On an Extended Hardy-Hilbert's Inequality and Some Reversed Form

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Abstract

By introducing a parameter λ and the β function, we give a new extended Hardy-Hilbert's inequality with a best constant factor and the equivalent form. Some reversed form for any particular parameter are considered.

Keywords: Hardy-Hilbert's inequality; Weight coefficient; Hölder's inequality

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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0 (n \in N)$, such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the well known Hardy- Hilbert's inequality and the equivalent form are as follows (see Hardy et al. [1]):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \quad (1.1)$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} a_n^p, \quad (1.2)$$

where the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. Inequality (1.1) is important in analysis and its applications (see Mitinovic [2]). In recent years, (1.1) had been strengthened by [3, 4]. By introducing a parameter, Yang [5, 6] gave two extensions of (1.1) as: If the series in the right of the following inequalities converge to some positive numbers, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B \left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (1.3)$$

where the constant factor $B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ ($2 - \min\{p, q\} < \lambda \leq 2$) is the best possible ($B(u, v)$ is the β function); and the other is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{\lambda} + n^{\lambda}} < \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}}, \quad (1.4)$$

where the constant factor $\pi/[\lambda \sin(\pi/p)]$ ($0 < \lambda \leq \min\{p, q\}$) is the best possible. For $\lambda = 1$, both (1.3) and (1.4) reduce to (1.1). In 2003, Yang et al. [7] summarized the way of weight coefficient on research for Hilbert-type inequalities. More recently, Zhao [8] considered some inverses of Pachpatte's inequalities.

The main objective of this paper is to build a new extended form of (1.1) with a best constant factor, related the left double series of (1.4). As applications, an equivalent form and some reversed form for $\lambda = 2$ are considered.

2 Some Lemmas and Main Results

We need the formula of the β function $B(p, q)$ as follows (see [9]):

$$B(p, q) = B(q, p) = \int_0^{\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du \quad (p, q > 0). \quad (2.1)$$

Lemma 2.1 If $r > 1, \frac{1}{r} + \frac{1}{s} = 1, 2 - \min\{r, s\} < \lambda \leq 2$, define $\varpi_{\lambda}(n, r)$ as

$$\varpi_{\lambda}(n, r) := n^{\lambda-1} \sum_{m=1}^{\infty} \frac{1}{m^{\lambda} + n^{\lambda}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{r}} \quad (n \in N). \quad (2.2)$$

Then one has

$$\varpi_{\lambda}(n, r) < \frac{1}{\lambda} B\left(\frac{r+\lambda-2}{r\lambda}, \frac{s+\lambda-2}{s\lambda}\right). \quad (2.3)$$

Proof Since $0 \leq 2 - \min\{r, s\} < \lambda \leq 2$, by (2.2), we have

$$\varpi_{\lambda}(n, r) < n^{\lambda-1} \int_0^{\infty} \frac{1}{x^{\lambda} + n^{\lambda}} \left(\frac{n}{x}\right)^{\frac{2-\lambda}{r}} dx.$$

Setting $t = (x/n)^{\lambda}$ in the above integral, we find

$$\varpi_{\lambda}(n, r) < \frac{1}{\lambda} \int_0^{\infty} \frac{1}{t+1} t^{\frac{r+\lambda-2}{r\lambda}-1} dt.$$

Then by (2.1), since $r + \lambda - 2 > 0$, $s + \lambda - 2 > 0$ and $\frac{r+\lambda-2}{r\lambda} + \frac{s+\lambda-2}{s\lambda} = 1$, we have (2.3). The lemma is proved.

