Malmquist Productivity Index with Interval and Fuzzy Data, An Application of Data Envelopment Analysis

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Abstract

Data envelopment analysis (DEA) measures the relative efficiency of decision making units (DMUs) with multiple inputs and multiple outputs. DEA-based Malmquist productivity index measures the productivity change over time. In this paper, unlike the original DEA models, it has been assumed that the levels of inputs and outputs are not known exactly. The true data be only known as in forms of bounded or fuzzy data. Hence, we provide a computational method for extension to the DEA-based Malmquist productivity index for all DMUs with interval and fuzzy data. A numerical examples for illustration purpose are presented.

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1 Introduction

Using linear programming techniques, data envelopment analysis (DEA) (Charnes et al., 1978) provides a suitable way to estimate a multiple inputs / multiple outputs empirical efficient function as described by Farrell (1957).

Fare et al. (1992, 1994a) developed a DEA-based Malmquist productivity index which measures the productivity change over time. The Malmquist index was first suggested by Malmquist (1953) as a quantity index for use in

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the analysis of consumption of inputs, Fare et al. combined ideas on the measurement of efficiency from Farrell and the measurement of productivity from Caves et al.(1982) to construct a Malmquist productivity index has proven itself to be a good tool for measuring the productivity change of DMUs.[10]

The original DEA-based Malmquist index assumes that inputs and outputs are measured by exact values on a ratio scale. However, this assumption may not be true, in the sense that some inputs and outputs may be only known as in forms of bounded or fuzzy data.

In this paper, we suggest a Malmquist productivity index for DMUs productivity evaluation with interval or fuzzy data.

2 Background

2.1 Data envelopment analysis

Consider n decision making units DMU_j , $j=1, \dots, n$, which each DMU consumes inputs levels x_{ij} , $i=1, \dots, m$, to produce outputs levels y_{rj} , $r=1, \dots, s$. Let $J = \{1, \dots, n\}$ and suppose that $X_j = (x_{1j}, \dots, x_{mj})^T$ and $Y_j = (y_{1j}, \dots, y_{sj})^T$ are the vectors of inputs and outputs values respectively, for DMU_j , in which it has been assumed that $X_j \geq 0$, $X_j \neq 0$ and $Y_j \geq 0$, $Y_j \neq 0$. The relative efficiency score of the DMU_o , $o \in \{1, \dots, n\}$, is obtained from the following model which is called input-oriented CCR envelopment model.

$$\begin{aligned}
 & \text{Min } \theta \\
 & \text{s.t. } \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io}, \quad i=1, \dots, m \\
 & \quad \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}, \quad r=1, \dots, s \\
 & \quad \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{1}$$

It can be proven that $0 < \theta^* \leq 1$ and DMU_o is efficient(technical) in the CCR model if and only if $\theta^* = 1$. Otherwise, the DMU_o is inefficient.

2.2 Interval data in DEA

Unlike the original DEA model, we assume here that the levels of inputs and outputs are not know exactly, the true input and output data know to lie within bounded intervals, i.e. $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$.

The worse case of DMU_o is define as $x_{io} = x_{io}^U$ ($i=1, \dots, m$), and $y_{ro} = y_{ro}^L$ ($r=1, \dots, s$). Similarly, the best case of DMU_o is define as $x_{io} = x_{io}^L$ ($i=1, \dots, m$), and $y_{ro} = y_{ro}^U$ ($r=1, \dots, s$).

In this case, the efficiency can be an interval. The upper limit of interval efficiency is obtained from the optimistic viewpoint and the lower limit is obtained from the pessimistic viewpoint. The following model provides such an upper bound for $DMU_o[8,9]$:

$$\begin{aligned}
 h_o^U = & \text{Min } \theta \\
 \text{s.t. } & \sum_{j \neq o} \lambda_j x_{ij}^U + \lambda_o x_{io}^L \leq \theta x_{io}^L, \quad i=1, \dots, m \\
 & \sum_{j \neq o} \lambda_j y_{rj}^L + \lambda_o y_{ro}^U \geq y_{ro}^U, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{2}$$

We denote by h_o^U the efficiency score attained by DMU_o in (2) and name the best case of efficiency score.

The model below provides a lower bound of the efficiency score for DMU_o :

$$\begin{aligned}
 h_o^L = & \text{Min } \theta \\
 \text{s.t. } & \sum_{j \neq o} \lambda_j x_{ij}^L + \lambda_o x_{io}^U \leq \theta x_{io}^U, \quad i=1, \dots, m \\
 & \sum_{j \neq o} \lambda_j y_{rj}^U + \lambda_o y_{ro}^L \geq y_{ro}^L, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{3}$$

We show that $h_o^L \leq h_o^U$. Suppose that S_o^U and S_o^L are the feasible regions in relation to (2) and (3), respectively. suppose that $(\lambda_1, \dots, \lambda_n, \theta)$ be an optimal solution for (2), since at least a constraint as $\sum_{j \neq o} \lambda_j x_{ij}^U + \lambda_o x_{io}^L \leq \theta x_{io}^L$ is binding

$$\sum_{j \neq o} \lambda_j x_{ij}^U = (\theta - \lambda_o) x_{io}^L$$

and since λ_j , x_{ij}^U and x_{io}^L are not negative, then $\theta - \lambda_o \geq 0$ and $\lambda_o \leq \theta \leq 1$. Therefore

$$\sum_{j \neq o} \lambda_j x_{ij}^L \leq \sum_{j \neq o} \lambda_j x_{ij}^U \leq (\theta - \lambda_o) x_{io}^L \leq (\theta - \lambda_o) x_{io}^U$$

$$\sum_{j \neq o} \lambda_j y_{rj}^U \geq \sum_{j \neq o} \lambda_j y_{rj}^L \geq (1 - \lambda_o) y_{ro}^U \geq (1 - \lambda_o) y_{ro}^L$$

then $(\lambda_1, \dots, \lambda_n, \theta)$ is a solution for (3), therefore $h_o^L \leq h_o^U$.

