Notes on the Graded Prime Submodules

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Abstract

Let G be a monoid with identity e , and let R be a G -graded commutative ring. Here we study the graded prime submodules of a graded R -module. While the bulk of this work is devoted to extending some results from prime submodules to graded prime submodules. A number of results concerning of these class of submodules are given.

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1 Introduction

Several authors have extended the notions of prime and primary ideals to modules (see [1]-[9]). In this paper, we continue this investigations for graded prime and graded primary submodules of a graded module over a G -graded ring. Various properties of such modules are considered (see section 2).

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary monoid with identity e . A commutative ring R with non-zero identity is G -graded if it has a direct sum decomposition (as an additive group) $R = \bigoplus_{g \in G} R_g$ such that $1 \in R_e$ and for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. If R is G -graded, then an R -module M is said to be G -graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be homogeneous element. A submodule of $N \subseteq M$, where M is G -graded, is called G -graded if $N = \bigoplus_{g \in G} (N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G -graded module with g -component $(M/N)_g = (M_g + N)/N$ for

$g \in G$. Clearly, 0 is a graded submodule of M . Also, we write $h(R) = \cup_{g \in G} R_g$ and $h(M) = \cup_{g \in G} M_g$. A graded ideal I of R is said to be graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I , denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A graded ideal I of R is said to be graded primary ideal if $I \neq R$; and whenever $a, b \in h(R)$ with $ab \in I$, then $a \in I$ or $b \in Gr(I)$. In this case, $Gr(I) = P$ is a graded prime ideal of R , and we say that I is a graded P -primary ideal of R (see [8, Lemma 1.8]). The graded radical of a graded submodule N of a graded module M denoted $Gr(N)$ and is defined to be intersection of all graded prime submodules of M containing N . Clearly, if N and K are graded submodules of M with $K \subseteq N$, then $Gr(K) \subseteq Gr(N)$.

2 Graded prime submodules

Our starting point is the following definition:

Definition 2.1 *Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M .*

- (i) *We say that M is a graded torsion-free R -module whenever $a \in h(R)$ and $m \in h(M)$ with $am = 0$ implies that either $a = 0$ or $m = 0$.*
- (ii) *N is a graded prime submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$.*
- (iii) *N is a graded primary submodule of M if $N \neq M$; and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a^k \in (N :_R M)$ for some k .*
- (iv) *N is a graded maximal submodule of M if $N \neq M$ and there is no graded submodule K of M such that $N \subsetneq K \subsetneq M$.*

The following lemma is known, but we write it here for the sake of references.

Lemma 2.2 *Let R be a G -graded ring, M a graded R -module and N, K graded R -submodules of M . The following hold:*

- (i) *If I and J are graded ideals of R , Then $(N : M)$, IJ , $I + J$ and $I \cap J$ are graded ideals of R .*
- (ii) *If $r \in h(R)$ and $x \in h(M)$, then Rx , IN and rN are graded submodule of M .*

Let R be a G -graded ring. We say that R is a graded integral domain whenever $a, b \in h(R)$ with $ab = 0$ implies that either $a = 0$ or $b = 0$. A G -graded ring R is said to be G -graded field if $0 \neq a \in h(R)$, then $ab = 1$ for some $b \in h(R)$.

Lemma 2.3 *Let R be a non-trivial G -graded ring, M a graded R -module and P a graded ideal of R . The following hold:*

- (i) R has at least one graded maximal ideal. In particular, if I is a proper graded ideal of R , then there exists a graded maximal ideal Q of R with $I \subseteq Q$.
- (ii) P is a graded maximal ideal of R if and only if R/P is a graded field.
- (iii) If R is a graded field, then M is a graded torsion-free.
- (iv) R is a graded field if and only if R has exactly two graded ideals.

Proof. The proof is straightforward. $\square$

Proposition 2.4 *Let R be a G -graded ring, M a graded R -module and N a graded R -submodule of M . The following hold:*

- (i) If $(N : M) = P$ is a graded maximal ideal of R , then N is a graded prime submodule of M .
- (ii) If Q is a graded maximal ideal of R with $QM \neq M$, then QM is graded prime.