Lemma 2.2 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - \min\{p, q\} < \lambda \leq 2$, and $0 < \varepsilon < \frac{q+\lambda-2}{2}$, then

$$I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^{\lambda} + n^{\lambda}} m^{\frac{\lambda-2-\varepsilon}{p}} n^{\frac{\lambda-2-\varepsilon}{q}}$$

$$> \frac{1}{\varepsilon\lambda} B\left(\frac{q+\lambda-2}{q\lambda} - \frac{\varepsilon}{q\lambda}, \frac{p+\lambda-2}{p\lambda} + \frac{\varepsilon}{q\lambda}\right) - \left(\frac{2q}{q+\lambda-2}\right). \quad (2.4)$$

Proof We have

$$I > \int_1^\infty x^{\frac{\lambda-2-\varepsilon}{p}} \left[\int_1^\infty \frac{1}{x^\lambda + y^\lambda} y^{\frac{\lambda-2-\varepsilon}{q}} dy \right] dx.$$

Setting $u = (y/x)^\lambda$ in the above integral, we obtain

$$\begin{aligned} I &> \frac{1}{\lambda} \int_1^\infty x^{-1-\varepsilon} \left[\int_{1/x^\lambda}^\infty \frac{1}{1+u} u^{\frac{q+\lambda-2-\varepsilon}{\lambda q}-1} du \right] dx \\ &= \frac{1}{\lambda} \int_1^\infty x^{-1-\varepsilon} \left[\int_0^\infty \frac{1}{1+u} u^{\frac{q+\lambda-2-\varepsilon}{\lambda q}-1} du \right] dx \\ &\quad - \frac{1}{\lambda} \int_1^\infty x^{-1-\varepsilon} \left[\int_0^{\frac{1}{x^\lambda}} \frac{1}{1+u} u^{\frac{q+\lambda-2-\varepsilon}{\lambda q}-1} du \right] dx \\ &= \frac{1}{\varepsilon\lambda} B\left(\frac{q+\lambda-2}{q\lambda} - \frac{\varepsilon}{\lambda q}, \frac{p+\lambda-2}{p\lambda} + \frac{\varepsilon}{\lambda q}\right) \\ &\quad - \frac{1}{\lambda} \int_1^\infty x^{-1-\varepsilon} \left[\int_0^{\frac{1}{x^\lambda}} \frac{1}{1+u} u^{\frac{q+\lambda-2-\varepsilon}{\lambda q}-1} du \right] dx. \end{aligned} \quad (2.5)$$

Since $0 \leq 2 - \min\{p, q\} < \lambda \leq 2$, and $0 < \varepsilon < (q + \lambda - 2)/2$, we find

$$\begin{aligned} &\frac{1}{\lambda} \int_1^\infty x^{-1-\varepsilon} \left[\int_0^{\frac{1}{x^\lambda}} \frac{1}{1+u} u^{\frac{q+\lambda-2-\varepsilon}{\lambda q}-1} du \right] dx \\ &< \frac{1}{\lambda} \int_1^\infty x^{-1} \left[\int_0^{\frac{1}{x^\lambda}} u^{\frac{q+\lambda-2}{2\lambda q}-1} du \right] dx = \left(\frac{2q}{q+\lambda-2}\right), \end{aligned}$$

then by (2.5), we have (2.4). The lemma is proved.

Theorem 2.3 If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - \min\{p, q\} < \lambda \leq 2, a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^\infty n^{1-\lambda} a_n^p < \infty$ and $0 < \sum_{n=1}^\infty n^{1-\lambda} b_n^q < \infty$, then we have

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{1}{\lambda} B\left(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}\right) \left\{ \sum_{n=1}^\infty n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \quad (2.6)$$

where the constant factor $\frac{1}{\lambda} B\left(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}\right)$ is the best possible.

Proof By Hölder's inequality and (2.2), one has

$$\begin{aligned} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^\lambda + n^\lambda} &= \sum_{n=1}^\infty \sum_{m=1}^\infty \left[\frac{a_m}{(m^\lambda + n^\lambda)^{\frac{1}{p}}} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{pq}} \right] \left[\frac{b_n}{(m^\lambda + n^\lambda)^{\frac{1}{q}}} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{pq}} \right] \\ &\leq \left\{ \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m^p}{m^\lambda + n^\lambda} \left(\frac{m}{n}\right)^{\frac{2-\lambda}{q}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{b_n^q}{m^\lambda + n^\lambda} \left(\frac{n}{m}\right)^{\frac{2-\lambda}{p}} \right\}^{\frac{1}{q}} \end{aligned}$$

$$= \left\{ \sum_{m=1}^{\infty} \varpi_{\lambda}(m, q) m^{1-\lambda} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \varpi_{\lambda}(n, p) n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}.$$

Hence by (2.3) and (2.1), we have (2.6).