The efficiency h_o^L attained by DMU_o in (3) serves as lower bound of its possible efficiency scores and name the worst case of efficiency score. Models

(2) and (3) provide each DMU with a bounded interval $[h_o^L, h_o^U]$ in which its possible efficiency scores lie from the worst to the best case.

On the basis of the above efficiency score intervals, DMUs can be classified in three subsets as follows:

$$\begin{aligned} E^{++} &= \{j \in J \mid h_j^L = 1\}, \\ E^+ &= \{j \in J \mid h_j^L < 1 \& h_j^U = 1\}, \\ E^- &= \{j \in J \mid h_j^U < 1\} \end{aligned} \quad (4)$$

2.3 DEA Malmquist productivity index

Fare et al. (1992) construct the DEA-based Malmquist productivity index as the geometric mean of two Malmquist productivity indexes of Caves et al. (1982), which are defined by a distance function $D(\cdot)$. Caves et al. assume $D^k(k) = 1$, that is, they assume the technology for fire k is efficient, and their distance function does not reveal inefficiency. By allowing for inefficiency and modeling the technology frontier as piecewise linear, Fare et al. decompose their Malmquist productivity index into two components, one measuring the change in efficiency and the other measuring the change in the frontier technology. The frontier technology determined by the efficient frontier is estimated using DEA for a set of DMUs. However, the frontier technology for a particular DMU under evaluation is only represented by a section of the DEA frontier or a facet.

Malmquist productivity index calculation requires two single period and two mixed period measures. The two single period measures can be obtained by using the CCR DEA model (Charnes et al., 1978)

$$\begin{aligned} D_o^t(t) = \quad & \text{Min } \theta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^t \leq \theta x_{io}^t, \quad i=1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj}^t \geq y_{ro}^t, \quad r=1, \dots, s \\ & \lambda_j \geq 0, \quad j=1, \dots, n \end{aligned} \quad (5)$$

where x_{io}^t is the i -th input and y_{ro}^t is the r -th output for DMU_o in time t . The efficiency ($D_o^t(t)$) determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $t+1$ instead of t for the above method, we get $D_o^{t+1}(t+1)$, the technical efficiency score for DMU_o in time period $t+1$.

The first of the mixed period measures, which is defined as $D_o^t(t+1)$ for each DMU_o , is computed as the optimal value to the following linear programming

problem:

$$\begin{aligned}
 D_o^t(t+1) = \quad & \text{Min} \quad \theta \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^t \leq \theta x_{io}^{t+1}, \quad i=1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj}^t \geq y_{ro}^{t+1}, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{6}$$

Similarly, the other mixed period measure, $D_o^{t+1}(t)$, which is needed in the computation of the Malmquist productivity index, is the optimal value to the following linear problem:

$$\begin{aligned}
 D_o^{t+1}(t) = \quad & \text{Min} \quad \theta \\
 \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij}^{t+1} \leq \theta x_{io}^t, \quad i=1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj}^{t+1} \geq y_{ro}^t, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{7}$$

Fare et al.'s input-oriented Malmquist productivity index, which measures the productivity change of a particular DMU_o , $o \in J = \{1, \dots, n\}$, in time $t+1$ and t is given as

$$M_o = \left[\frac{D_o^t(t+1)}{D_o^t(t)} \cdot \frac{D_o^{t+1}(t+1)}{D_o^{t+1}(t)} \right]^{1/2} \tag{8}$$

It can be seen that the above measure actually is the geometric mean of two Caves et al.'s Malmquist productivity indexes. Thus, following Caves et al.'s suit, Fare et al. define that $M_o > 1$ indicates productivity gain; $M_o < 1$ indicates productivity loss; and $M_o = 1$ means no change in productivity from time t to $t+1$. [10]

3 Malmquist productivity index with interval data

In this section, we define the Malmquist productivity index for DMUs with interval data. Hence, suppose that the true input and output lie within bounded intervals, as follows

$$\begin{aligned}
 x_{ij}^t & \in [x_{ij}^{t,L}, x_{ij}^{t,U}], & y_{rj}^t & \in [y_{rj}^{t,L}, y_{rj}^{t,U}] \\
 x_{ij}^{t+1} & \in [x_{ij}^{t+1,L}, x_{ij}^{t+1,U}], & y_{rj}^{t+1} & \in [y_{rj}^{t+1,L}, y_{rj}^{t+1,U}]
 \end{aligned}$$

with upper and lower bounds of the intervals given are as constants and assumed to be strictly positive.

When the level of inputs and outputs for DMU_o are not know exactly, we expect the value of Malmquist productivity index for DMU_o and its components in (8) is not know exactly. Since the true input and output data know to lie within intervals, in this cases, the Malmquist productivity index (and its component) lie within interval, which is called interval of productivity index.

Interval productivity evaluation requires two single period and two mixed period measures. The two single period measures, $D_o^t(t)$ and $D_o^{t+1}(t+1)$, can be lie within bounded interval measures as interval efficiency mention in subsection (2.2), as $[D_o^{t,L}(t), D_o^{t,U}(t)]$ and $[D_o^{t+1,L}(t+1), D_o^{t+1,U}(t+1)]$. which the lower and upper bounds are obtain from the optimistic and pessimistic viewpoint, as follows

$$\begin{aligned}
 D_o^{t,L}(t) = \quad & \text{Min} \quad \theta \\
 \text{s.t.} \quad & \sum_{j \neq o, j=1}^n \lambda_j x_{ij}^{t,L} + \lambda_o x_{io}^{t,U} \leq \theta x_{io}^{t,U}, \quad i=1, \dots, m \\
 & \sum_{j \neq o, j=1}^n \lambda_j y_{rj}^{t,U} + \lambda_o y_{ro}^{t,L} \geq y_{ro}^{t,L}, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{9}$$

and

$$\begin{aligned}
 D_o^{t,U}(t) = \quad & \text{Min} \quad \theta \\
 \text{s.t.} \quad & \sum_{j \neq o, j=1}^n \lambda_j x_{ij}^{t,U} + \lambda_o x_{io}^{t,L} \leq \theta x_{io}^{t,L}, \quad i=1, \dots, m \\
 & \sum_{j \neq o, j=1}^n \lambda_j y_{rj}^{t,L} + \lambda_o y_{ro}^{t,U} \geq y_{ro}^{t,U}, \quad r=1, \dots, s \\
 & \lambda_j \geq 0, \quad j=1, \dots, n
 \end{aligned} \tag{10}$$

and using $t+1$ instead of t for the above models, we get $D_o^{t+1,L}(t+1)$ and $D_o^{t+1,U}(t+1)$, respectively, for lower and upper bounds of the interval.

