Characters and quasi-permutation representations  
of finite $p$-groups with few non-normal subgroups  

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Abstract. Let $G$ be a finite group and let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of $G$. In [1], we gave algorithms to calculate $c(G)$, $q(G)$ and $p(G)$ for a finite group $G$. In this paper, we will calculate irreducible characters of finite $p$-groups $G$, with $\nu(G) = 3$. Also we will calculate $c(G)$, $q(G)$ and $p(G)$. Finally we will show that $\text{cd}(G) = \{1, p\}$, where $\text{cd}(G)$ denote the degree of irreducible characters of $G$.

Keywords: Quasi-permutation representations, Finite groups, Finite $p$-groups, Character theory.

Mathematics Subject Classification: Primary 20C15, Secondary 20B05

1. Introduction

Let $G$ be a finite group. Following [3] and [7], we let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of $G$. Clearly $\nu(G) = 0$ if and only if $G$ is a hamiltonian. In [3], R. Brandl characterizes the finite groups $G$ whose non-normal subgroups are all conjugate. In [7], H. Mousavi characterizes the finite groups $G$ with $\nu(G) = 2$. In [8], H. Mousavi characterizes the finite $p$-groups $G$ with $\nu(G) = 3$, and proves the following theorem:
Theorem 1.1. Let $G$ be a finite $p$-group. Then $\nu(G) = 3$ if and only if $G$ is isomorphic to one of the following groups:

(a) $G_1 = Q_8 \rtimes \mathbb{Z}_4$;
(b) $G_2 = \langle x, y : x^2 = y^4 = [x, y^2] = 1, [x, y]^2 = y^2 \rangle$;
(c) $G_3 = \langle x, y : x^8 = y^8 = 1, x^4 = y^4, yxy = x \rangle$;
(d) $G_4 = \langle x, y : x^4 = y^8 = 1, [y, x] = y^4x^2 \rangle$;
(e) $G_5 = \langle x, y, z : x^2 = y^2 = z^{2^n} = [x, z] = [y, z] = 1, [x, y] = z^{2^n-1} \rangle, n \geq 2$;
(f) $G_6 = \langle x, y : x^{3^2} = y^{3^n} = 1, [x, y] = y^{3^2} \rangle, n \geq 2$.

Now let define the quantities $c(G)$, $q(G)$ and $p(G)$. By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group $G$, let $p(G)$ denote the minimal degree of a faithful permutation representation of $G$ (or of a faithful representation of $G$ by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ denote the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices. See [1]. It is easy to see that

$$c(G) \leq q(G) \leq p(G)$$

where $G$ is a finite group.

In this paper, we will calculate irreducible characters of finite $p$-groups $G$, with $\nu(G) = 3$. Also we will calculate $c(G)$, $q(G)$ and $p(G)$. Finally we will show that $\text{cd}(G) = \{1, p\}$, where $\text{cd}(G)$ denote the degree of irreducible characters of $G$.

2. Quasi permutation representation of groups $G_1, G_2, G_3, G_4$

We will number our groups as they appear in the library of small groups in GAP.

In order to calculate $q(G)$ we will need the Schur index of irreducible characters. Hence we will need the following lemmas.

Lemma 2.1. Let $G$ be a 2-group and $\chi \in \text{Irr}(G)$. Then $m_\mathbb{Q}(\chi) = m_\mathbb{F}(\chi)$.

Proof : See [[9], Satz 1].
Lemma 2.2. Let $G$ be a finite group and $\chi \in \text{Irr}(G)$. Let

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Then

$$\nu(\chi) = \begin{cases} 
1 & \text{if } \chi = \bar{\chi} \text{ and } m_R(\chi) = 1 \\
-1 & \text{if } \chi = \bar{\chi} \text{ and } m_R(\chi) = 2 \\
0 & \text{if } \chi \neq \bar{\chi}
\end{cases}.$$  

Proof: See [[4], page 191, Lemma 33.4].

Note: By Lemmas 2.1 and 2.2 one can calculate the Schur index of any irreducible character of a 2-group by calculating $\nu(\chi)$. Note that calculating $\nu(\chi)$ it is not so easy.

Lemma 2.3. Let $G$ be a 2-group with an irreducible character of degree 2. Then $\det \chi$ is the principle character if and only if the Schur index $m_Q(\chi) = 2$.

Proof: See [[5], Theorem 3].

Theorem 2.4. The following table hold.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$c(G)$</th>
<th>$q(G)$</th>
<th>$p(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1 \cong (32,26)$</td>
<td>8</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$G_2 \cong (16,8)$</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$G_3 \cong (32,15)$</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$G_4 \cong (32,12)$</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Proof: We will use the GAP for the character tables and the subgroups and the core of subgroups. Also we will use Lemmas 2.2 and 2.3 and $\nu(\chi)$ for Schur indices. Finally we will use [[1], Corollaries 2.4 and 3.11] for groups of cyclic center or [[1], Theorems 2.2 and 3.6] for groups of non-cyclic center, in order to calculate $c(G)$, $q(G)$ and $p(G)$.

