Sufficient conditions of analytic functions involving the generalized fractional integral operator

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Abstract

In this paper, we determine the sufficient conditions that make the generalized fractional integral operator $J_{0,z}^{\beta,\gamma,\delta}$ to be in the comprehensive family $\mathcal{E}(\Phi, \Psi; \alpha)$ of analytic functions of order α which contains various well-known classes of analytic functions as well as many new ones.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions $f \in \mathcal{A}$ of order α ($0 \leq \alpha < 1$) satisfying

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}) \quad (2)$$

and let $\mathcal{C}(\alpha)$ be the class of convex functions $f \in \mathcal{A}$ of order α ($0 \leq \alpha < 1$) such that $zf'(z) \in \mathcal{S}^*(\alpha)$. Further, let $\mathcal{P}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ such that

$$\operatorname{Re} (f'(z)) > \alpha \quad (0 \leq \alpha < 1, z \in \mathcal{U}). \quad (3)$$

Given two functions $f, h \in \mathcal{A}$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ their convolution or Hadamard product $f(z) * h(z)$, is defined by

$$f(z) * h(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n \quad (z \in \mathcal{U}). \quad (4)$$

Ruscheweyh [3] using the convolution techniques, introduced and studied an important subclass of $\mathcal{A}$, the class of prestarlike functions of order α , which is denoted by $\mathcal{R}(\alpha)$. Thus $f \in \mathcal{A}$ is said to be prestarlike function of order α ($0 \leq \alpha < 1$) if $f * s_\alpha \in \mathcal{S}^*(\alpha)$ where $s_\alpha(z) = z(1-z)^{2\alpha-2} = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n$; $c(\alpha, n) = \prod_{k=2}^n \frac{k-2\alpha}{(n-1)!}$ ($n \geq 2$). It may be noted that $\mathcal{R}(0) = \mathcal{C}(0)$ and $\mathcal{R}(1/2) = \mathcal{S}^*(1/2)$.

Juneja *et al.* [1] define the family $\mathcal{E}(\Phi, \Psi; \alpha)$ consisting of the functions $f \in \mathcal{A}$ so that

$$\operatorname{Re} \left(\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} \right) > \alpha \quad (z \in \mathcal{U}) \quad (5)$$

where α ($0 \leq \alpha < 1$), $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic in $\mathcal{U}$ with the conditions $\lambda_n \geq 0$, $\mu_n \geq 0$, $\lambda_n \geq \mu_n$ for $n \geq 2$, and $f(z) * \Psi(z) \neq 0$.

It is easy to check that various subclasses of $\mathcal{A}$ referred to above can be represented as $\mathcal{E}(\Phi, \Psi; \alpha)$ for suitable choices of Φ, Ψ . For example

- (i) $\mathcal{E} \left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha \right) = \mathcal{S}^*(\alpha)$;
- (ii) $\mathcal{E} \left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha \right) = \mathcal{C}(\alpha)$
- (iii) $\mathcal{E} \left(\frac{z}{(1-z)^2}, z; \alpha \right) = \mathcal{P}(\alpha)$;
- (iv) $\mathcal{E} \left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha \right) = \mathcal{R}(\alpha)$.

In fact many new subclasses of $\mathcal{A}$ can be defined and studied by suitably choosing $\Phi(z)$ and $\Psi(z)$. Thus

- (v) $\mathcal{E} \left(\frac{z+z^2}{(1-z)^3}, z; \alpha \right) = \{f \in \mathcal{A} : \operatorname{Re}((zf'(z))') > \alpha\}$
- (vi) $\mathcal{E} \left((1-\delta)\frac{z}{(1-z)^2} + \delta\frac{z+z^2}{(1-z)^3}, z; \alpha \right)$
 $= \{f \in \mathcal{A} : \operatorname{Re}((1-\delta)f'(z) + \delta(zf'(z))') > \alpha\}$ and so on.

Let $F(a, b; c; z)$ be the Gauss hypergeometric function defined, for $z \in \mathcal{U}$, by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}, \quad (6)$$

where $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu + 1)(\nu + 2) \cdots (\nu + n - 1) & (n \in \mathbb{N}) \end{cases} \quad (7)$$

We need the following definitions of fractional integral operator given by Srivastava *et al.*[2].

