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An approximation to the solution of parabolic equation by Adomian decomposition method and comparing the result with Crank-Nicolson method

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Abstract

Mathematical modeling of many phenomena in applied science leads to parabolic equations. So the solutions of such equations are of interest. Numerical solutions such as finite difference approach needs a large size of computation. Adomian decomposition method which needs less computation is employed to solve parabolic partial differential equations and results are compared with the results of Crank- Nicolson method.

Keywords: decomposition , Crank- Nicolson method, parabolic equations.

1 Introduction

Consider the second-order quasi linear partial differential equation:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + e = 0 \tag{1}$$

Where a,b,c and e may be functions of x,y,u,$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ but not of $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$nor $\frac{\partial^2 u}{\partial y^2}$, and the second-order derivatives occur with the degree at almost one. If $b^2 - 4ac = 0$, the Eq. (1) is of parabolic kind. In numerical methods, such as finite difference methods which are commonly used for solving these equations, large size of computations are needed and usually the round-off error causes the loss of accuracy.

2 The Adomian decomposition method applied to parabolic equation:

For applying Adomian decomposition method to solve Eq. (1), this equation must be rewritten in a special form which is called canonical form. Canonical form of Eq. (1) depends on the indicated initial or boundary conditions, and will be discussed in the following.

I) Consider the parabolic Eq. (1) with the following initial conditions:

$$u(x,0) = f(x) \tag{2}$$

$$\frac{\partial u(x,0)}{\partial y} = g(x) \tag{3}$$

Regarding these conditions, let's rewrite the equation (1) as the following:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{e}{c} - \frac{a}{c}\frac{\partial^2 u}{\partial x^2} - \frac{b}{c}\frac{\partial^2 u}{\partial x \partial y} \tag{4}$$

Using the operator$L_{yy} = \frac{\partial^2}{\partial y^2}$, (4) can be written as:

$$L_{yy}u = -\frac{e}{c} - \frac{a}{c}\frac{\partial^2 u}{\partial x^2} - \frac{b}{c}\frac{\partial^2 u}{\partial x \partial y} \tag{5}$$

Applying the inverse operator$L_{yy}^{-1} = \int_0^y \int_0^y (.)dydy$ to both sides of (5), we get:

$$u(x,y) = u(x,0) + \frac{\partial u(x,0)}{\partial y}y - \int_0^y \int_0^y (\frac{e}{c} + \frac{a}{c}\frac{\partial^2 u}{\partial x^2} + \frac{b}{c}\frac{\partial^2 u}{\partial x \partial y})dydy. \tag{6}$$

Considering initial conditions, we have:

$$u(x,y) = f(x) + g(x)y - \int_0^y \int_0^y (\frac{e}{c} + \frac{a}{c}\frac{\partial^2 u}{\partial x^2} + \frac{b}{c}\frac{\partial^2 u}{\partial x \partial y})dydy. \tag{7}$$

Eq.(7) is a canonical form of Eq. (1). To solve Eq. (7) by Adomian decomposition method let's consider, as usual in this method, the solution u as the summation of a series

$$u = \Sigma_{n=0}^{\infty} u_n \tag{8}$$

And the integrand on the right side as the summation of a series:

$$\frac{e}{c} + \frac{a}{c}\frac{\partial^2 u}{\partial x^2} + \frac{b}{c}\frac{\partial^2 u}{\partial x \partial y} = \Sigma_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) \tag{9}$$

Where $A_n(u_0, u_1, \ldots, u_n)$, well defined in [3], are called Adomian polynomials and should be computed. Substituting (8) and (9) into (7) leads to:

$$\Sigma_{n=0}^{\infty} u_n = f(x) + g(x)y - \Sigma_{n=0}^{\infty} \int_0^y \int_0^y (A_n(u_0, u_1, \ldots, u_n))dydy \qquad (10)$$

From which The following Adomian procedure can be defined:

$$u_0 = f(x) + g(x)y$$

$$u_{n+1} = \int_0^y \int_0^y (A_n(u_0, u_1, \ldots, u_n))dydy \qquad n = 0, 1, 2, \ldots$$

We can determine the components u_n as many as is necessary to enhance the desired accuracy for the approximation. So, the n-terms approximation $\varphi_n = \Sigma_{i=0}^{n-1} u_i$ can be used to approximate the solution.