For $0 < \varepsilon < (q + \lambda - 2)/2$, setting $\tilde{a}_n, \tilde{b}_n$ as: $\tilde{a}_n = n^{(\lambda-2-\varepsilon)/p}$; $\tilde{b}_n = n^{(\lambda-2-\varepsilon)/q}$, $n \in N$, then we find

$$\begin{aligned} & \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} \tilde{a}_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} \tilde{b}_n^q \right\}^{\frac{1}{q}} = 1 + \sum_{n=2}^{\infty} n^{-1-\varepsilon} \\ & < 1 + \int_1^{\infty} x^{-1-\varepsilon} dx = 1 + \frac{1}{\varepsilon}. \end{aligned} \quad (2.7)$$

If there exists a parameter λ , such that the constant factor $\frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})$ in (2.6) is not the best possible, then there exists a positive number k with $k < \frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})$, such that (2.6) is still value if one replaces $\frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})$ by k . In particular, one has

$$\varepsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{m^{\lambda} + n^{\lambda}} < \varepsilon k \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} \tilde{a}_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} \tilde{b}_n^q \right\}^{1/q}.$$

Hence by (2.4) and (2.7), we obtain

$$\frac{1}{\lambda} B\left(\frac{q+\lambda-2}{q\lambda} - \frac{\varepsilon}{\lambda q}, \frac{p+\lambda-2}{p\lambda} + \frac{\varepsilon}{\lambda q}\right) - \varepsilon \left(\frac{2q}{q+\lambda-2}\right) < k(1+\varepsilon),$$

and $\frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}) \leq k$ ($\varepsilon \rightarrow 0^+$). This contradicts that $k < \frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})$. Hence the constant factor $\frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})$ in (2.6) is the best possible. The theorem is proved.

Theorem 2.4 If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $a_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^p < \left[\frac{1}{\lambda} B\left(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}\right) \right]^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p, \quad (2.8)$$

where the constant factor $[\frac{1}{\lambda} B(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda})]^p$ is the best possible; Inequality (2.8) is equivalent to (2.6). In particular, for $\lambda = 2$, one has

$$\sum_{n=1}^{\infty} n^{p-1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^p < \left(\frac{\pi}{2}\right)^p \sum_{n=1}^{\infty} \frac{a_n^p}{n}. \quad (2.9)$$

Proof Setting b_n as: $b_n = n^{(p-1)(\lambda-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^{p-1}$, then by (2.6), we find

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q = \sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^p = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{\lambda} + n^{\lambda}}$$

$$\leq \frac{1}{\lambda} B\left(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}\right) \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}; \quad (2.10)$$

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \leq \left[\frac{1}{\lambda} B\left(\frac{p+\lambda-2}{p\lambda}, \frac{q+\lambda-2}{q\lambda}\right) \right]^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty. \quad (2.11)$$

Hence (2.10) takes the form of strict inequality using (2.6); so is (2.11). Which shows (2.8).

If (2.8) is valid, then by Hölder's inequality, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^{\lambda} + n^{\lambda}} &= \sum_{n=1}^{\infty} [n^{(\lambda-1)/q} \sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}}] [n^{(1-\lambda)/q} b_n] \\ &\leq \left\{ \sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\lambda} + n^{\lambda}} \right)^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (2.12)$$

Hence by (2.8), we have (2.6). It follows that (2.8) is equivalent to (2.6).

If the constant factor in (2.8) is not the best possible, we can get a contradiction from (2.12) that the constant factor in (2.6) is not the best possible. The theorem is proved.