But, two mixed period measures, $D_o^{t+1}(t)$ and $D_o^t(t+1)$, are also can be lie within bounded interval measures, as $[D_o^{t,L}(t+1), D_o^{t,U}(t+1)]$ and $[D_o^{t+1,L}(t), D_o^{t+1,U}(t)]$, which the lower and upper bounds are obtain from the

optimistic and pessimistic viewpoint, as follows:

$$\begin{aligned}
 D_o^{t+1,L}(t) = & \text{Min } \theta \\
 \text{s.t. } & \sum_{j=1}^n \lambda_j x_{ij}^{t+1,L} \leq \theta x_{io}^{t,U}, \quad i=1,\dots,m \\
 & \sum_{j=1}^n \lambda_j y_{rj}^{t+1,U} \geq y_{ro}^{t,L}, \quad r=1,\dots,s \\
 & \lambda_j \geq 0, \quad j=1,\dots,n
 \end{aligned} \tag{11}$$

and

$$\begin{aligned}
 D_o^{t+1,U}(t) = & \text{Min } \theta \\
 \text{s.t. } & \sum_{j=1}^n \lambda_j x_{ij}^{t+1,U} \leq \theta x_{io}^{t,L}, \quad i=1,\dots,m \\
 & \sum_{j=1}^n \lambda_j y_{rj}^{t+1,L} \geq y_{ro}^{t,U}, \quad r=1,\dots,s \\
 & \lambda_j \geq 0, \quad j=1,\dots,n
 \end{aligned} \tag{12}$$

and using $t + 1$ instead of t and vice versa, for the above models, we get $D_o^{t,L}(t + 1)$ and $D_o^{t,U}(t + 1)$, respectively, for lower and upper bounds of the interval measure for DMU_o in time period $t + 1$ with respect to time period t .

Theorem 1. Suppose that the true inputs and outputs are given as

$$x_{ij}^{t+k} \in [x_{ij}^{t+k,L}, x_{ij}^{t+k,U}]$$

$$y_{ij}^{t+k} \in [y_{ij}^{t+k,L}, y_{ij}^{t+k,U}]$$

which

$$i = 1, \dots, m, \quad r = 1, \dots, s, \quad j = 1, \dots, n, \quad k = 0, 1.$$

then the true value of the Malmquist productivity index components will be

$$D_o^{t+k}(t + h) \in [D_o^{t+k,L}(t + h), D_o^{t+k,U}(t + h)], \quad k, h = 0, 1.$$

Proof. As mentioned in section (2.2), we have

$$D_o^{t+k}(t + k) \in [D_o^{t+k,L}(t + k), D_o^{t+k,U}(t + k)], \quad k = 0, 1$$

Now, we show that

$$D_o^{t+1}(t) \in [D_o^{t+1,L}(t), D_o^{t+1,U}(t)]$$

Suppose that $S_o^{t+1}(t)$, $S_o^{t+1,L}(t)$ and $S_o^{t+1,U}(t)$ are the feasible regions in relation to $D_o^{t+1}(t)$, $D_o^{t+1,L}(t)$ and $D_o^{t+1,U}(t)$, respectively. It suffices to show that

$$S_o^{t+1,U}(t) \subseteq S_o^{t+1}(t) \subseteq S_o^{t+1,L}(t)$$

in this case, we have

$$D_o^{t+1,L}(t) \leq D_o^{t+1}(t) \leq D_o^{t+1,U}(t).$$

Suppose that $(\lambda_1, \dots, \lambda_n, \theta)$ be a solution for (7), therefore we have

$$\sum_{j=1}^n \lambda_j x_{ij}^{t+1} \leq \theta x_{io}^t, \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \lambda_j y_{rj}^{t+1} \geq y_{ro}^t, \quad r = 1, \dots, s$$

Since $\theta > 0$, we have

$$\sum_{j=1}^n \lambda_j x_{ij}^{t+1,L} \leq \sum_{j=1}^n \lambda_j x_{ij}^{t+1} \leq \theta x_{io}^t \leq \theta x_{io}^{t,U}, \quad i = 1, \dots, m$$

and

$$\sum_{j=1}^n \lambda_j y_{rj}^{t+1,U} \geq \sum_{j=1}^n \lambda_j y_{rj}^{t+1} \geq y_{ro}^t \geq y_{ro}^{t,L}, \quad r = 1, \dots, s$$

therefore, $S_o^{t+1}(t) \subseteq S_o^{t+1,L}(t)$.

Now, we show that $S_o^{t+1,U}(t) \subseteq S_o^{t+1}(t)$. Suppose that $(\lambda_1, \dots, \lambda_n, \theta)$ be a feasible solution for (12), therefore we have;

$$\sum_{j=1}^n \lambda_j x_{ij}^{t+1,U} \leq \theta x_{io}^{t,L}, \quad i = 1, \dots, m$$

$$\sum_{j=1}^n \lambda_j y_{rj}^{t+1,L} \geq y_{ro}^{t,U}, \quad r = 1, \dots, s$$

hence;

$$\sum_{j=1}^n \lambda_j x_{ij}^{t+1} \leq \sum_{j=1}^n \lambda_j x_{ij}^{t+1,U} \leq \theta x_{io}^{t,L} \leq \theta x_{io}^t, \quad i = 1, \dots, m$$

and

$$\sum_{j=1}^n \lambda_j y_{rj}^{t+1} \geq \sum_{j=1}^n \lambda_j y_{rj}^{t+1,L} \geq y_{ro}^{t,U} \geq y_{ro}^t, \quad r = 1, \dots, s$$

therefore, $S_o^{t+1,U}(t) \subseteq S_o^{t+1}(t)$.

