Interpolatory quadrature formulas
for the Fejér kernel

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Abstract

Interpolatory quadrature formulas with uniformly distributed nodes over the unit circle in the complex plane are studied for the approximation of integrals over the unit circle with the Fejér kernel as weight function. Computable expression for the coefficients of this quadrature and numerical experiments comparing with Szegö quadrature are given.

Mathematics Subject Classification: 41, 65D

Keywords: Fejér kernel, interpolatory quadrature, uniformly distributed nodes, Szegö quadrature

1 Introduction

This paper fits into the general framework of the approximation of integrals of the form

\[ I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t)dt \]

where \( w(t) \) is a weight function on \( t \in [-\pi, \pi] \), that is, \( w(t) \geq 0 \) in \( t \in [-\pi, \pi] \), and \( 0 < \int_{-\pi}^{\pi} w(t)dt < \infty \). For the approximation we use Szegö and interpolatory quadrature formulas. These last of the form

\[ I_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n}), \; z_{j,n} \in T, \; z_{\ell,n} \neq z_{j,n} \text{ if } \ell \neq j, \]

where \( T = \{ z \in C : |z| = 1 \} \) denotes the unit circle in the complex plane \( C \).

Let \( p \) and \( q \) be integers with \( p \leq q \). We denote by \( \Lambda_{p,q} \) the linear space of functions of the form

\[ \sum_{j=p}^{q} \alpha_j z^j, \; \alpha_j \in C. \]
We call this functions Laurent polynomials.

Let an integer \( n \geq 1 \) be given and \( \{z_{j,n}\}_{j=1}^{n} \) be a set of distinct points on \( T \). Consider non-negative integers \( p_n \) and \( q_n \) such that \( p_n + q_n = n - 1 \). Then there is a unique Laurent polynomial \( R_{p_n,q_n} \in \Lambda_{-p_n,q_n} \) that interpolates a given function \( f(z) \) at \( \{z_{j,n}\}_{j=1}^{n} \), that is, \( R_{p_n,q_n}(z_{j,n}) = f(z_{j,n}) \), \( j = 1, 2, \ldots, n \). It holds that

\[
R_{p_n,q_n}(z) = \sum_{j=1}^{n} L_{j,n}(z)f(z_{j,n})
\]

where \( L_{j,n}(z) \in \Lambda_{-p_n,q_n} \) are called the fundamental Lagrange Laurent polynomials and they are given by

\[
L_{j,n}(z) = \frac{V_n(z)}{(z - z_{j,n})V'_n(z_{j,n})}
\]

where

\[
V_n(z) = \frac{N_n(z)}{z^{p_n}}, \quad N_n(z) = \prod_{j=1}^{n} z - z_{j,n}.
\]

Thus

\[
\int_{-\pi}^{\pi} R_{p_n,q_n}(e^{it})w(t)dt = \sum_{j=1}^{n} \left( \int_{-\pi}^{\pi} L_{j,n}(e^{it})w(t)dt \right)f(z_{j,n})
\]

is a quadrature formula of the form (2).

**Definition 1.1** A quadrature formula \( I_n(f) \) of the form (2) for the approximation of the integral (1) is of interpolatory type in \( \Lambda_{-p_n,q_n} \) if the coefficients \( \{c_{j,n}\}_{j=1}^{n} \) of the quadrature formula are given by

\[
c_{j,n} = \int_{-\pi}^{\pi} L_{j,n}(e^{it})w(t)dt, \quad j = 1, 2, \ldots, n
\]

where \( L_{j,n}(z) \) is given by (3).

Thus, interpolatory quadrature formulas are those precisely constructed from interpolation.

