{1, 2, \ldots, n\}$

$$0 = \psi_{is} - \psi_{it} = \bar{c}_s - \bar{c}_t + \text{per}(A|i|s) - \text{per}(A|i|t)$$

we obtain

$$\text{per}(A|i|s) - \text{per}(A|i|t) = \left(\frac{1}{c_s} - \frac{1}{c_t}\right) \prod_{j=1}^{n} c_j.$$

By the (average) Lemma 2 [5], if $B$ is the matrix obtained from $A$ by averaging columns $s$ and $t$ then $\psi(A) = \psi(B)$. The similar result holds for rows. Consequently, after a sequence of averaging on columns and rows of $A$, we obtain $A = J_n$.

**LEMMA 2.3** Let $A$ be a ψ-maximizing matrix on $K_n$ with (3). Then

$$0 < \text{per}(A) \leq \text{per}(A(1|1)),$$

with equality holds if $r_s = r_t$.

**Proof.** The $0 < \text{per}(A)$ follows from (2). By Lemma 2.1, we get

$$\psi(A) - \psi_{11}(A) = r + c - \text{per}(A) - (\bar{r}_1 + \bar{c}_1 - \text{per}(A(1|1)))$$

$$= \bar{r}_1(r_1 - 1) + \bar{c}_1(c_1 - 1) + \text{per}(A(1|1)) - \text{per}(A) \geq 0,$$

which implies that

$$\bar{r}_1(r_1 - 1) + \bar{c}_1(c_1 - 1) \geq \text{per}(A) - \text{per}(A(1|1)).$$

Since $r_1 \leq 1$ and $c_1 \leq 1$ we have $\text{per}(A) \leq \text{per}(A(1|1))$, which completes the proof.

**LEMMA 2.4** Let $A = [a_{ij}]$ be a ψ-maximizing matrix on $K_n$ with (3) and let $k$ be an integer such that $1 \leq k \leq n$. If $a_{sk} > 0$ and $a_{tk} > 0$ for any integers $s$ and $t$ such that $1 \leq s < t \leq n$, then

$$\text{per}(A(s|k)) \geq \text{per}(A(t|k)),$$
with equality holds if \( r_s = r_t \).

**Proof.** By Lemma 2.1, we get

\[
\psi_{sk}(A) - \psi_{tk}(A) = (\bar{r}_s + \bar{c}_k - \text{per}(A(s|k))) - (\bar{r}_t + \bar{c}_k - \text{per}(A(t|k)))
\]

\[
= \left( \frac{r}{r_s} - \frac{r}{r_t} \right) - (\text{per}(A(s|k)) - \text{per}(A(t|k)))
\]

\[
= \frac{r(r_t - r_s)}{r_s r_t} - (\text{per}(A(s|k)) - \text{per}(A(t|k))) = 0.
\]

Since \( r_t \geq r_s \), we have \( \text{per}(A(s|k)) \geq \text{per}(A(t|k)) \), and if \( r_s = r_t \) then the equality holds. Thus the proof is completed.

The following is an immediate consequence of Lemma 2.4.

**COROLLARY 2.5** Let \( A = [a_{ij}] \) be a \( \psi \)-maximizing matrix on \( K_n \) with (3) and let \( k \) be an integer such that \( 1 \leq k \leq n \). If \( a_{ik} > 0 \) for all \( i = 1, 2, \ldots, n \), then

\[ \text{per}(A(1|k)) \geq \text{per}(A(2|k)) \geq \cdots \geq \text{per}(A(n|k)), \]

with all equalities hold if \( A \) is a row stochastic matrix.

**THEOREM 2.6** Let \( A \) be a \( \psi \)-maximizing matrix on \( K_n \) with (3). If

\[ \text{per}(A) = \text{per}(A(1|1)), \] (4)

then the Dittert conjecture holds.

**Proof.** Note that \( r_1 \leq 1 \) and \( c_1 \leq 1 \). Suppose that \( r_1 < 1 \) or \( c_1 < 1 \). Then we get

\[
\psi(A) = r + c - \text{per}(A)
\]

\[
= r_1 \cdot \bar{r}_1 + c_1 \cdot \bar{c}_1 - \text{per}(A(1|1)) + \text{per}(A(1|1)) - \text{per}(A)
\]

\[
< \bar{r}_1 + \bar{c}_1 - \text{per}(A(1|1)) + \text{per}(A(1|1)) - \text{per}(A)
\]

\[
= \psi_{11}(A) + \text{per}(A(1|1)) - \text{per}(A).
\]

If \( \text{per}(A) = \text{per}(A(1|1)) \) then \( \psi(A) < \psi_{11}(A) \), which contradicts to Lemma 2.1. Thus \( r_1 = 1 \) and \( c_1 = 1 \). It follows that \( r_1 = \cdots = r_n = 1 \) and \( c_1 = \cdots = c_n = 1 \). From the van der Waerden-Egoryčev-Falikman Theorem we have \( A = J_n \), and the proof is completed.

Now, we obtain a sufficient condition for which (4) holds. To prove our next theorem, we shall use Alexandrov’s inequality for permanents:

\[
(\text{per}(A))^2 \geq \text{per}[a_1, a_1, a_2, \ldots, a_{n-1}]\text{per}[a_2, \ldots, a_{n-1}, a_n, a_n]
\]
for any nonnegative $n \times n$ matrix $A = [a_1, a_2, \ldots, a_n]$, which is a reformulation of the original one due to Egoryčev [2], where $a_i$ is the $i$-th column vector of $A$.

**THEOREM 2.7** Let $A = [a_{ij}]$ be a $\psi$-maximizing matrix on $K_n$ with (3). If $\operatorname{per}(A(1|1)) \leq \operatorname{per}(A(i|j))$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, n\}$, where $c_1c_n \geq 1$ then

$$\operatorname{per}(A) = \operatorname{per}(A(1|1)).$$

**Proof.** From Lemma 2.3, suppose that $\operatorname{per}(A) < \operatorname{per}(A(1|1))$. Now Alexandrov’s inequality (also see [2]) implies that

$$\begin{align*}
(\operatorname{per}(A))^2 & \geq \left[ \sum_{i=1}^{n} a_{i1} \operatorname{per}(A(i|n)) \right] \left[ \sum_{i=1}^{n} a_{in} \operatorname{per}(A(1|i)) \right] \\
& \geq \left( \sum_{i=1}^{n} a_{i1} \right) \operatorname{per}(A(1|1)) \left( \sum_{i=1}^{n} a_{in} \right) \operatorname{per}(A(1|1)) \\
& = c_1c_n (\operatorname{per}(A(1|1)))^2 \\
& > c_1c_n (\operatorname{per}(A))^2.
\end{align*}$$

Since $c_1c_n \geq 1$ we have $(\operatorname{per}(A))^2 > (\operatorname{per}(A))^2$, which is a contradiction. Hence $\operatorname{per}(A) = \operatorname{per}(A(1|1))$. This completes the proof.

**References**


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