Similarly, we can show that

$$D_o^{t+1}(t) \in [D_o^{t+1,L}(t), D_o^{t+1,U}(t)]. \square$$

Considering the above mentioned theorem and (8), we can deduce that;

$$M_o \geq \left[\frac{D_o^{t,L}(t+1)}{D_o^{t,U}(t)} \cdot \frac{D_o^{t+1,L}(t+1)}{D_o^{t+1,U}(t)} \right]^{1/2}$$

and

$$M_o \leq \left[\frac{D_o^{t,U}(t+1)}{D_o^{t,L}(t)} \cdot \frac{D_o^{t+1,U}(t+1)}{D_o^{t+1,L}(t)} \right]^{1/2}$$

in which M_o is Malmquist productivity index with true value of of input and output data for specific DMU_o .

Therefore, the lower and upper bounded productivity for DMU_o denote as:

$$M_o^L = \left[\frac{D_o^{t,L}(t+1)}{D_o^{t,U}(t)} \cdot \frac{D_o^{t+1,L}(t+1)}{D_o^{t+1,U}(t)} \right]^{1/2} \quad (13)$$

$$M_o^U = \left[\frac{D_o^{t,U}(t+1)}{D_o^{t,L}(t)} \cdot \frac{D_o^{t+1,U}(t+1)}{D_o^{t+1,L}(t)} \right]^{1/2} \quad (14)$$

we have

$$M_o \in [M_o^L, M_o^U]$$

that provide for each specific DMU_o , with a bounded interval productivity $[M_o^L, M_o^U]$, in which its possible value of Malmquist productivity index lie from the worst to the best case. It should be considered that $M_o \neq M_o^L$ and $M_o \neq M_o^U$.

On the basis of the original Malmquist productivity index, we can deduce that, $M_o^L > 1$ indicates a progress; $M_o^U < 1$ indicates a regress in productivity from time t to $t+1$ for DMU_o , otherwise, no things can be said.

4 Malmquist productivity index with fuzzy data

4.1 Preliminary definitions [12]

A classical (crisp) set is normally defined as a collection of elements or objects $x \in X$. Each single element can either belong to or not belong to a set A , $A \subseteq X$. but for a fuzzy set, a element of X can be belong to the set A , by a degree of membership.

Definition 1. If X is a collection of objects denoted generically by x , then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$. The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. If $\sup_x \mu_{\tilde{A}}(x) = 1$, the fuzzy set $\tilde{A}$ is called normal. A nonempty fuzzy set $\tilde{A}$ can always be normalized by dividing $\mu_{\tilde{A}}(x)$ by $\sup_x \mu_{\tilde{A}}(x) = 1$. As a matter of convenience, we will generally assume that fuzzy sets are normalized and elements with a zero degree of membership are normally not listed.

A more general and even more useful notion is that of an α – level set.

Definition 2. The (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α ($\alpha > 0$) is called the α – level set:

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

Theorem 2. If $\tilde{A}$ is a fuzzy set and $0 < \alpha_1 < \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$.

Convexity also plays a role in fuzzy set theory.

Definition 3. A fuzzy set $\tilde{A}$ is convex if

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \quad x_1, x_2 \in X, \lambda \in [0, 1]$$

Alternatively, a fuzzy set is convex if all α – level sets are convex.

Definition 4. A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line $\mathbb{R}$ such that

1. It exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{M}}(x_0) = 1$.
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Dubois and Prade [1979] suggest a special type of representation for fuzzy numbers of the following type: They call L and R , which map $\mathbb{R}^+ \rightarrow [0, 1]$, shape functions if $L(0) = R(0) = 1$, and are decreasing.

Definition 5. A fuzzy number $\tilde{M}$ is of LR-type if there exist reference function L (for left), R (for right), and scalars $\alpha > 0, \beta > 0$ with

$$\mu_{\tilde{M}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & x \leq m \\ R(\frac{x-m}{\beta}), & x \geq m \end{cases}$$

m , called the mean value of $\tilde{M}$, is a real number, and α and β are called the left and right spreads, respectively. Symbolically, $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Example 1. Suppose

$$L(x) = R(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

then $\tilde{M} = (m, \alpha, \beta)_{LR}$ is called a triangular fuzzy number. (see Fig. 1.)

Fig. 1. The triangular fuzzy number $\tilde{M} = (m, \alpha, \beta)_{LR}$

4.2 Malmquist productivity index

In this section, we are in purpose to evaluate Malmquist productivity index for DMUs with fuzzy data. Therefore, assume that fuzzy numbers, $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are the i -th input and the r -th output for DMU_j , with continuous membership function.

The α -level set of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ is the interval $[x_{ij,\alpha}^L, x_{ij,\alpha}^U]$ and $[y_{rj,\alpha}^L, y_{rj,\alpha}^U]$, where $x_{ij,\alpha}^L$ and $y_{rj,\alpha}^L$ are lower and $x_{ij,\alpha}^U$ and $y_{rj,\alpha}^U$ are upper points of any α -level sets, respectively, and we suppose that are finite. Then for a fixed $\alpha \in (0, 1]$, we can suppose that the true i -th input and the r -th output of DMU_j are follow as:

$$x_{ij} \in [x_{ij,\alpha}^L, x_{ij,\alpha}^U]$$

and

$$y_{rj} \in [y_{rj,\alpha}^L, y_{rj,\alpha}^U]$$

that is, DMU_j ($j=1, \dots, n$) consumes a value x_{ij} of the i -th ($i=1, \dots, m$) input with at least degree membership α , to product a value y_{rj} of the r -th ($r=1, \dots, s$) output with at least degree membership α . Therefore to evaluation Malmquist productivity index for DMUs with fuzzy data, we can use some way from the method applied for DMUs with interval data.