Observe that when \( \{c_{j,n}\}_{j=1}^{n} \) are given by (4) then \( I_n(f) = I(f), \quad f \in \Lambda_{-p_n,q_n} \) since \( R_{p_n,q_n}(z) = f(z) \). Conversely, if \( I_n(f) = I(f), \quad f \in \Lambda_{-p_n,q_n} \) then taking \( f(z) = L_{j,n}(z) \) in \( I_n(f) \) gives \( c_{j,n} = \int_{-\pi}^{\pi} L_{j,n}(e^{it})w(t)dt \). Hence, the following Theorem holds.

**Theorem 1.2** A quadrature formula \( I_n(f) \) of the form (2) for the approximation of integrals (1) is of interpolatory type in \( \Lambda_{-p_n,q_n} \) if and only if \( I_n(f) = I(f), \quad f \in \Lambda_{-p_n,q_n} \).
In a similar way as for integrals on the real line one can construct for integrals \((1)\) on the unit circle, quadrature formulas with a large domain of validity via orthogonalization. Indeed, let \(\Lambda\) be the linear space of all Laurent polynomials. Consider the inner product on \(\Lambda\times\Lambda\) given by

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{it})g(e^{it})w(t)dt.
\]  

(5)

Let \(\{\varrho_n\}_0^\infty\) be the sequence of monic polynomials obtained by orthogonalization of \(\{z^n\}_0^\infty\) with respect to the inner product \((5)\). The sequence \(\{\varrho_n\}_0^\infty\) is called the sequence of Szegö polynomials with respect to the weight function \(w\). It is well known, see, e.g. \([6, Theorem 11.4.1]\) that \(\varrho_n\) has its zeros in the region \(|z| < 1\). Thus, they are not adequate as nodes for a general purpose quadrature formula to approximate integrals over the unit circle.

**Theorem 1.3** (See [1, 2]) Let \(\{\varrho_n\}_0^\infty\) be the sequence of monic orthogonal polynomials with respect to the weight function \(w\). Let \(\{\kappa_n\}_0^\infty\) be a sequence of complex numbers satisfying \(|\kappa_n| = 1, n \geq 1\). Let the para-orthogonal polynomials be defined by \(B_n(z, \kappa_n) = \varrho_n(z) + \kappa_n \varrho_n^*(z)\) where \(\varrho_n^*(z) = z^n \overline{\varrho}_n(1/z)\), and \(\overline{\varrho}_n\) denotes the operation of conjugate the coefficients to the polynomials \(\varrho_n\). Then \(B_n(z, \kappa_n)\) has \(n\) distinct zeros \(\zeta_m^{(n)}(\kappa_n), m = 1, 2, \ldots, n, n \geq 1,\) located on \(T\). Let

\[
\lambda_m^{(n)}(\kappa_n) = \int_{-\pi}^{\pi} \frac{B_n(e^{it}, \kappa_n)}{(e^{it} - \zeta_m^{(n)}(\kappa_n)) B'_n(\zeta_m^{(n)}(\kappa_n), \kappa_n)} w(t)dt, 1 \leq m \leq n, n \geq 1.
\]

(6)

Then

\[
I(f) = \int_{-\pi}^{\pi} f(e^{it})w(t)dt = I_n(f) = \sum_{m=1}^{n} \lambda_m^{(n)}(\kappa_n)f(\zeta_m^{(n)}(\kappa_n))
\]

(7)

for all \(f \in \Lambda_{+(n-1),n-1} = \text{span}\{ \frac{1}{z^{n-1}}, \ldots, \frac{1}{z}, 1, z, \ldots, z^{n-1}, z^n - \frac{\varrho_n(0) + \kappa_n}{\varrho_n(0) + \kappa_n} \} \).

It holds that \(\lambda_m^{(n)}(\kappa_n) > 0, 1 \leq m \leq n, n \geq 1\), and there cannot exist an \(n\)-point quadrature formula \(G(f) = \sum_{m=1}^{n} \lambda_m f(\alpha_m)\), \(\alpha_m \in T\) which correctly integrates every function \(f \in \Lambda_{-(n-1),n}\) or every function \(f \in \Lambda_{-n,n-1}\).