II) If the initial conditions are:

$$u(0, y) = f(y) \qquad (11)$$

$$\frac{\partial u(0, y)}{\partial x} = g(y). \qquad (12)$$

(13) The operator $L_{xx} = \frac{\partial^2}{\partial x^2}$ is suitable and we rewrites the Eq.(1) as the following

$$\frac{\partial^2 u}{\partial x^2} = -\frac{e}{a} - \frac{b}{a}\frac{\partial^2 u}{\partial x \partial y} - \frac{c}{a}\frac{\partial^2 u}{\partial y^2}. \qquad (13)$$

Applying the inverse operator $L_{xx}^{-1} = \int_0^x \int_0^x (.)dxdx$ to both sides of (13), we get:

$$u(x, y) = u(0, y) + \frac{\partial u(0, y)}{\partial x}x - \int_0^x \int_0^x (\frac{e}{a} + \frac{b}{a}\frac{\partial^2 u}{\partial x \partial y} + \frac{c}{a}\frac{\partial^2 u}{\partial y^2})dxdx. \qquad (14)$$

Substituting (11) and (12) into (14), we have:

$$u(x, y) = f(y) + g(y)x - \int_0^x \int_0^x (\frac{e}{a} + \frac{b}{a}\frac{\partial^2 u}{\partial x \partial y} + \frac{c}{a}\frac{\partial^2 u}{\partial y^2})dxdx. \qquad (15)$$

III) If the initial conditions are:

$$u(x, 0) = f(x) \qquad (16)$$

$$u(0, y) = g(y) \qquad (17)$$

Then we use the operator $L_{xy} = \frac{\partial^2}{\partial x \partial y}$. The Eq.(1) can be written as:

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{e}{b} - \frac{a}{b}\frac{\partial^2 u}{\partial x^2} - \frac{c}{b}\frac{\partial^2 u}{\partial y^2}. \tag{18}$$

Applying the inverse operator $L_{xy}^{-1} = \int_0^y \int_0^x (.)dxdy$ to both sides of (18), we get:

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) - \int_0^y \int_0^x (\frac{e}{b} + \frac{a}{b}\frac{\partial^2 u}{\partial x^2} + \frac{c}{b}\frac{\partial^2 u}{\partial y^2})dxdy. \tag{19}$$

Substituting conditions (16) and (17) into (19), we derive:

$$u(x,y) = f(x) + g(y) - u(0,0) - \int_0^y \int_0^x (\frac{e}{b} + \frac{a}{b}\frac{\partial^2 u}{\partial x^2} + \frac{c}{b}\frac{\partial^2 u}{\partial y^2})dxdy. \tag{20}$$

IV) Another form of parabolic equation which worths to mention is:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2} \tag{21}$$

With the following initial condition,

$$u(x,y,z,0) = f(x,y,z). \tag{22}$$

Applying the inverse operator$L_t^{-1} = \int_0^t (.)dt$ to both sides of (21), results,

$$u(x,y,z,t) = u(x,y,z,0) + \int_0^t (\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2})dt. \tag{23}$$

Substituting initial condition (22) into (23), we have:

$$u(x,y,z,t) = f(x,y,z) + \int_0^t (\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} + c\frac{\partial^2 u}{\partial z^2})dt. \tag{24}$$

Equations (16), (20) and (24) are canonical forms of Eq. (1) which are suitable forms, regarding different initial or boundary conditions. Applying Adomian method is almost the same as what was done for Eq. (6).

