ON A PROBLEM OF EL-Metwally et al.

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Abstract
This paper is an attempt to provide a solution to the open problem raised by EL-Metwally et al., (2001): that is; to determine the boundedness character of the positive solution of equation (1) when the population growth rate is an exponential function of the immigration rate. We make use of a dimensionless quantity, $\epsilon$, say, which has the characteristics of a threshold parameter.

Keywords: Boundedness, difference equation, population dynamics

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Introduction:
El-Metwally et al., in [1] studied the global stability, the boundedness nature, and the periodic character of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}. \quad (1)$$

This equation may be viewed as describing the dynamic of a population, in which case $\alpha$ represents the immigration rate, while $\beta$ is the population growth rate. This paper is an attempt to provide a solution to the open problem raised

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therein, that is; to determine the boundedness character of the positive solution of equation (1) when $\beta = e^\alpha$.

Equation (1) is interesting in its own right, and thus worthy of study for the purpose of mathematical elucidation. Our analysis follows basically the approach adopted [1], with appropriate modifications.

The Model Equation

As pointed out by Metwally et al., in [1], if $\beta = e^\alpha$, this implies that the population growth rate is an exponential function of the immigration rate. Hence, an equation of the form

$$x_{n+1} = \alpha + x_{n-1}e^{\alpha-x_n}, \quad n = 0, 1, 2, \ldots$$

may model the dynamics of insects population such as locusts invading a lowly population area.

Difference equations arise in modelling various life situation because animals and many plants are counted in discrete units. For instance, the well-known logistic map arises in a variety of systems such as chemical reactions electrical circuits, hydrodynamical flows, and population dynamics. Another example is the modified Henon map

$$H_{n+1} = a + y_n - bH_n^2$$
$$y_{n+1} = cH_n$$

(3)

For the classical Henon map, $a = 1$, $b = 1$ and $c = 1.3$.

In general, stability is the necessary and sufficient condition for convergence of a properly posed initial value problem (and a finite difference approximation that satisfies the consistency condition).

Let $I$ be an interval of real numbers, and $f : I \times I \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 1, 2, \ldots$$

where the initial condition $x_0, x_1 \in I$.

Since we are dealing with population dynamics, we do not consider $x_{-1}$, as in [1]. Also, $x_n \geq 0$, negative values of $x_n$ are biologically meaningless [3] and no one cares what these values do when $n$ is negative [2]. Also, negative time steps are also irrelevant.

The following comparison Theorem (which we state without proof) will be useful in the sequel.

**Theorem 1:** Assume that $\alpha \in \mathbb{R}^+$. Let $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be two sequences of real numbers, such that:

$$x_0 \leq z_0 \quad \text{and} \quad x_1 \leq z_1;$$
and
\[
\begin{cases}
  x_{n+1} \leq \alpha x_{n-1} + e^\alpha \\
  z_{n+1} = \alpha z_{n-1} + e^\alpha
\end{cases}
\]  
(4)

Then \(x_n \leq z_n\) for \(n \geq 0\).

Now the main result.

**Theorem 2:** Suppose \(e^\alpha < \infty\), \(0 < \alpha < 1\), and let \(\{x_n\}_{n=0}^\infty\) be a positive solution of equation (2) for \(n \geq 1\), then \(x_{n+1} \leq \alpha + x_{n-1}\)

**Proof:** Consider \(z_{n+1} = \alpha + \epsilon z_{n-1}, \epsilon > 0\) is a dimensionless scaling parameter, \(n = 1, 2, \ldots (5)\) with \(z_0 = x_0\) and \(z_1 = x_1\). Then \(x_n \leq z_n\) for all \(n \geq 0\).

Now,
\[
\lim_{n \to 0} z_n = \lim_{\epsilon \to 1} \frac{\alpha}{1 - \epsilon}
\]  
(5)

\(\epsilon\) here is a threshold parameter. If \(\epsilon = 1\), there is a blow-up solution. Thus, if \(\epsilon < 1\), then, every positive solution of equation (2) is bounded. For \(\epsilon > 1\), positive solution of equation (2) will be negative since \(\lim_{n \to \infty} z_n < 0\), a contradiction. Thus, \(\epsilon\) cannot exceed unity. By neglecting terms of order greater than or equal to 2, we have that
\[
\lim_{\epsilon \to 1} \frac{\alpha}{1 - \epsilon} = \lim_{\epsilon \to 1} \alpha \left( 1 + \epsilon - \frac{\epsilon^2}{2} - \cdots \right) \leq 2\alpha
\]  
(6)

This solution is locally stable but globally unstable as a small perturbation will disrupt the system. This crude approximation tells us that the sequence in (2) is bounded, if we can find a scaling parameter \(\epsilon > 0\), such that \(z_{n+1} = \alpha + \epsilon z_{n-1}\). A possible guess for \(\epsilon\) to be an optimum control is when \(0 < \epsilon < 1\). Thus, every positive solution of (5) is bounded if \(\epsilon < 1\).

**References**


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