Proof. (i) By Lemma 2.3, M/N is a graded torsion-free over the graded field R/P , so N is prime by [2, Theorem 2.11].

(ii) since $Q \subseteq (QM : M)$, we must have $(QM : M) = Q$. Now the assertion follows from (i). $\square$

Theorem 2.5 *Let R be a G -graded ring, M a graded R -module and $N, N_1, \dots, N_k$ graded R -submodules of M . If N is graded prime with $\bigcap_{i=1}^k N_i \subseteq N$, then either $N_i \subseteq N$ or $(N_i : M) \subseteq (N : M)$ for some i .*

Proof. Suppose not. Then there exists a homogeneous element $a \in N_1$ and for each $i \neq 1$, there are homogeneous elements $b_i \in (N_i : M)$ such that $b_i \notin (N : M)$ and $a \notin N$, so $b_2 b_3 \dots b_k a \in \bigcap_{i=1}^k N_i \subseteq N$; hence N graded prime gives $b_2 b_3 \dots b_k \in (N : M)$ which is a contradiction, as required. $\square$

Let M be a graded module over a G -graded ring R . Now consider the subset $T(M)$ of M is defined by

$$T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in h(R)\}$$

If R is a graded integral domain, then $T(M)$ is a graded R -submodule of M (see [2, Proposition 2.5]).

Proposition 2.6 *If R is a graded integral domain and M a graded R -submodule with $T(M) \neq M$, then $T(M)$ is a graded prime submodule of M and $(T(M) : M) = 0$.*

Proof. By [2, Proposition 2.5 and Theorem 2.11], $T(M)$ is graded prime and $Q = (T(M) : M)$ is a graded prime ideal of R . Now we will show that $Q = 0$. Let $a = a_{g_1} + \dots + a_{g_k} \in Q$ with $a_{g_i} \neq 0$. By assumption, there is a homogeneous element m_g of M with $m_g \notin T(M)$. Since Q is graded ideal, we must have $a_{g_i}m_g \in T(M)$ for all i , so there are elements $0 \neq s_{h_i} \in h(R)$ such that $s_{h_i}a_{g_i}m_g = 0$ for $i = 1, \dots, k$; hence for each i , $s_{h_i}a_{g_i} = 0$ since $m_g \notin T(M)$. As R is graded integral domain, we get that $a = 0$, as needed. $\square$

Let R be a G -graded ring, M a graded R -module, K a graded R -submodule of M . A graded prime R -submodule N of M is said to be a minimal graded prime of K if $K \subseteq N$ and if N' is another graded prime submodule with $K \subseteq N' \subseteq N$, then $N = N'$. A graded R -module M is said to be graded finitely generated if there exist $x_{g_1}, \dots, x_{g_n} \in h(M)$ such that $M = Rx_{g_1} + \dots + Rx_{g_n}$.

Lemma 2.7 *Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . Then the following hold:*

- (i) *If N is prime, then it contains a minimal graded prime submodule.*
- (ii) *If M is a graded finitely generated R -module, then it has a graded maximal submodule.*

Proof. (i) Since $N \subseteq N$, the set Δ of all graded prime submodules K of M such that $K \subseteq N$ is not empty. Of course, the relation of reverse inclusion, $\supseteq$, is a partial order on Δ . Let Λ be a non-empty subset of Δ which is totally ordered with respect to the above partial order. Then $P = \bigcap_{K \in \Lambda} K$ is a proper graded submodule of M since $N \neq M$. We show that P is a graded prime submodule of M . Let $rm \in P$ where $r \in h(R)$ and $m \in h(M) - P$, so there exists $Q \in \Lambda$ such that $m \notin Q$. Let $S \in \Lambda$. Then either $Q \subseteq S$ or $S \subseteq Q$. In the first case, the facts that $rm \in Q$ implies that $r \in (Q : M) \subseteq (S : M)$, so $r \in \bigcap_{K \in \Lambda} (K : M) = (P : M)$; in the second case $r \in (S : M)$; hence $r \in (P : M)$. Thus, P is an upper bound for Λ . We now use Zorn's lemma to complete the proof.