3 Numerical results

To illustrate the method some examples are provided, for different cases.

example1:consider the following equation [2]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \qquad 0 \le x \le 1$$

$$u(x,0) = 1 - x^2 \qquad 0 \le x \le 1$$

$$\frac{\partial u(0,t)}{\partial x} = 0 \quad , \quad u(1,t) = 0 \qquad t > 0$$

To solve this second-order partial differential equation let's rewrite the equation as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}$$

In this case the operator $L = \frac{\partial}{\partial t}$, with inverse $L_t^{-1} = \int_0^t (.)dt$, can be used. Applying inverse operator results

$$u(x,t) = 1 - x^2 + int_0^t(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x})dt$$

And the solution by Adomian decomposition method consists of following scheme:

$$u_0 = 1 - x^2$$

$$u_{n+1} = \int_0^t (\frac{\partial^2 u_n}{\partial x^2} + \frac{2}{x}\frac{\partial u_n}{\partial x})dt \qquad n = 0, 1, 2, \ldots$$

From which:

$$u_1 = -6t$$

$$u_2 = 0$$

$$\vdots$$

$$u_n = 0$$

Therefore the exact solution will be derived:

$$u(x,t) = 1 - x^2 - 6t$$

In table 1 the results of Adomian method and crank-Nicolson method are compared, for some specified value of x and t.

*table*1

The solution of u(x,t) for different values of x and t

x	t	u(x,t)(Adomian method)	u(x,t)(Crank-Nicolson method)
0.1	0.000	0.9900	0.9900
0.3	0.001	0.9040	0.9026
0.5	0.002	0.7380	0.7384
0.7	0.003	0.4920	0.4920
0.9	0.003	0.1720	0.1745

example2: Here is another example for case IV, [2]:

$$\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t} \qquad 0 \le x \le 1$$

$$u(x,0) = x^2 \qquad 0 \le x \le 1$$

$$u(0,t) = 0, \quad u(1,t) = 1 \qquad t > 0$$

Which can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

And without any more details

$$u(x,t) = x^2 + \int_0^t \left(\frac{\partial^2 u}{\partial x^2} - u\right) dt$$

By using an algorithm for computing Adomian polynomials [4] the Adomian scheme would be as follows:

$$u_0 = x^2$$

$$u_{n+1} = \int_0^t \left(\frac{\partial^2 u_n}{\partial x^2} - u_n\right) dt \qquad n = 0, 1, 2, \ldots$$

First few terms are

$$u_1 = (2 - x^2)t$$

$$u_2 = (x^2 - 4)\frac{t^2}{2!}$$

$$u_3 = (x^2 - 6)\frac{t^3}{3!}$$

$$\vdots$$

The general term:

$$u_n = (-1)^n (x^2 - 2n)\frac{t^n}{n!}$$

So

$$u(x,t) = x^2 + \Sigma_{n=1}^{\infty}(-1)^n (x^2 - 2n)\frac{t^n}{n!} = \Sigma_{n=0}^{\infty}(-1)^n \frac{x^2 t^n}{n!}(-2)\Sigma_{n=0}^{\infty}(-1)^n \frac{t^n}{(n-1)!}$$

$$= x^2 e^{-t} + 2te^{-t}.$$

Which is the exact solution.

For some specified value of and, the results of Adomian method and Crank-Nicolson method are compared in table 2.

table2

The solution of $u(x,t)$ for different values of x and t

x	t	u(x,t)(Adomian method)	u(x,t)(Crank-Nicolson method)
0.0	0.001	0.0020	0.0000
0.3	0.002	0.0938	0.0938
0.6	0.003	0.3649	0.3649
0.9	0.004	0.8147	0.8141
0.8	0.002	0.6454	0.6453

example3:consider the following equation [2]:

$$\frac{\partial u}{\partial t} = 2x\frac{2}{x}\frac{\partial u}{\partial x} + (1+x^2)\frac{\partial^2 u}{\partial x^2} \qquad -1 \le x \le 1$$

$$u(x,0) = e^x \qquad -1 \le x \le 1$$

$$u(-1,t) = 0 \qquad t > 0$$

By using $L_t^{-1} = \int_0^t (.)dt$ we have

$$u(x,t) = e^x + \int_0^t (2x\frac{2}{x}\frac{\partial u}{\partial x} + (1+x^2)\frac{\partial^2 u}{\partial x^2})dt$$

From (11) we get:

$$u_0 = e^x$$

$$u_{n+1} = \int_0^t (2x\frac{2}{x}\frac{\partial u_n}{\partial x} + (1+x^2)\frac{\partial^2 u_n}{\partial x^2})dt \qquad n = 0, 1, 2, \ldots$$