Remark 2.5 For $\lambda = 1$, (2.6) and (2.8) reduce respectively to (1.1) and (1.2). It follows that (2.6) is a generalization of (1.1), and (2.8) is a generalization of (1.2). Although inequalities (2.6), (1.3) and (1.4) are with respectively a parameter λ , but they are different.

3 Some Reversed Inequalities for $\lambda = 2$

Lemma 3.1 If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < \varepsilon < p$, then

$$J := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{-\varepsilon/p} n^{-\varepsilon/q}}{m^2 + n^2} < \frac{1}{2} B\left(\frac{p-\varepsilon}{2p}, \frac{p+\varepsilon}{2p}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}. \quad (3.1)$$

Proof Since $0 < p < 1$, we find

$$\sum_{m=1}^{\infty} \frac{m^{-\varepsilon/p} n^{-\varepsilon/q}}{m^2 + n^2} < \int_0^{\infty} \frac{x^{-\varepsilon/p} n^{-\varepsilon/q}}{x^2 + n^2} dx.$$

Setting $y = (x/n)^2$ in the above integral, we obtain $dx = \frac{n}{2} y^{-1/2} dy$ and

$$J < \sum_{n=1}^{\infty} \int_0^{\infty} \frac{x^{-\varepsilon/p} n^{-\varepsilon/q}}{x^2 + n^2} dx = \frac{1}{2} \int_0^{\infty} \frac{1}{1+y} y^{\frac{p-\varepsilon}{2p}-1} dy \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}.$$

Hence by (2.1), we have (3.1). The lemma is proved.

Theorem 3.2 If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} \frac{a_n^p}{n} < \infty$ and $0 < \sum_{n=1}^{\infty} \frac{b_n^q}{n} < \infty$, then we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^2 + n^2} > \frac{\pi}{2} \left\{ \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{a_n^p}{n} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{b_n^q}{n} \right\}^{\frac{1}{q}}, \quad (3.2)$$

where the constant factor $\frac{\pi}{2}$ is the best possible.

Proof By the reverse of Hölder's inequality, one has

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^2 + n^2} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(m^2 + n^2)^{\frac{1}{p}}} \right] \left[\frac{b_n}{(m^2 + n^2)^{\frac{1}{q}}} \right] \\ &\geq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^p}{m^2 + n^2} \right\}^{1/p} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_n^q}{m^2 + n^2} \right\}^{1/q} \\ &= \left\{ \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{m}{m^2 + n^2} \right) \frac{a_m^p}{m} \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{n}{m^2 + n^2} \right) \frac{b_n^q}{n} \right\}^{1/q} \\ &= \left\{ \sum_{m=1}^{\infty} \omega(m) \frac{a_m^p}{m} \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \omega(n) \frac{b_n^q}{n} \right\}^{1/q}, \end{aligned} \quad (3.3)$$

where, the weight coefficient $\omega(j)$ is defined by $\omega(j) := \sum_{k=1}^{\infty} \frac{j}{j^2 + k^2}$ ($j = m, n$).

Since we find

$$\omega(n) < \int_0^{\infty} \frac{n}{n^2 + x^2} dx = [\arctan \frac{x}{n}]_0^{\infty} = \frac{\pi}{2}; \quad (3.4)$$

$$\omega(m) > \int_1^{\infty} \frac{m}{m^2 + x^2} dx = \frac{\pi}{2} - \arctan \frac{1}{m} > \frac{\pi}{2} - \frac{1}{m}, \quad (3.5)$$

then in view of $0 < p < 1, q < 0$ and (3.3), we have (3.2).