The quadrature formula \(I_n\) given by the two last terms of \((7)\) is called the \(n\)-point Szegö quadrature formula with respect to the weight function \(w\) and the parameter \(\kappa_n\) or briefly the \(n\)-point Szegö quadrature formula. From here on, we consider that the weight function \(w(t)\) is given by \(w(t) = K_N(t)\) where

\[
K_N(t) = \sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right)e^{ijt}, N = 0, 1, 2, \ldots, -\pi \leq t \leq \pi
\]

(8)

is the well known Fejér kernel.
Theorem 1.4 (See [5]) The Szegő quadrature formula with respect to the Fejé kernel $K_N(t)$, $N = 0, 1, 2, \ldots$, is characterised as follows.

The monic orthogonal polynomials $g_n(z)$ of degree $n$, $n = 0, 1, 2, \ldots$, is the reciprocal polynomial of the numerator of the classical Padé approximant of type $(n, N)$ to the function $\left(\frac{z - 1}{z^{N+1} - 1}\right)^2$.

The coefficients $\lambda_{m}^{(n)}(\kappa_n)$, $1 \leq m \leq n$, $n \geq 1$, of the $n$-point Szegő quadrature formula admit the representation

$$\lambda_{m}^{(n)}(\kappa_n) = -\frac{2\pi}{(N + 1)B'_{n}(\zeta_{m}^{(n)}(\kappa_n), \kappa_n)} \sum_{j=0}^{N} (j + 1) \sum_{k=0}^{N-j} \frac{b_{N-j-k}}{\left(\zeta_{m}^{(n)}(\kappa_n)\right)^{k+1}}$$

where

$$B_{n}(z, \kappa_n) = b_{n}z^{n} + b_{n-1}z^{n-1} + \cdots + b_{1}z + b_{0}$$

is the para-orthogonal polynomial of degree $n$ and $b_{j} = 0$ if $j \geq n + 1$.

The paper is organised as follows. In section 2 we obtain computable expressions for the coefficients $c_{j,n}$ of the quadrature formula of interpolatory type with uniformly distributed nodes over the unit circle. In section 3 we compare through several numerical examples the quadrature formula of interpolatory type with uniformly distributed nodes over the unit circle and the Szegő quadrature formula.

2 Calculation of the coefficients

In this section we obtain a computable expression for the coefficients $c_{j,n}$ of the quadrature formula of interpolatory type with uniformly distributed nodes $z_{j,n}$ over the unit circle for the approximation of integrals (1) with the Fejé kernel (8) as the weight function.

When the nodes are uniformly distributed over the unit circle then the coefficients $c_{j,n}$ can be computed by the Fast Fourier Transform algorithm, see [4]. This can be deduced from the expression

$$c_{j,n} = \frac{1}{n} \sum_{\ell=-p_{n}}^{q_{n}} m_{\ell} \frac{\omega^{(1-j)\ell}}{z_{1,n}^{\ell}}, \ 1 \leq j \leq n, \ n \geq 1,$$

where the moments $m_{\ell}$ are given by $m_{\ell} = \int_{-\pi}^{\pi} e^{i\ell t} w(t)dt$, $\ell = 0, 1, \pm 2, \ldots$ and $\omega = e^{2\pi i/n}$. As is well known, this algorithm uses a number of arithmetic operations of order $O(n \log(n))$. The expression of the coefficients $c_{j,n}$ that we obtain in this section need for their computation a less order $O(n)$ in the number of operations.
**Theorem 2.1** The coefficients \( c_{j,n} \) of the \( n \)-point quadrature formula of interpolatory type in \( \Lambda_{p_n,q_n} \) with uniformly distributed nodes \( z_{j,n} = e^{i(t_n+2\pi(j-1))/n} \), \( 1 \leq j \leq n, -\pi \leq t_n \leq \pi \) over the unit circle are given by

\[
c_{j,n} = \frac{1}{n} \sum_{\ell=-p_n}^{q_n} \frac{m_\ell}{z_{j,n}^\ell}, \quad 1 \leq j \leq n, \quad n \geq 1.
\]

**Proof.** It follows from (9) taking into account that the nodes satisfy \( z_{j,n} = \omega^{j-1}z_{1,n}, \ j = 2, 3, \ldots, n, \) and \( \omega = e^{2\pi i/n} \).