And we have:

$$u_0 = e^x$$

$$u_1 = e^x t(x^2 + 2x + 1)$$

$$u_2 = \frac{1}{2}e^x t^2(x^4 + 8x^3 + 16x^2 + 12x + 7)$$

$$u_3 = \frac{1}{6}e^x t^3(x^6 + 18x^5 + 101x^4 + 220x^3 + 227x^2 + 162x + 63)$$

$$u_4 = \frac{1}{24}e^x t^4(x^8 + 32x^7 + 360x^6 + 1800x^5 + 4318x^4 + 5552x^3 + 4832x^2 + 2840x + 841)$$

$$u_5 = \frac{1}{120}e^x t^5(x^10 + 50x^9 + 945x^8 + 8680x^7 + 41902x^6 + 110196x^5 + 168726x^4 + 175552x^3$$
$$+ 131153x^2 + 62842x + 16185)$$

$$u_6 = \frac{1}{720}e^x t^6(x^12+72x^11+2056x^10+30260x^9+250757x^8+1210304x^7+3467304x^6+ 6145272x^5 + 74298835x^4 + 6672664x^3 + 425648x^2 + 1792820x + 404175)$$

$$\vdots$$

Six-terms approximation to the solution will be as follows:

$$u(x,t) = e^x(1 + (x^2 + 2x + 1)t + \frac{1}{2}(x^4 + 8x^3 + 16x^2 + 12x + 7)t^2 + \frac{1}{6}(x^6 + 18x^5 + 101x^4+220x^3+227x^2+162x+63)t^3+\frac{1}{24}(x^8+32x^7+360x^6+1800x^5+4318x^4+5552x^3 +4832x^2+2840x+841)t^4+\frac{1}{120}(x^10+50x^9+945x^8+8680x^7+41902x^6+110196x^5+ 168726x^4+175552x^3+131153x^2+62842x+16185)t^5+\frac{1}{720}(x^12+72x^11+2056x^10+30260x^9 +250757x^8+1210304x^7+3467304x^6+6145272x^5+74298835x^4+6672664x^3+425648x^2 +1792820x + 404175)t^6)$$

The results of Adomian method and Crank-Nicolson method are compared, for some specified value of and , in table 3.

table3

The solution of $u(x,t)$ for different values of x and t

x	t	u(x,t)(Adomian method)	u(x,t)(Crank-Nicolson method)
-0.8	0.001	0.4493	0.4499
-0.4	0.003	0.6711	0.6755
0.0	0.005	1.0051	1.0206
0.4	0.002	1.4977	1.5093
0.8	0.004	2.2550	2.2881

4 Conclusion

The aim of this article was to derive an approximation to the solution of parabolic equations. We have achieved this aim by applying Adomian decomposition method. In example 1 after some steps the remaining terms would vanish and the solution will be derived. In some cases as example 2, we can recognize the solution from the series resulted by applying Adomian decomposition method. In other cases as example 3, an approximation can be obtained by calculating as many terms as desired to increase the accuracy, the more terms the more accuracy. The small size of computations in comparison with the computational size required in Crank-Nicolson method is one of the advantages of Adomian method. Another point worth to mention is the rapid of convergence of Adomian decomposition method. The reliability, simplicity and accuracy of this method have been pointed by many authors who have applied this method for solving different functional equations. This method has been extended for solving some systems of functional equations by the first author, the problem of convergence of the method in applying the method for various systems are a reach source for young mathematicians to work on.

5 Reference

[1] Smith, Numerical Method for Partial Difference Equation, Oxford press, (1978).

[2] G. Evens, J.Blackledge, p.Yardley, Numerical Method for Partial Difference Equations, Springer, (2000).

[3] G. Adomian, Solving Frontier problem of Physics: The Decomposition Method, Kluwer Academic press,(1994)

[4] J.Biazar, E. Babolian, A. Nouri, R. Islam, An alternate algorithm for computing Adomian Decomposition method in special cases, Applied Mathematics and Computation 28(2-3),PP.523-529,(2003).

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