Characters and quasi-permutation representations of finite p -groups with few non-normal subgroups

Houshang Behravesht and Sebar Ghadery

Department of Mathematics
University of Urmia
Urmia, Iran
h.behravesht@mail.urmia.ac.ir

Abstract. Let G be a finite group and let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of G . In [1], we gave algorithms to calculate $c(G)$, $q(G)$ and $p(G)$ for a finite group G . In this paper, we will calculate irreducible characters of finite p -groups G , with $\nu(G) = 3$. Also we will calculate $c(G)$, $q(G)$ and $p(G)$. Finally we will show that $\text{cd}(G) = \{1, p\}$, where $\text{cd}(G)$ denote the degree of irreducible characters of G .

Keywords : Quasi-permutation representations, Finite groups, Finite p -groups, Character theory.

Mathematics Subject Classification: Primary 20C15, Secondary 20B05

1. INTRODUCTION

Let G be a finite group. Following [3] and [7], we let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of G . Clearly $\nu(G) = 0$ if and only if G is a hamiltonian. In [3], R. Brandl characterizes the finite groups G whose non-normal subgroups are all conjugate. In [7], H. Mousavi characterizes the finite groups G with $\nu(G) = 2$. In [8], H. Mousavi characterizes the finite p -groups G with $\nu(G) = 3$, and proves the following theorem:

Theorem 1.1. *Let G be a finite p -group. Then $\nu(G) = 3$ if and only if G is isomorphic to one of the following groups:*

- (a) $G_1 = Q_8 \rtimes \mathbb{Z}_4$;
- (b) $G_2 = \langle x, y : x^2 = y^4 = [x, y^2] = 1, [x, y]^2 = y^2 \rangle$;
- (c) $G_3 = \langle x, y : x^8 = y^8 = 1, x^4 = y^4, yxy = x \rangle$;
- (d) $G_4 = \langle x, y : x^4 = y^8 = 1, [y, x] = y^4 x^2 \rangle$;
- (e) $G_5 = \langle x, y, z : x^2 = y^2 = z^{2^n} = [x, z] = [y, z] = 1, [x, y] = z^{2^{n-1}} \rangle$,
 $n \geq 2$;
- (f) $G_6 = \langle x, y : x^{3^2} = y^{3^n} = 1, [x, y] = y^{3^{n-1}} \rangle$, $n \geq 2$.

Now let define the quantities $c(G)$, $q(G)$ and $p(G)$. By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G , let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or of a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [1]. It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

In this paper, we will calculate irreducible characters of finite p -groups G , with $\nu(G) = 3$. Also we will calculate $c(G)$, $q(G)$ and $p(G)$. Finally we will show that $\text{cd}(G) = \{1, p\}$, where $\text{cd}(G)$ denote the degree of irreducible characters of G .

2. QUASI PERMUTATION REPRESENTATION OF GROUPS G_1, G_2, G_3, G_4

We will number our groups as they appear in the library of small groups in GAP.

In order to calculate $q(G)$ we will need the Schur index of irreducible characters. Hence we will need the following lemmas.

Lemma 2.1. *Let G be a 2-group and $\chi \in \text{Irr}(G)$. Then $m_{\mathbb{Q}}(\chi) = m_{\mathbb{R}}(\chi)$.*

Proof : See [[9], Satz 1].

Lemma 2.2. *Let G be a finite group and $\chi \in \text{Irr}(G)$. Let*

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Then

$$\nu(\chi) = \begin{cases} 1 & \text{if } \chi = \bar{\chi} \text{ and } m_{\mathbb{R}}(\chi) = 1 \\ -1 & \text{if } \chi = \bar{\chi} \text{ and } m_{\mathbb{R}}(\chi) = 2 \\ 0 & \text{if } \chi \neq \bar{\chi} \end{cases}.$$

Proof : See [[4], page 191, Lemma 33.4].

Note : By Lemmas 2.1 and 2.2 one can calculate the Schur index of any irreducible character of a 2-group by calculating $\nu(\chi)$. Note that calculating $\nu(\chi)$ it is not so easy.

Lemma 2.3. *Let G be a 2-group with an irreducible character of degree 2. Then $\det \chi$ is the principle character if and only if the Schur index $m_{\mathbb{Q}}(\chi) = 2$.*

Proof : See [[5], Theorem 3].

Theorem 2.4. *The following table hold.*

G	$c(G)$	$q(G)$	$p(G)$
$G_1 \cong (32, 26)$	8	12	12
$G_2 \cong (16, 8)$	8	8	8
$G_3 \cong (32, 15)$	16	16	16
$G_4 \cong (32, 12)$	12	12	12

Proof : We will use the GAP for the character tables and the subgroups and the core of subgroups. Also we will use Lemmas 2.2 and 2.3 and $\nu(\chi)$ for Schur indices. Finally we will use [[1], Corollaries 2.4 and 3.11] for groups of cyclic center or [[1], Theorems 2.2 and 3.6] for groups of non-cyclic center, in order to calculate $c(G)$, $q(G)$ and $p(G)$.

3. CHARACTERS AND QUASI-PERMUTATION REPRESENTATIONS OF G_5

Let

$$G = G_5 = \langle x, y, z : x^2 = y^2 = z^{2^n} = [x, z] = [y, z] = 1, [x, y] = z^{2^{n-1}} \rangle.$$

Then it is easy to see that $Z(G) = \langle z \rangle$ and $G' = \langle z^{2^{n-1}} \rangle$. Also $|G| = 2^{n+2}$ and $|G : Z(G)| = 4$. Since G is non-abelian, so by [[6], Corollary 2.30], $\text{cd}(G) = \{1, 2\}$. By [[6], Corollary 2.6], we have $|G : G'| = 2^{n+1}$ linear characters. Also by [[6], Corollary 2.7] we have 2^{n-1} irreducible characters of degree 2.

