ON A PROBLEM OF EL-Metwally *et al.*

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Abstract

This paper is an attempt to provide a solution to the open problem raised by EL-Metwally *et al.*, (2001): that is; to determine the boundedness character of the positive solution of equation (1) when the population growth rate is an exponential function of the immigration rate. We make use of a dimensionless quantity, ϵ , say, which has the characteristics of a threshold parameter.

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Introduction:

El-Metwally *et al.*, in [1] studied the global stability, the boundedness nature, and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_n e^{-x_n}. \quad (1)$$

This equation may be viewed as describing the dynamic of a population, in which case α represents the immigration rate, while β is the population growth rate. This paper is an attempt to provide a solution to the open problem raised

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therein, that is; to determine the boundedness character of the positive solution of equation (1) when $\beta = e^\alpha$.

Equation (1) is interesting in its own right, and thus worthy of study for the purpose of mathematical elucidation. Our analysis follows basically the approach adopted [1], with appropriate modifications.

The Model Equation

As pointed out by Metwally *et al.*, in [1], if $\beta = e^\alpha$, this implies that the population growth rate is an exponential function of the immigration rate. Hence, an equation of the form

$$x_{n+1} = \alpha + x_{n-1}e^{\alpha-x_n}, \quad n = 0, 1, 2, \dots \quad (2)$$

may model the dynamics of insects population such as locusts invading a lowly population area.

Difference equations arise in modelling various life situation because animals and many plants are counted in discrete units. For instance, the well-known logistic map arises in a variety of systems such as chemical reactions electrical circuits, hydrodynamical flows, and population dynamics. Another example is the modified Henon map

$$\begin{aligned} H_{n+1} &= a + y_n - bH_n^2 \\ y_{n+1} &= cH_n \end{aligned} \quad (3)$$

For the classical Henon map, $a = 1$, $b = 1$ and $c = 1.3$.

In general, stability is the necessary and sufficient condition for convergence of a properly posed initial value problem (and a finite difference approximation that satisfies the consistency condition).

Let I be an interval of real numbers, and $f : I \times I \rightarrow I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 1, 2, \dots$$

where the initial condition $x_0, x_1 \in I$.

Since we are dealing with population dynamics, we do not consider x_{-1} , as in [1]. Also, $x_n \geq 0$, negative values of x_n are biologically meaningless [3] and no one cares what these values do when n is negative [2]. Also, negative time steps are also irrelevant.

The following comparison Theorem (which we state without proof) will be useful in the sequel.

Theorem 1: Assume that $\alpha \in \mathbb{R}^+$. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be two sequences of real numbers, such that:

$$x_0 \leq z_0 \quad \text{and} \quad x_1 \leq z_1;$$

and

$$\begin{cases} x_{n+1} \leq \alpha x_{n-1} + e^\alpha \\ z_{n+1} = \alpha z_{n-1} + e^\alpha \end{cases} \quad (4)$$

Then $x_n \leq z_n$ for $n \geq 0$.

Now the main result.

Theorem 2: Suppose $e^\alpha < \infty$, $0 < \alpha < 1$, and let $\{x_n\}_{n=0}^\infty$ be a positive solution of equation (2) for $n \geq 1$, then $x_{n+1} \leq \alpha + x_{n-1}$

Proof: Consider $z_{n+1} = \alpha + \epsilon z_{n-1}$, $\epsilon > 0$ is a dimensionless scaling parameter, $n = 1, 2, \dots$ (5) with $z_0 = x_0$ and $z_1 = x_1$. Then $x_n \leq z_n$ for all $n \geq 0$.

Now,

$$\lim_{n \rightarrow \infty} z_n = \lim_{\epsilon \rightarrow 1} \frac{\alpha}{1 - \epsilon} \quad (5)$$

ϵ here is a threshold parameter. If $\epsilon = 1$, there is a blow-up solution. Thus, if $\epsilon < 1$, then, every positive solution of equation (2) is bounded. For $\epsilon > 1$, positive solution of equation (2) will be negative since $\lim_{n \rightarrow \infty} z_n < 0$, a contradiction. Thus, ϵ cannot exceed unity. By neglecting terms of order greater than or equal to 2, we have that

$$\lim_{\epsilon \rightarrow 1} \frac{\alpha}{1 - \epsilon} = \lim_{\epsilon \rightarrow 1} \alpha \left(1 + \epsilon - \frac{\epsilon^2}{2} - \dots \right) \leq 2\alpha \quad (6)$$

This solution is locally stable but globally unstable as a small perturbation will disrupt the system. This crude approximation tells us that the sequence in (2) is bounded, if we can find a scaling parameter $\epsilon > 0$, such that $z_{n+1} = \alpha + \epsilon z_{n-1}$. A possible guess for ϵ to be an optimum control is when $0 < \epsilon < 1$. Thus, every positive solution of (5) is bounded if $\epsilon < 1$.

References

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