The moments \( m_\ell, \ \ell = 0, \pm 1, \pm 2, \ldots \), for the Fejér kernel are given by

\[
m_\ell = \begin{cases} 2\pi \left(1 - \frac{|\ell|}{N+1}\right), & \text{if } |\ell| \leq N, \\ 0, & \text{if } |\ell| > N. \end{cases}
\]

Let \( a_n = \min\{p_n, N\} \) and \( b_n = \min\{q_n, N\} \). Then

\[
c_{j,n} = \frac{2\pi}{n} \left( s_{j,n} + \overline{s}_{j,n} \right)
\]

where

\[
s_{j,n} = \sum_{\ell=-a_n}^{-1} \left(1 - \frac{|\ell|}{N+1}\right) \frac{1}{z_{j,n}^\ell}
\]

and

\[
\overline{s}_{j,n} = \sum_{\ell=0}^{b_n} \left(1 - \frac{\ell}{N+1}\right) \frac{1}{z_{j,n}^\ell}
\]

Both sums \( s_{j,n} \) and \( \overline{s}_{j,n} \) are arithmetic-geometric sums if \( z_{j,n} \neq 1 \), else they are arithmetic sums. Thus, if \( z_{j,n} \neq 1 \) then

\[
s_{j,n} = \frac{z_{j,n} \left(N(1 - z_{j,n}) + z_{j,n}^{a_n} - z_{j,n} + (N+1-a_n)z_{j,n}^{a_n}(z_{j,n} - 1)\right)}{(N+1)(z_{j,n} - 1)^2}
\]

and

\[
\overline{s}_{j,n} = \frac{(N + 1)(z_{j,n} - 1)z_{j,n}^{b_n+1} + (1 - z_{j,n})z_{j,n} + (N + 1 - b_n)(1 - z_{j,n})}{(N + 1)(z_{j,n} - 1)^2z_{j,n}^{b_n}}
\]

Hence the coefficients \( c_{j,n} \) admit the expression
\[ c_{j,n} = \frac{2\pi z_{j,n}^{a_n+1}((N - a_n)(z_{j,n} - 1) + z_{j,n}) - 2z_{j,n} - z_{j,n}^{-b_n}((N - b_n)(z_{j,n} - 1) - 1)}{(N + 1)(z_{j,n} - 1)^2} \]  

(10)

If \( z_{j,n} = 1 \) then

\[ s_{j,n} = \frac{a_n(2N + 1 - a_n)}{2(N + 1)} \]

and

\[ \overline{s}_{j,n} = \frac{(b_n + 1)(2(N + 1) - b_n)}{2(N + 1)}. \]

Hence

\[ c_{j,n} = \frac{\pi a_n(2N + 1 - a_n) + (b_n + 1)(2(N + 1) - b_n)}{N + 1}. \]

(11)

Thus one can establish the following result.

**Theorem 2.2** The quadrature formula

\[ I_n(f) = \sum_{j=1}^{n} c_{j,n} f(z_{j,n}) \]

of interpolatory type in \( \Lambda_{-p_n,q_n} \), \( p_n \) and \( q_n \) non-negative integers such that \( p_n + q_n = n - 1, n \geq 1 \) with uniformly distributed nodes

\[ z_{j,n} = e^{i(t_n + 2\pi(j-1))/n}, -\pi \leq t_n \leq \pi \]

over the unit circle for the approximation of integrals

\[ I(f) = \int_{-\pi}^{\pi} f(e^{it})K_N(t)dt \]

with the Fejér kernel \( K_N(t) \) as the weight function has coefficients \( c_{j,n} \) given by (10) for nodes \( z_{j,n} \neq 1 \) and by (11) if \( z_{j,n} = 1 \).