Theorem 3.1. (1) $G/G' \cong C_2 \times C_2 \times C_{2^{n-1}}$.

(2) All characters of degree 2 are faithful. Also if χ_i denote an irreducible non-linear characters of G , then we have:

$$\begin{aligned}\chi_i(x) &= 0 \text{ for all } x \in G - Z(G) \\ \chi_i(z^j) &= 2\epsilon^{ij},\end{aligned}$$

where $1 \leq j \leq 2^n$, $(2^n, i) = 1$, $1 \leq i \leq 2^n - 1$ and ϵ is an 2^n -th primitive root of 2^n .

$$(3) \ c(G) = q(G) = p(G) = 2^{n+1}$$

Proof : (1) It is easy to see that $G/G' \cong \langle xG', yG', zG' \rangle$. So the result follows.

(2) By [[6], Theorem 2.32 (b)], there exists a faithful irreducible character of G . By (1), G has no linear and faithful character. So the faithful character has degree 2. Also by [[6], Corollary 2.30], we know that this character vanishes on $G - Z(G)$ and on any element of $Z(G)$ is equal to 2 times the value of an irreducible character of $Z(G)$ on that element. As $Z(G)$ has only 2^{n-1} faithful linear characters and they are all in one Galois orbit, so G has at least 2^{n-1} faithful and irreducible character of degree 2. Now by the first paragraph of this section the result follows.

(3) $\langle x \rangle$ is a subgroup of G and its corefree. So by [[1], Corollary 2.4] and [[1], Theorem 4.12], the result follows.

4. QUASI-PERMUTATION REPRESENTATIONS OF G_6

It is easy to prove that

$$G_6 = \langle x^2, y : (x^2)^9 = y^{3^n} = 1, x^{-2}yx^2 = y^{1+3^n} \rangle.$$

So

$$G_6 \cong \langle z, y : z^9 = y^{3^n} = 1, z^{-1}yz = y^{1+3^n} \rangle.$$

Now it is easy to see that

$$\langle z, y : z^{3^2} = y^{3^n} = 1, z^{-1}yz = y^{1+3^n} \rangle$$

satisfy the conditions of metacyclic p -groups stated in [[2], page 347]. So we are able to calculate the irreducible characters of G_6 and also show that

$$c(G_6) = q(G_6) = p(G_6) = 3^2 + 3^n.$$

Also here we will state some results without using of [2].

Lemma 4.1. (Ito) *Let G be a finite group and let H be an abelian subgroup of G . Then for all $\chi \in \text{Irr}(G)$*

$$\chi(1) \mid |G : H|.$$

Moreover if G be a finite non-abelian p -group and $|G : H| = p$, then $\text{cd}(G) = \{1, p\}$.

Proof : See [[6], Theorem 6.15].

Lemma 4.2. *Let $G = G_6 = \langle x, y : x^9 = y^{3^n} = 1, [x, y] = y^{3^{n-1}} \rangle$. Then*

$$(1) Z(G) = \langle x^3, y^3 \rangle, |Z(G)| = 3^n \text{ and } |G : Z(G)| = 9.$$

$$(2) G' = \langle y^{3^{n-1}} \rangle.$$

$$(3) \text{cd}(G) = \{1, 3\}.$$

Proof : (1) and (2) are easy to prove.

(3) Let $H = \langle x^3, y \rangle$. Then H is an abelian group and $H \triangleleft G$. Since $\chi(1) \mid |G|$ for all irreducible characters of G , so $\chi(1)$ is a power of p for non-linear and irreducible characters of G . Now by Lemma 4.1, degree of non-linear and irreducible characters of G are 3.

Also one can use the fact that $\chi(1) \leq |G : Z(G)|^{1/2} = 3$ for all irreducible characters of G , so the result (3) follows.

Corollary 4.3. *Let G be a finite p -group and $\nu(G) = 3$. Then $\text{cd}(G) = \{1, p\}$.*

REFERENCES

- [1] **H. Behraves**. *Quasi-permutation representations of p -groups of class 2*. J. London Math. Soc. (2) 55 (1997) 251-260.
- [2] **H. Behraves**. *Quasi-permutation representations of metacyclic p -groups with non-cyclic center*. Southeast Asian Bulletin of Mathematics (2000) 24: 345-353.

- [3] **R. Brandel.** *Groups with few non-normal subgroups.* Comm. Algebra 23 (1995) 2091-2098.
- [4] **L. Dornhoff.** *Group representation theory, part A.* Dekker, New York, 1971.
- [5] **Y. Iida.** *A note on the Schur index of an irreducible character of a 2-group.* Soochow J. Math. Vol. 24, No. 2 (1998) 163-165.
- [6] **I. M. Isaacs.** *Character theory of finite groups.* Dover Publications, Inc., New York, (1994).
- [7] **H. Mousavi.** *On finite groups with few non-normal subgroups.* Comm. Algebra 27(7) (1999), 3143-31151.
- [8] **H. Mousavi.** *Finite p -groups with non-normal subgroups having the same order.* Will appear.
- [9] **P. Roquette.** *Realisierung von Darstellungen endlicher nilpotenter Gruppen.* Archiv. der Math. 9 (1958) 241-250.
- [10] **M. Schonert et al.** *GAP: Groups, Algorithms, and Programming.* Lehrstuhl D für Mathematik, RWTH Aachen, 1994.

Received: February 1, 2006