3. Characters and quasi-permutation representations of $G_5$

Let

$$G = G_5 = \langle x, y, z : x^2 = y^2 = z^{2^n} = [x, z] = [y, z] = 1, [x, y] = z^{2^{n-1}} \rangle.$$
Then it is easy to see that $Z(G) = \langle z \rangle$ and $G' = \langle z^{2^{n-1}} \rangle$. Also $|G| = 2^{n+2}$ and $|G : Z(G)| = 4$. Since $G$ is non-abelian, so by [[6], Corollary 2.30], $\text{cd}(G) = \{1, 2\}$. By [[6], Corollary 2.6], we have $|G : G'| = 2^{n+1}$ linear characters. Also by [[6], Corollary 2.7] we have $2^{n-1}$ irreducible characters of degree 2.

**Theorem 3.1.** (1) $G/G' \cong C_2 \times C_2 \times C_{2^{n-1}}$.

(2) All characters of degree 2 are faithful. Also if $\chi_i$ denote an irreducible non-linear characters of $G$, then we have:

$$\chi_i(x) = 0 \text{ for all } x \in G - Z(G)$$

$$\chi_i(z^j) = 2\epsilon^j,$$

where $1 \leq j \leq 2^n$, $(2^n, i) = 1$, $1 \leq i \leq 2^n - 1$ and $\epsilon$ is an $2^n$-th primitive root of $2^n$.

(3) $c(G) = q(G) = p(G) = 2^{n+1}$

Proof: (1) It is easy to see that $G/G' \cong \langle xG', yG', zG' \rangle$. So the result follows.

(2) By [[6], Theorem 2.32 (b)], there exists a faithful irreducible character of $G$. By (1), $G$ has no linear and faithful character. So the faithful character has degree 2. Also by [[6], Corollary 2.30], we know that this character vanishes on $G - Z(G)$ and on any element of $Z(G)$ is equal to 2 times the value of an irreducible character of $Z(G)$ on that element. As $Z(G)$ has only $2^{n-1}$ faithful linear characters and they are all in one Galios orbit, so $G$ has at least $2^{n-1}$ faithful and irreducible character of degree 2. Now by the first paragraph of this section the result follows.

(3) $\langle x \rangle$ is a subgroup of $G$ and its corefree. So by [[1], Corollary 2.4] and [[1], Theorem 4.12], the result follows.

4. **Quasi-permutation representations of $G_6$**

It is easy to prove that

$$G_6 = \langle x^2, y : (x^2)^9 = y^{3^n} = 1, x^{-2}yx^2 = y^{1+3^n} \rangle.$$

So

$$G_6 \cong \langle z, y : z^9 = y^{3^n} = 1, z^{-1}yz = y^{1+3^n} \rangle.$$
Now it is easy to see that
\[
\left\langle z, y : z^{3^2} = y^{3^n} = 1, z^{-1}yz = y^{1+3^n} \right\rangle
\]
satisfy the conditions of metacyclic $p$-groups stated in [[2], page 347]. So we
are able to calculate the irreducible characters of $G_6$ and also show that
\[
c(G_6) = q(G_6) = p(G_6) = 3^2 + 3^n.
\]
Also here we will state some results without using of [2].

**Lemma 4.1. (Ito)** Let $G$ be a finite group and let $H$ be an abelian subgroup
of $G$. Then for all $\chi \in \text{Irr}(G)$
\[
\chi(1) \mid |G : H|.
\]
Moreover if $G$ be a finite non-abelian $p$-group and $|G : H| = p$, then $\text{cd}(G) = \{1, p\}$.

*Proof:* See [[6], Theorem 6.15].

**Lemma 4.2.** Let $G = G_6 = \langle x, y : x^9 = y^{3^n} = 1, [x, y] = y^{3^n-1} \rangle$. Then
(1) $Z(G) = \langle x^3, y^3 \rangle$, $|Z(G)| = 3^n$ and $|G : Z(G)| = 9$.
(2) $G' = \langle y^{3^n-1} \rangle$.
(3) $\text{cd}(G) = \{1, 3\}$.

*Proof:* (1) and (2) are easy to prove.
(3) Let $H = \langle x^3, y \rangle$. Then $H$ is an abelian group and $H \triangleleft G$. Since $\chi(1) \mid |G|$ for all irreducible characters of $G$, so $\chi(1)$ is a power of $p$ for non-linear and irreducible characters of $G$. Now by Lemma 4.1, degree of non-linear and irreducible characters of $G$ are 3.
Also one can use the fact that $\chi(1) \leq |G : Z(G)|^{1/2} = 3$ for all irreducible characters of $G$, so the result (3) follows.

**Corollary 4.3.** Let $G$ be a finite $p$-group and $\nu(G) = 3$. Then $\text{cd}(G) = \{1, p\}$.

**References**


Received: February 1, 2006