Let below definition represent Malmquist productivity index for any α -level set of DMU's data.

Let fuzzy numbers, $\tilde{x}_{ij}^k$ and $\tilde{y}_{rj}^k$ are the $i - th$ input and the $r - th$ output for DMU_j in k period ($k=t, t+1$), with continuous membership function. Then for a fixed $\alpha \in (0, 1]$, the true inputs and outputs in k period as follows

$$x_{ij}^k \in [x_{ij,\alpha}^{k,L}, x_{ij,\alpha}^{k,U}]$$

and

$$y_{rj}^k \in [y_{rj,\alpha}^{k,L}, y_{rj,\alpha}^{k,U}]$$

where

$$i = 1, \dots, m, \quad r = 1, \dots, s, \quad j = 1, \dots, n, \quad k = t, t + 1$$

Definition 7. α - interval productivity for DMU_o , is defined by $[M_{o,\alpha}^L, M_{o,\alpha}^U]$, where $M_{o,\alpha}^L$ and $M_{o,\alpha}^U$ can obtained by equations (13-14) with interval data attained of the $\alpha - levels$.

Theorem 3. If $\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 < \alpha_2$, then

$$[M_{o,\alpha_2}^L, M_{o,\alpha_2}^U] \subseteq [M_{o,\alpha_1}^L, M_{o,\alpha_1}^U]$$

Proof . From theorem 2, we have

$$x_{ij,\alpha_2}^{t+k,L} \in [x_{ij,\alpha_1}^{t+k,L}, x_{ij,\alpha_1}^{t+k,U}], \quad k = 0, 1$$

$$y_{rj,\alpha_2}^{t+k,L} \in [y_{rj,\alpha_1}^{t+k,L}, y_{rj,\alpha_1}^{t+k,U}], \quad k = 0, 1$$

and

$$x_{ij,\alpha_2}^{t+k,U} \in [x_{ij,\alpha_1}^{t+k,L}, x_{ij,\alpha_1}^{t+k,U}], \quad k = 0, 1$$

$$y_{rj,\alpha_2}^{t+k,U} \in [y_{rj,\alpha_1}^{t+k,L}, y_{rj,\alpha_1}^{t+k,U}], \quad k = 0, 1$$

Therefore, from theorem 1, we can result that

$$M_{o,\alpha_2}^L, M_{o,\alpha_2}^U \in [M_{o,\alpha_1}^L, M_{o,\alpha_1}^U]. \square$$

From above theorem, we deduce that if a DMU has a progress in its performance with $\alpha_2 - level$ and $\alpha_1 > \alpha_2$, then it has also progress with $\alpha_1 - level$. For example, if α indicates amount of trust of decision maker to select inputs and outputs levels for DMUs then we can see that above result is agree with each other. Similarly if a DMU has a regress in its performance with $\alpha_2 - level$, then it has also regress with $\alpha_1 - level$ ($\alpha_1 > \alpha_2$).

The following, we present a numerical example to illustrate the result.

4.3 Malmquist productivity index of commercial bank branches

In the current section, we employ the Malmquist index for evaluating 24 bank branches for two period times by using the methodology developed above. This section describes the data and results. Table 1 and 2 show inputs and outputs of these banks with symmetric triangular fuzzy data (m, β, β) , denoted by (m, β) , at time periods t and $t + 1$, respectively. Each bank contains three inputs and five outputs. For determining Malmquist productivity index, we use (13-14) and definition 7, then we measure the productivity change between period t and $t + 1$.

Table 3 reports the results obtained from α – levels for $\alpha = 0.1, 0.2, \dots, 1$ and $\alpha = 0$ that means interval data $[m - \beta, m + \beta]$ and relative to Malmquist productivity index obtain by (13-14). As shown in Table 3, DMU_{12} indicate a progress and DMUs 2,3,7,8,9,10,11,13,14,15,18,19,22,23 and 24 do not indicate any progress for each level. Other DMUs indicate a progress for some level that are agreement with theorem 3.

Table 3 also reports the results obtained from α – levels. As shown, DMU_{11} indicate a regress and DMUs 1,4,5,6,12,16,17,19,,20 and 21 do not indicate any regress for each level. Other DMUs indicate a progress for some level that are agreement with theorem 3.