(ii) Let $x_{g_1}, \dots, x_{g_n} \in h(M)$ be a minimal spanning set for M and $N = Rx_{g_2} + \dots + Rx_{g_n}$ with $x_{g_1} \notin N$. Then the set Ω of all graded submodules N

of M such that $K \subseteq N$ and $x_{g_1} \notin N$ is not empty since $K \in \Omega$. We therefore apply Zorn's lemma to the partially ordered set $(\Omega, \subseteq)$. $\square$

Proposition 2.8 *Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . Then $Gr(N)$ is the intersection of the minimal graded prime of N .*

Proof. Let K be a graded prime submodule of M with $N \subseteq K$. Then K/N is a graded prime submodule of M/N by [2, Lemma 2.8]. By Lemma 2.7, there exists some graded prime submodule S of M such that S/N is a minimal graded prime submodule M/N and $S/N \subseteq K/N$; hence $N \subseteq S \subseteq K$. If T is another graded prime submodule of M with $N \subseteq T \subseteq S$, then in M/N , we must have $S/N = T/N$ so that $T = S$ in M . Thus, S is a minimal graded prime of N .

Assume that Δ is the set of all graded prime submodules of M containing N and Λ is the set of all minimal graded prime submodules of N . then the above consideration thus leads us $Gr(N) = \bigcap_{K \in \Delta} K = \bigcap_{S \in \Lambda} S$. $\square$

Let R be a G -graded ring. Graded dimension R is defined by supremum of all numbers n for which there exists a chain of graded prime ideals $P_0 \subset P_1 \subset \dots \subset P_n$ in R , and it is denoted by $Gdim R$.

Theorem 2.9 *If R is a G -graded integral domain with $Gdim R = 1$ and M is a graded R -module with 0 graded prime, then either $(0 : M) = 0$ or every proper graded submodule of M is graded prime.*

Proof. Suppose $(0; M) \neq 0$. Since 0 is graded prime, we must have $P = (0 : M)$ is a graded prime ideal of R by [2, theorem 2.11]. It follows that P is a graded maximal ideal of R since $Gdim R = 1$. If N is a proper graded submodule of M , then $P \subseteq (N : M) \neq R$ and so $P = (N : M)$. Now the assertion follows from Proposition 2.4. $\square$

Let R be a G -graded ring and let M a graded R -module. Given an element a of R , we say that a divides M if $aM = M$, and we say that a is nilpotent on M if $a^n M = 0$ for some n . A non-zero graded module M is graded secondary if every homogeneous element of R either divides M or is nilpotent on M , in which case $Gr(ann M) = P$ is a graded prime ideal of R , and M is said to be graded P -secondary (see [9, Proposition 2.2]). For example, as discussed in Sharp ([9, p. 215]), if $R = k[x]$ is a polynomial ring in one variable with the natural $\mathbb{Z}$ -graded ring and $M = k[x, 1/x]$, then M is graded secondary but is

not secondary. So the graded secondary and secondary modules are different concepts.

Theorem 2.10 *If R is a G -graded integral domain with $G\dim R = 1$ and M is a graded torsion R -module with 0 graded prime, then M is graded secondary.*

Proof. Since M is a graded torsion R -module with 0 graded prime, we must have $(0 : M) = P \neq 0$ is a graded prime ideal. Let $a \in h(R)$. If $a \in P$, then a is nilpotent on M . Now we show that a divides M . Suppose not. Then by Theorem 2.9, $(aM : M) = P$, so $a \in P$ which is a contradiction. $\square$

Let M be a graded module over a G -graded ring R . Note that some graded modules M have no graded prime submodules and we call such modules M G -primeless. There are many examples as we show next. A graded R -module M is called G -divisible (resp. graded torsion module) if $aM = M$ for all $0 \neq a \in h(R)$ (resp. $T(M) = M$).