3 Test examples

In the next Examples we compare the estimations provided by the interpolatory quadrature formula with uniformly distributed nodes on \( T \) and the Szegö quadrature formula for integrals with the Fejér kernel as the weight function.

In the following Examples the function \( f(z) \) to be integrated satisfies the property \( f(\overline{z}) = \overline{f(z)} \). For these functions the corresponding integral (1) for
deduces that the coefficient \( \lambda_{para-orthogonal\ polynomials} \) is real. Thus, one should choose the parameters of the interpolatory and the Szegö formula taking into account this property.

The interpolatory quadrature formula with uniformly distributed nodes on \( T \) is characterised in Theorem 2.2. This quadrature depends on the parameters \( p_n \) and \( t_n \). (Recall that \( q_n = n-1-p_n \).) The value of \( p_n \) is appropriately selected in each of the following Examples. Nevertheless we take \( t_n = 0 \) in all of them. Next we explain this selection. If \( t_n = 0 \) then \( z_{1,n} = 1 \) and the corresponding coefficient \( c_{1,n} \) given by (11) is real. From \( z_{j,n} = \omega^{j-1}z_{1,n}, j = 2, 3, \ldots, n \), where \( \omega = e^{2\pi i/n} \), and from (10) it follows that \( \overline{c}_{n-j,n} = c_{j+2,n}, j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \). Hence, the interpolatory quadrature formula with uniformly distributed nodes on \( T \) for \( t_n = 0 \) takes the form

\[
\begin{align*}
c_{1,n} + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_{j+2,n} f(z_{j+2,n}) + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} \overline{c}_{j+2,n} f(\overline{z}_{j+2,n}).
\end{align*}
\]

The notation \( \sum' \) means that the last term (the one corresponding to \( j = \lfloor \frac{n}{2} \rfloor - 1 \)) should be divided by two if \( n \) is even. Furthermore, if the function \( f \) satisfies \( f(\overline{z}) = \overline{f(z)} \) then the interpolatory quadrature admits the expression

\[
\begin{align*}
c_{1,n} + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} c_{j+2,n} f(z_{j+2,n}) + c_{j+2,n} \overline{f(z_{j+2,n})}.
\end{align*}
\]

Hence, we achieve that the estimations provided by the interpolatory quadrature formula will be real numbers like the value of the integral.

Similarly, since for the Fejér kernel the moments \( m_t \) are real then the orthogonal polynomials \( g_n(z) \) have real coefficients. If \( \kappa_n \) is real then also the para-orthogonal polynomials \( B_n(z, \kappa_n) = g_n(z) + \kappa_n e_n^*(z) \) have real coefficients. Thus, if \( \zeta \) is a zero of \( B_n(z, \kappa_n) \) then \( \overline{\zeta} \) is also a zero. Then from (6) one deduces that the coefficient \( \lambda^{(n)}_{m}(\kappa_n) \) corresponding to the zero \( \zeta \) and the one corresponding to \( \overline{\zeta} \) are equal. Thus, the estimations provided by the Szegö quadrature formula are also real number for functions satisfying \( f(\overline{z}) = \overline{f(z)} \). Thus we take the value \( \kappa_n = 1, n \geq 1 \) for the Szegö quadrature formula in the following Examples.

All Tables list the absolute value of the difference between the integral and the quadrature formula.