	Input 1 (m, β)	Input 2 (m, β)	Input 3 (m, β)	Output 1 (m, β)	Output 2 (m, β)	Output 3 (m, β)	Output 4 (m, β)	Output 5 (m, β)
1	(3070.68, 34.37)	(36.16, 0.69)	(122623, 713)	(3056411, 53257.5)	(202125, 2295.7)	(1729763, 223516)	(61896.9, 50376.2)	(1030.79, 57)
2	(1089.38, 24.53)	(20.87, 0.36)	(12979, 1331)	(159036.5, 13590.5)	(10240.5, 934.5)	(317989, 5475)	(4743.84, 2840.73)	(542.28, 317.18)
3	(1471.73, 4.55)	(19.27, 0.73)	(10750, 15)	(383498.5, 48006.5)	(112348.5, 48886.5)	(319677.5, 2557.5)	(20432.94, 13762.21)	(132.80, 35.01)
4	(1832.77, 129.71)	(13.53, 0.05)	(2109.5, 0.5)	(293016, 27371)	(21073, 6118)	(254867, 3263)	(2179.95, 961.73)	(13.53, 2.41)
5	(4489.14, 31.89)	(26.39, 0.31)	(40488.5, 1401.5)	(818393, 16303)	(205462.5, 29535.5)	(1791451.5, 26443.5)	(29093.7, 22076.06)	(603.26, 69.56)
6	(542.19, 10.22)	(21.73, 0.15)	(18998.5, 63.5)	(1609642, 99461)	(225462.5, 29535.5)	(517322, 2398)	(5551.64, 4513.4)	(58.19, 30.8)
7	(4225.23, 69.55)	(14.63, 0.17)	(5249, 1264)	(381571, 10249)	(9300.5, 4509.5)	(182155.5, 1341.5)	(2120.51, 194.85)	(78.28, 55.85)
8	(7644.81, 1082.46)	(18.43, 0.06)	(42398.5, 6284.5)	(1209657, 37628)	(104228.5, 2133.5)	(671674.5, 31239.5)	(27387.03, 19046.43)	(852.81, 444.24)
9	(305.89, 3.79)	(28.46, 0.05)	(34544.5, 1725.5)	(244006, 26415)	(313700.5, 48065.5)	(447592.5, 8498.5)	(2842.93, 279.77)	(331.5, 133.95)
10	(901.73, 42.77)	(17.11, 1.31)	(264, 0)	(266914.5, 29258.5)	(57168.5, 8165.5)	(1048095, 20526)	(17029.59, 11683.39)	(628.01, 173.19)
11	(2456.88, 40.4)	(26.27, 0.76)	(9549, 186)	(425312, 9048)	(709016, 57520)	(1144691, 25518)	(9087.34, 2433.19)	(1805.66, 298.04)
12	(5720.37, 5.8)	(12.14, 0.35)	(1006, 15)	(600784, 6515)	(8181.5, 558.5)	(2137153.5, 64259.5)	(64335.39, 6527.05)	(13.57, 8.77)
13	(1302.96, 286.51)	(21.77, 1.03)	(289966.5, 9114.5)	(440736.5, 16878.5)	(477393, 51903)	(2317901, 190320)	(78235.43, 32575.89)	(2383.17, 434.29)
14	(10139.28, 37.51)	(15.07, 0.46)	(12608.5, 50.5)	(814840.5, 11832.5)	(8282.5, 1072.5)	(2745254.5, 12160.5)	(30453.73, 5789.16)	(5.96, 1.56)
15	(2706.34, 34.93)	(20.16, 0.98)	(997, 15)	(673772.5, 1010.5)	(47341.5, 6070.5)	(2659016.5, 268741.5)	(30820.80, 27532.41)	(597.12, 86.02)
16	(3484.62, 43.66)	(22.14, 1.03)	(36806.5, 16081.5)	(454592.5, 30.5)	(52664, 19068)	(327083.5, 39378.5)	(4823.74, 172.93)	(99.84, 24.14)
17	(4826.61, 6.48)	(20.29, 1.52)	(20537, 79)	(565225, 14648)	(24857, 3980)	(450233, 3006)	(4914.04, 2831.04)	(100.13, 2.66)
18	(2535.04, 11.73)	(22.27, 0.95)	(18978, 60)	(355386, 1975)	(27408, 886)	(182012, 319)	(919.28, 357.34)	(412.92, 59.55)
19	(2114.91, 19.79)	(38.19, 1.22)	(52817.5, 4476.5)	(342975, 3978)	(216919.5, 18419.5)	(595863.5, 6814.5)	(8801.51, 2314.22)	(141.83, 25.27)
20	(3726.91, 20.15)	(27.47, 0.95)	(25643, 3719)	(358964, 489)	(105528, 4949)	(348602, 3417)	(1997.52, 346.66)	(58.58, 10.56)
21	(2985.11, 93.26)	(26.49, 0.73)	(41104, 0)	(774749.5, 2939.5)	(479584, 4140)	(1788331.5, 39002.5)	(1774.67, 611.83)	(1061.06, 136.06)
22	(492.03, 0.03)	(13.39, 0.24)	(19451, 250)	(320154, 18668)	(19653, 15886)	(28402, 1169)	(1017.55, 541.79)	(58.35, 38.76)
23	(3266.93, 50.57)	(927.98, 0.57)	(18337, 762)	(620923, 13849)	(799447, 23177)	(871844.5, 172427.5)	(52013.98, 37563.43)	(1375.39, 192.56)
24	(775.66, 4.79)	(29.24, 0.28)	(31568.5, 76.5)	(137327, 977)	(13637, 611)	(297046, 623)	(3329.65, 505.8)	(95.55, 16.46)