For $0 < \varepsilon < p$, setting $\tilde{a}_n, \tilde{b}_n$ as: $\tilde{a}_n = n^{-\varepsilon/p}; \tilde{b}_n = n^{-\varepsilon/q}$, then we find

$$\begin{aligned} &\left\{ \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{\tilde{a}_n^p}{n} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{\tilde{b}_n^q}{n} \right\}^{\frac{1}{q}} = \left\{ \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{1}{n^{1+\varepsilon}} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right\}^{\frac{1}{q}} \\ &> \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right\}^{\frac{1}{q}} = (1 - o(1))^{\frac{1}{p}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \quad (\varepsilon \rightarrow 0^+). \end{aligned} \quad (3.6)$$

If the constant factor $\frac{\pi}{2}$ in (3.2) is not the best possible, then there exists a positive number $k > \frac{\pi}{2}$, such that (3.2) is still valid if one replaces $\frac{\pi}{2}$ by k . In particular, one has

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{m^2 + n^2} > k \left\{ \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n}\right) \frac{\tilde{a}_n^p}{n} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{\tilde{b}_n^q}{n} \right\}^{\frac{1}{q}}. \quad (3.7)$$

Then by (3.1) and (3.6), it follows that

$$\frac{1}{2}B\left(\frac{p-\varepsilon}{2p}, \frac{p+\varepsilon}{2p}\right) \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} > k(1-o(1))^{\frac{1}{p}} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}},$$

and then $\frac{\pi}{2} \geq k(\varepsilon \rightarrow 0^+)$. This contradicts the fact that $k > \frac{\pi}{2}$. Hence the constant factor $\frac{\pi}{2}$ in (3.2) is the best possible. The theorem is proved.

Theorem 3.3 If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} \frac{a_n^p}{n} < \infty$, then

$$\sum_{n=1}^{\infty} n^{p-1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^p > \left(\frac{\pi}{2} \right)^p \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n} \right) \frac{a_n^p}{n}, \quad (3.8)$$

where the constant factor $(\frac{\pi}{2})^p$ is the best possible.

Proof By the reverse of Hölder's inequality and (3.4), one has

$$\begin{aligned} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^p &= \left\{ \sum_{m=1}^{\infty} \left[\frac{a_m}{(m^2 + n^2)^{\frac{1}{p}}} \right] \left[\frac{1}{(m^2 + n^2)^{\frac{1}{q}}} \right] \right\}^p \\ &\geq \left\{ \sum_{m=1}^{\infty} \frac{a_m^p}{m^2 + n^2} \right\} \left\{ \frac{\omega(n)}{n} \right\}^{p-1} > \left(\frac{\pi}{2n} \right)^{p-1} \sum_{m=1}^{\infty} \frac{a_m^p}{m^2 + n^2}. \end{aligned}$$

Hence we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^p &> \left(\frac{\pi}{2} \right)^{p-1} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m^p}{m^2 + n^2} \\ &= \left(\frac{\pi}{2} \right)^{p-1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 + n^2} a_m^p = \left(\frac{\pi}{2} \right)^{p-1} \sum_{m=1}^{\infty} \omega(m) \frac{a_m^p}{m}. \end{aligned}$$

By (3.5), we have (3.8).

By the reverse of Hölder's inequality, one has

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^2 + n^2} &= \sum_{n=1}^{\infty} \left[n^{1/q} \sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right] [n^{-1/q} b_n] \\ &\geq \left\{ \sum_{n=1}^{\infty} n^{p-1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^2 + n^2} \right)^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \frac{b_n^q}{n} \right\}^{1/q}. \end{aligned} \quad (3.9)$$

If the constant factor in (3.8) is not the best possible, one can get a contradiction that the constant factor in (3.2) is not the best possible by using (3.9). Hence the constant factor in (3.8) is the best possible. Thus we complete the proof of the theorem.

Remark 3.4 Following (2.6) in $\lambda = 2$ and (3.2), one can get a two-sides inequality as:

$$\left\{ \sum_{n=1}^{\infty} \left(1 - \frac{2}{\pi n} \right) \frac{a_n^p}{n} \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{b_n^q}{n} \right\}^{\frac{1}{q}} < \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^2 + n^2} < \left\{ \sum_{n=1}^{\infty} \frac{a_n^r}{n} \right\}^{\frac{1}{r}} \left\{ \sum_{n=1}^{\infty} \frac{b_n^s}{n} \right\}^{\frac{1}{s}}, \quad (3.10)$$

where $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$.

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