**Example 3.1** Consider the computation of the integral

\[
I(f) = \int_{-\pi}^{\pi} f(e^{it})K_N(t)dt
\]

where \( f(z) = e^z \) and \( N = 10 \). It holds \( I(f) = 4697191\pi/950400 \). Since \( f(z) \) is analytic in the region \( |z| \leq 1 \) then we take \( p_n = 0 \) and hence \( q_n = n-1, n \geq 1 \) in the interpolatory quadrature formula as recommended in [3]. See Table 1.
Table 1: $f(z) = e^z$, $N = 10$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n=4$</th>
<th>$n=8$</th>
<th>$n=12$</th>
<th>$n=16$</th>
<th>$n=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Szegö</td>
<td>$0.62 \times 10^{-1}$</td>
<td>$0.30 \times 10^{-4}$</td>
<td>$0.36 \times 10^{-8}$</td>
<td>$0.42 \times 10^{-13}$</td>
<td>$0.10 \times 10^{-17}$</td>
</tr>
<tr>
<td>Inter.</td>
<td>$0.11$</td>
<td>$0.12 \times 10^{-3}$</td>
<td>$0.14 \times 10^{-7}$</td>
<td>$0.31 \times 10^{-12}$</td>
<td>$0.10 \times 10^{-17}$</td>
</tr>
</tbody>
</table>

Table 2: $f(z) = \log(3 - z)$, $N = 3$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n=4$</th>
<th>$n=8$</th>
<th>$n=12$</th>
<th>$n=16$</th>
<th>$n=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Szegö</td>
<td>$0.11 \times 10^{-1}$</td>
<td>$0.69 \times 10^{-4}$</td>
<td>$0.56 \times 10^{-6}$</td>
<td>$0.51 \times 10^{-8}$</td>
<td>$0.50 \times 10^{-10}$</td>
</tr>
<tr>
<td>Inter.</td>
<td>$0.24 \times 10^{-1}$</td>
<td>$0.15 \times 10^{-3}$</td>
<td>$0.12 \times 10^{-5}$</td>
<td>$0.11 \times 10^{-7}$</td>
<td>$0.11 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

**Example 3.2** Let

$$I(f) = \int_{-\pi}^{\pi} f(e^{it})K_N(t)dt$$

where $f(z) = \log(3 - z)$ (principal branch) and $N = 3$. One has $I(f) = 2\pi(\log(3) - \frac{91}{324})$. Again, since $f(z)$ is analytic in the region $|z| \leq 1$ then we take $p_n = 0$ and hence $q_n = n - 1$, $n \geq 1$ in the interpolatory quadrature formula. See Table 2.

**Example 3.3** Consider the integral

$$I(f) = \int_{-\pi}^{\pi} f(e^{it})K_N(t)dt$$

where $f(z) = \frac{1}{z - 1/4}$ and $N = 8$. One finds $I(f) = \frac{18609\pi}{8192}$. Since $f(z)$ is a rational function we take the parameters $p_n$ and $q_n$ according to the indications given in [3] for this type of functions. The values for this particular rational function are $p_n = n - 1$ and hence $q_n = 0$, $n \geq 1$. See Table 3.

In the three Tables we observe that the interpolatory quadrature formula with uniformly distributed nodes on the unit circle for the Fejér kernel is

Table 3: $f(z) = \frac{1}{z - \frac{1}{4}}$, $N = 8$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n=4$</th>
<th>$n=8$</th>
<th>$n=12$</th>
<th>$n=16$</th>
<th>$n=20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Szegö</td>
<td>$0.31 \times 10^{-1}$</td>
<td>$0.11 \times 10^{-3}$</td>
<td>$0.39 \times 10^{-6}$</td>
<td>$0.14 \times 10^{-8}$</td>
<td>$0.59 \times 10^{-11}$</td>
</tr>
<tr>
<td>Inter.</td>
<td>$0.58 \times 10^{-1}$</td>
<td>$0.44 \times 10^{-3}$</td>
<td>$0.19 \times 10^{-5}$</td>
<td>$0.75 \times 10^{-8}$</td>
<td>$0.29 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
competitive with the Szegő quadrature. Furthermore, take into account that it has the advantage that its "total computational effort" is much more less than the one needed for the Szegő quadrature.

References


Received: February 9, 2006