Table1.The symmetric triangular fuzzy data for 24 bank branches at time period t

	Input 1 (m, β)	Input 2 (m, β)	Input 3 (m, β)	Output 1 (m, β)	Output 2 (m, β)	Output 3 (m, β)	Output 4 (m, β)	Output 5 (m, β)
1	(3076.08, 184.33)	(35.49, 0.02)	(117360, 2270)	(3123693, 7570)	(402137, 105056)	(2472732, 80311)	(6028.94, 2096.2)	(1461.55, 536.71)
2	(1162.36, 2.42)	(20.73, 0.5)	(14325.5, 15.5)	(184708, 2701)	(23567, 3030)	(325115.5, 1109.5)	(2895.74, 38.95)	(289.82, 54.85)
3	(1427.87, 35.81)	(19.48, 0.3)	(10354, 17)	(491902, 13077)	(128127.5, 10264.5)	(343029, 1016)	(7212.37, 539.37)	(188.39, 113.99)
4	(3308.89, 387.88)	(13.77, 0.04)	(2100, 8)	(472349.5, 43983.5)	(29838.5, 4537.5)	(324215.5, 3537.5)	(2705.88, 701.15)	(68.5, 56.78)
5	(4567.35, 80.35)	(26.39, 0.42)	(40690.5, 88.5)	(878364.5, 47394.5)	(504721, 21288)	(1786134.5, 24562.5)	(6408.04, 2585.41)	(165.49, 17.38)
6	(569.72, 29.56)	(21.34, 0.34)	(18757, 60)	(2045804, 128284)	(190330.5, 13997.5)	(521353.5, 11492.5)	(19024.12, 17781.61)	(70.51, 13.42)
7	(4233.19, 225.98)	(14.5, 0.04)	(11092.5, 4554.5)	(366456.5, 30784.5)	(13361.5, 7508.5)	(188902.5, 1566.5)	(2437.77, 458.28)	(7.56, 0.46)
8	(8820.11, 6.41)	(19.11, 0.11)	(137599, 22824)	(1339185.5, 21196.5)	(139101.5, 31451.5)	(709574.5, 38696.5)	(18339.18, 8859.75)	(451.2, 159)
9	(319.12, 4.79)	(28.07, 0.42)	(42618, 83)	(185151.5, 5022.5)	(392412, 12139)	(492942.5, 1024.5)	(3963.14, 2624.45)	(152.76, 14.21)
10	(827.18, 13.17)	(18.52, 0.14)	(264, 0)	(214154, 4783)	(58978.5, 860.5)	(1023902.5, 28267.5)	(26901.23, 12258.51)	(331.61, 10.34)
11	(2741, 35.36)	(27.82, 0.06)	(9992, 217)	(441397, 4892)	(783990.5, 8018.5)	(1182139, 15985)	(8920.65, 3130.23)	(852.98, 211.1)
12	(6662.5, 265.6)	(13.49, 0.53)	(991, 0)	(780258.5, 28107.5)	(39822, 1709)	(2291294, 82675)	(106026.08, 2079.96)	(8.25, 4.38)
13	(839.66, 4)	(22.12, 0.4)	(262101.5, 6891.5)	(337976.5, 7931.5)	(603573, 75743)	(2824090.5, 129295.5)	(12932.45, 2140.66)	(2751.95, 713.06)
14	(10804.5, 1.29)	(16.59, 0.13)	(11878, 76)	(878035.5, 4690.5)	(7210.5, 1237.5)	(2812140.5, 20117.5)	(33357.48, 4977.63)	(0.85, 0.15)
15	(2204.06, 13.68)	(20.17, 0.48)	(1018.5, 131.5)	(606918, 23383)	(35233.5, 8883.5)	(2466430.5, 100952.5)	(13335.65, 7976.88)	(440.94, 40.57)
16	(3578.06, 39.3)	(22.11, 0.27)	(52610, 232)	(463424.5, 1065.5)	(121656, 1705)	(382798, 5115)	(831.75, 303.7)	(66.4, 6.36)
17	(3841.24, 443.76)	(22.38, 0.57)	(23311.5, 2699.5)	(620548.5, 35526.5)	(31335, 9947)	(528085.5, 16405.5)	(2453.93, 1427.6)	(126.38, 7.99)
18	(2517.19, 39.16)	(21.19, 0.38)	(18844.5, 60.5)	(348629.5, 2964.5)	(68996.5, 39941.5)	(266081, 48348)	(1480.08, 207.69)	(313.93, 182.8)
19	(2391.27, 126.83)	(38.41, 1.44)	(55733.5, 3951.5)	(355011, 8603)	(194151.5, 13916.5)	(641793.5, 7644.5)	(5713.87, 625.4)	(263.46, 142.01)
20	(3832.75, 63.9)	(27.27, 1.15)	(30005, 573)	(382754, 6988)	(124359, 483)	(382840.5, 4446.5)	(5369.37, 356.66)	(105.06, 28)
21	(3927.21, 504.01)	(25.47, 0.79)	(41429, 0)	(1019794, 62580)	(410531, 6744)	(1909089.5, 25648.5)	(12067.66, 7450.88)	(1225.03, 1086.43)
22	(495.55, 1.57)	(15.2, 0.5)	(21783, 499)	(257099, 6710)	(28052.5, 243.5)	(28769.5, 2759.5)	(688.9, 236.96)	(8.04, 1.48)
23	(3433.88, 36.39)	(28.47, 0.07)	(17182.5, 66.5)	(641042, 23757)	(813874, 41205)	(1151291.5, 34243.5)	(5900.76, 3137.43)	(1242.65, 205.38)
24	(833.64, 7.52)	(28.98, 0.14)	(30615.5, 703.5)	(155628.5, 22.5)	(13891, 480)	(311337.5, 42356.5)	(3272.5, 559.45)	(95.35, 27.47)

Table 2. The symmetric triangular fuzzy data for 24 bank branches at time period t+1