Theorem 2.11 *Let R be a G -graded integral domain. Then every graded torsion G -divisible R -module is G -primeless.*

Proof. Assume that M is a G -divisible graded torsion R -module. Suppose that M is not G -primeless; we show that $M = N$. Let N be a graded prime submodule of M with $(N : M) = P$. First, we show that $P = 0$. Otherwise, there is an element $r = \sum_{i=1}^n r_{g_i} \in P$ with $r_{g_i} \neq 0$, so for each i , we must have $r_{g_i} \in P$ since P is graded ideal; hence by assumption, we get $M = r_{g_i}M \subseteq N$ which is a contradiction. Therefore, by [2, Theorem 2.11], M/N is a graded torsion-free R -module. Next, we show that M/N graded torsion-free gives $M \subseteq N$. Suppose that $M \not\subseteq N$. Then there is a homogeneous element m_g of M such that $m_g \notin N$, so $r(m_g + N) \neq 0$ for all non-zero element r of $h(R)$; hence $rm_g \notin N$ for all $0 \neq r \in h(R)$. On the other hand, as M is graded torsion, there is a non-zero homogeneous element s of R such that $sm_g = 0 \in N$. Thus, $M = N$ which is a contradiction. $\square$

Theorem 2.12 *Let M be a G -primeless module over a G -graded integral domain R . Then M is graded torsion, $PM = M$ for every graded maximal ideal P of R and M is not a graded finitely generated R -module.*

Proof. If $T(M) \neq M$, then $T(M)$ is a graded prime submodule of M by Proposition 2.6. Thus M is graded torsion. By Proposition 2.4, we must have $PM = M$ for every graded maximal ideal P of R . Finally, M is not graded

finitely generated by Lemma 2.7 (since every graded maximal submodule is graded prime by [4, Proposition 2.6]). $\square$

Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . We say that N has a reduced graded primary decomposition if there are finitely many graded primary submodules N_i such that $N = \bigcap_{i=1}^n N_i$ with $\bigcap_{i \neq j} N_i \not\subseteq N_j$ for each i and the graded prime ideals $Gr(N_i : M)$ are distinct. In this case, we say that $Gr(N_i)$ is an isolated prime submodule of N if it is minimal the set $\{Gr(N_1), \dots, Gr(N_i)\}$.

Theorem 2.13 *Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . If N has a reduced graded primary decomposition $N = \bigcap_{i=1}^n N_i$ such that all the graded prime ideals associated with N are isolated, then $(N : M) = \sum_{i=1}^n (N_i : M)$ is a reduced graded primary decomposition of the graded ideal $(N : M)$ in R .*

Proof. Suppose not. As the graded ideals $Gr(N_i : M)$ are distinct, there exists integer i such that $\bigcap_{i \neq j} (N_j : M) \subseteq (N_i : M)$, so by [8, Proposition 1.2], $\bigcap_{i \neq j} Gr(N_j : M) \subseteq Gr(N_i : M)$; hence $Gr(N_j : M) \subseteq (N_i : M)$ for some $j \neq i$ by [8, Lemma 1.8 and proposition 1.4]. However, this final inclusion contradicts the assumption that $Gr(N_i : M)$ is an isolated graded prime ideal of N . $\square$

Proposition 2.14 *Let R be a G -graded ring, M a graded R -module, N a graded R -submodule of M . If N has a reduced graded primary decomposition $N = \bigcap_{i=1}^n N_i$ such that all the graded prime ideals associated with N are isolated, then the following hold:*

- (i) N is graded primary if and only if $(N : M)$ is graded primary.
- (ii) N is graded prime if and only if $(N : M)$ is graded prime.

Proof. (i) If N is graded primary, then $(N : M)$ is graded primary by [4, Proposition 2.7]. conversely, assume that $(N : M)$ is a graded primary ideal of R . By Theorem 2.13, $(N : M) = \sum_{i=1}^n (N_i : M)$ is a reduced graded primary decomposition of $(N : M)$, so we must have $n = 1$ and so $N = N_1$ is primary.