DMUs	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
1	(0.65, 1.34)	(0.69, 1.32)	(0.72, 1.29)	(0.76, 1.27)	(0.80, 1.24)	(0.84, 1.21)	(0.88, 1.18)	(0.95, 1.15)	(0.98, 1.17)	(1.04, 1.10)	(1.07, 1.07)
2	(0.21, 2.23)	(0.22, 1.9)	(0.24, 1.60)	(0.26, 1.34)	(0.28, 1.15)	(0.30, 0.99)	(0.34, 0.86)	(0.39, 0.74)	(0.42, 0.67)	(0.47, 0.59)	(0.53, 0.53)
3	(0.35, 2.88)	(0.37, 2.57)	(0.39, 2.24)	(0.40, 1.95)	(0.42, 1.66)	(0.45, 1.40)	(0.47, 1.21)	(0.51, 1.01)	(0.53, 0.90)	(0.60, 0.79)	(0.69, 0.69)
4	(0.94, 2.01)	(0.99, 1.93)	(1.03, 1.86)	(1.07, 1.80)	(1.11, 1.73)	(1.15, 1.67)	(1.2, 1.61)	(1.26, 1.53)	(1.29, 1.50)	(1.34, 1.44)	(1.39, 1.39)
5	(0.81, 1.23)	(0.83, 1.21)	(0.86, 1.19)	(0.88, 1.16)	(0.90, 1.14)	(0.91, 1.11)	(0.92, 1.09)	(0.95, 1.06)	(0.96, 1.05)	(0.98, 1.02)	(1.01, 1.01)
6	(0.75, 2.59)	(0.81, 2.46)	(0.88, 2.33)	(0.94, 2.21)	(0.98, 2.06)	(1.01, 1.92)	(1.05, 1.79)	(1.17, 1.62)	(1.20, 1.56)	(1.28, 1.46)	(1.37, 1.37)
7	(0.43, 1.21)	(0.45, 1.14)	(0.47, 1.07)	(0.49, 1.01)	(0.51, 0.95)	(0.54, 0.90)	(0.56, 0.85)	(0.59, 0.79)	(0.61, 0.75)	(0.64, 0.71)	(0.67, 0.67)
8	(0.52, 1.48)	(0.54, 1.40)	(0.56, 1.31)	(0.58, 1.23)	(0.61, 1.15)	(0.63, 1.07)	(0.66, 1.01)	(0.71, 0.92)	(0.72, 0.91)	(0.76, 0.85)	(0.80, 0.80)
9	(0.71, 1.42)	(0.73, 1.36)	(0.76, 1.30)	(0.78, 1.25)	(0.81, 1.20)	(0.83, 1.16)	(0.86, 1.11)	(0.90, 1.07)	(0.92, 1.04)	(0.95, 1.01)	(0.98, 0.98)
10	(0.57, 1.97)	(0.58, 1.86)	(0.59, 1.76)	(0.61, 1.67)	(0.62, 1.55)	(0.64, 1.41)	(0.67, 1.28)	(0.74, 1.15)	(0.78, 1.08)	(0.85, 0.99)	(0.92, 0.92)
11	(0.56, 0.92)	(0.58, 0.89)	(0.59, 0.87)	(0.60, 0.84)	(0.62, 0.82)	(0.63, 0.80)	(0.65, 0.78)	(0.67, 0.76)	(0.68, 0.74)	(0.69, 0.73)	(0.71, 0.71)
12	(1.11, 1.53)	(1.13, 1.51)	(1.15, 1.48)	(1.17, 1.46)	(1.19, 1.44)	(1.21, 1.41)	(1.23, 1.39)	(1.25, 1.37)	(1.27, 1.35)	(1.29, 1.33)	(1.31, 1.31)
13	(0.44, 1.76)	(0.48, 1.64)	(0.52, 1.52)	(0.56, 1.42)	(0.60, 1.32)	(0.64, 1.23)	(0.68, 1.14)	(0.74, 1.04)	(0.78, 0.99)	(0.83, 0.93)	(0.88, 0.88)
14	(0.85, 1.05)	(0.86, 1.03)	(0.87, 1.02)	(0.88, 1.01)	(0.89, 1.00)	(0.90, 0.99)	(0.91, 0.98)	(0.92, 0.97)	(0.93, 0.97)	(0.94, 0.96)	(0.95, 0.95)
15	(0.66, 1.27)	(0.68, 1.22)	(0.69, 1.17)	(0.71, 1.11)	(0.73, 1.06)	(0.75, 1.01)	(0.78, 0.99)	(0.80, 0.95)	(0.82, 0.93)	(0.85, 0.90)	(0.87, 0.87)
16	(0.86, 1.68)	(0.9, 1.64)	(0.94, 1.61)	(0.98, 1.57)	(1.02, 1.53)	(1.063, 1.50)	(1.10, 1.46)	(1.16, 1.42)	(1.19, 1.40)	(1.25, 1.34)	(1.30, 1.30)
17	(0.68, 1.52)	(0.70, 1.46)	(0.73, 1.41)	(0.76, 1.36)	(0.79, 1.31)	(0.82, 1.26)	(0.85, 1.21)	(0.90, 1.14)	(0.92, 1.11)	(0.96, 1.06)	(1.01, 1.01)
18	(0.40, 1.89)	(0.44, 1.76)	(0.47, 1.62)	(0.51, 1.50)	(0.55, 1.39)	(0.60, 1.29)	(0.65, 1.20)	(0.73, 1.07)	(0.76, 1.04)	(0.82, 0.96)	(0.88, 0.88)
19	(0.5, 1.41)	(0.53, 1.32)	(0.56, 1.24)	(0.61, 1.18)	(0.66, 1.12)	(0.70, 1.07)	(0.74, 1.04)	(0.79, 0.99)	(0.82, 0.97)	(0.87, 0.94)	(0.91, 0.91)
20	(0.87, 1.42)	(0.89, 1.38)	(0.92, 1.35)	(0.94, 1.31)	(0.96, 1.27)	(0.99, 1.25)	(1.014, 1.22)	(1.04, 1.19)	(1.06, 1.17)	(1.09, 1.14)	(1.11, 1.11)
21	(0.76, 1.55)	(0.78, 1.49)	(0.80, 1.43)	(0.83, 1.38)	(0.85, 1.33)	(0.87, 1.28)	(0.90, 1.23)	(0.96, 1.16)	(0.97, 1.14)	(1.02, 1.09)	(1.05, 1.05)
22	(0.36, 1.10)	(0.39, 0.99)	(0.41, 0.92)	(0.44, 0.88)	(0.46, 0.84)	(0.49, 0.81)	(0.51, 0.78)	(0.55, 0.73)	(0.57, 0.70)	(0.60, 0.67)	(0.63, 0.63)
23	(0.60, 1.14)	(0.62, 1.11)	(0.64, 1.05)	(0.66, 1.02)	(0.68, 0.99)	(0.70, 0.96)	(0.72, 0.93)	(0.75, 0.89)	(0.77, 0.87)	(0.79, 0.84)	(0.82, 0.82)
24	(0.60, 1.44)	(0.67, 1.35)	(0.73, 1.26)	(0.79, 1.22)	(0.82, 1.18)	(0.85, 1.15)	(0.88, 1.11)	(0.92, 1.06)	(0.93, 1.05)	(0.96, 1.02)	(0.99, 0.99)

Table3.Interval Malmquist productivity index for 24 DMUs with fuzzy data

5 Conclusion

The purpose of this study was to develop the Malmquist productivity index to DMUs with interval and fuzzy data that since the level of inputs and outputs for DMU_o are not known exactly, we expect the value of Malmquist productivity index for DMU_o and its components is not known exactly, when the true input and output data known to lie within bonded intervals, then, in this cases, the Malmquist productivity index (and its component) can known to lie within an interval and when input and output data is fuzzy, then, in this cases, the Malmquist productivity index (and its component) can not known exactly that in this paper, we try by using of concept α – level to approach level of Malmquist productivity index and applied to a numerical example.

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