(ii) The necessity follows from [4, Proposition 2.7]. If $(N : M)$ is graded prime, then $N = N_1$ is a primary submodule, so $(N : M) = Gr(N : M)$ by [8, proposition 1.2]. Suppose $ax \in N$ with $a \in h(R)$ and $x \in h(M) - N$. Then N graded primary gives $a \in Gr(N : M) = (N : M)$; hence N is a prime submodule of M . $\square$

Lemma 2.15 *Let R be a G -graded ring. Then R is a graded field if and only if every proper graded ideal is graded prime.*

Proof. If R is a graded field, then the result follows from Lemma 2.3. Conversely, Let c be a non-zero element of $h(R)$. By assumption, Rc^2 is a graded prime ideal of R , so $c^2 \in Rc^2$ implies $c \in Rc^2$; hence there exists $a = a_{g_1} + \dots + a_{g_k} \in R$ with $a_{g_i} \neq 0$ such that $c = c^2a$. It follows that $c = c^2a_{g_j}$ for some j . As the graded ideal $\{0\}$ is prime, we must have R is a G -graded integral domain, so we can cancel c to obtain $ca_{g_j} = 1$, showing that R is G -graded field. $\square$

Theorem 2.16 *Let R be a G -graded ring and $0 \neq M$ a graded R -module, then R is a graded field if and only if every proper graded submodule of M is a graded prime submodule of M and $T(M) \neq M$.*

Proof. Let R be a graded field. First, suppose that $T(M) = M$ and $m = m_{g_1} + \dots + m_{g_n} \in M$ with $m_{g_i} \neq 0$. There are non-zero homogeneous elements r_{g_i} of R such that $r_{g_i}m_{g_i} = 0$ for $i = 1, \dots, n$, so for each i , $m_{g_i} = 0$ since R is graded field; hence $M = 0$ which is a contradiction. Thus $T(M) \neq M$. Next, assume that N is a proper graded submodule of M and let $ax \in N$ where $a \in h(R)$ and $x \in h(M)$. We can assume that $a \neq 0$. Then $x \in N$ since R is graded field. Conversely, let $c \in M - T(M)$, so there is a homogeneous component $c_g \neq 0$ of c such that $c_g \notin T(M)$; hence $\text{Ann}(c_g) = 0$. In view of the assumption, it is easy to see that every proper graded submodule of the graded R -module Rc_g is a graded prime submodule of Rc_g and $Rc_g \cong R$ as R -modules. Therefore, every proper graded ideal of R is a graded prime ideal; hence R is graded field by Lemma 2.15. $\square$

Corollary 2.17 *If R is a G -graded integral domain with $\text{Gdim}R = 1$ and M is a graded R -module with $(0 : M) \neq 0$ and 0 graded prime, then $R/(0 : M)$ is graded field.*

Proof. This follows from Theorem 2.9 and Lemma 2.3. $\square$

A G -graded ring R is said to be G -graded quasi-local if it has a unique graded maximal ideal.

Lemma 2.18 *If R is a G -graded ring in which every proper graded ideal is graded primary, then R is a graded quasi-local.*

Proof. Let P and I be graded R -ideals with P graded prime. We show that either $P \subseteq I$ or $I \subseteq P$. Assume that $I \not\subseteq P$ and choose $r \in I - P$. Then there exists a homogeneous component $r_g \neq 0$ of r such that $r_g \notin P$. Let $a = a_{h_1} + \dots, a_{h_k} \in P$ with $a_{h_i} \neq 0$. Then for each i , $r_g a_{h_i} \in IP$ and IP is graded primary ideal, so we must have either $a_{h_i} \in IP$ or $r_g^m \in IP$ for some m . But if $r_g^m \in IP \subseteq P$, then $r_g \in P$ which is a contradiction. Thus we must have for each i , $a_{h_i} \in IP$; hence $a \in IP \subseteq I$. Thus, $P \subseteq I$. Let M be any graded maximal ideal (so it is a graded prime ideal) of R . Then it is comparable to any proper graded ideal. Now if Q is any maximal ideal, then M and Q are comparable, so $M = Q$, as needed. $\square$

Theorem 2.19 *Let R be a G -graded ring and $0 \neq M$ a graded R -module. If every proper graded submodule of M is a graded primary submodule of M and $T(M) \neq M$, then R is graded quasi-local.*

Proof. As $T(M) \neq M$, there is a homogeneous element $0 \neq b_g \in h(M)$ such that $b_g \notin T(M)$ and $\text{Ann}(b_g) = 0$; hence $Rb_g \cong R$ as R -modules. Clearly, every proper graded submodule of the graded R -module $Rb_g \cong R$ is a graded primary submodule; hence R is graded quasi-local by lemma 2.18. $\square$

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