Definition 1. For real number $\lambda > 0$, γ and δ , the fractional integral operator $I_{0,z}^{\lambda,\gamma,\delta}$ is defined by

$$I_{0,z}^{\beta,\gamma,\delta} f(z) = \frac{z^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^z (z-t)^{\lambda-1} F(\beta + \gamma, -\delta; \beta; 1-t/z) f(t) dt, \quad (8)$$

where a function $f(z)$ is analytic in a simply-connected region of the z -plane containing the origin with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0),$$

with $\varepsilon > \max\{0, \gamma - \delta\} - 1$, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Definition 2. Under the hypotheses of Definition 1, let

$$\beta > 0, \quad \min\{\beta + \delta, -\gamma + \delta, -\gamma\} > -2, \quad \text{and} \quad 3\beta \geq \gamma(\beta + \eta). \quad (9)$$

Then the fractional integral operator $J_{0,z}^{\beta,\gamma,\delta}$ is defined by

$$J_{0,z}^{\beta,\gamma,\delta} f(z) = \frac{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)}{\Gamma(2-\gamma+\eta)} z^\gamma I_{0,z}^{\beta,\gamma,\delta} f(z) \quad (10)$$

2 Main Results

In order to derive our results, we need the following lemmas.

Lemma 1 ([2]). *If $\beta > 0$ and $n > \gamma - \delta - 1$, then*

$$I_{0,z}^{\beta,\gamma,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\gamma+\delta+1)}{\Gamma(n-\gamma+1)\Gamma(n+\beta+\delta+1)} z^{n-\gamma}. \quad (11)$$

Lemma 2([1]) . Let the functions $f(z)$ defined by (1) satisfies

$$\sum_{n=2}^{\infty} \frac{\lambda_n - \alpha\mu_n}{1-\alpha} a_n \leq 1. \quad (12)$$

then $f(z) \in \mathcal{E}(\Phi, \Psi; \alpha)$. The result (12) is sharp.

Now we prove

Theorem 1. Under the constraints (9), if the function $f(z)$ defined by (1) satisfies

$$\sum_{n=2}^{\infty} \frac{\lambda_n - \alpha\mu_n}{1-\alpha} a_n \leq \frac{(2-\gamma)(2+\beta+\delta)}{2(2-\gamma+\delta)}, \quad (13)$$

then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{E}(\Phi, \Psi; \alpha)$.

Proof. With the aid of Lemma 1 and Definition 2, we have

$$J_{0,z}^{\beta,\gamma,\delta} f(z) = z + \sum_{n=2}^{\infty} \Delta(n) a_n z^n \quad (14)$$

where

$$\Delta(n) = \frac{(2-\gamma+\delta)_{n-1}(1)_n}{(2-\gamma)_{n-1}(2+\beta+\delta)_{n-1}} \quad (n \geq 2),$$

we can see that the function $\Delta(n)$ is non-increasing for integers $n \geq 2$, and we have

$$0 < \Delta(n) \leq \Delta(2) = \frac{2(2-\gamma+\delta)}{(2-\gamma)(2+\beta+\delta)}. \quad (15)$$

Therefore, by using (13) and (15), we have

$$\sum_{n=2}^{\infty} \frac{\lambda_n - \alpha\mu_n}{1-\alpha} \Delta(n) a_n \leq \Delta(2) \sum_{n=2}^{\infty} \frac{\lambda_n - \alpha\mu_n}{1-\alpha} \leq 1. \quad (16)$$

Hence, by Lemma 2, $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{E}(\Phi, \Psi; \alpha)$.

Putting $\Phi(z) = z/(1-z)^2$ and $\Psi(z) = z/(1-z)$ in Theorem 1, we have

Corollary 1. Under the constraints (9), if the function $f(z)$ defined by (1) satisfies

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq \frac{(1 - \alpha)(2 - \gamma)(2 + \beta + \delta)}{2(2 - \gamma + \delta)}, \quad (17)$$

then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{S}^*(\alpha)$.

Putting $\Phi(z) = (z + z^2)/(1 - z)^3$ and $\Psi(z) = z/(1 - z)^2$ in Theorem 1, we have

Corollary 2. *Under the constraints (9), if the function $f(z)$ defined by (1) satisfies*

$$\sum_{n=2}^{\infty} n(n - \alpha) a_n \leq \frac{(1 - \alpha)(2 - \gamma)(2 + \beta + \delta)}{2(2 - \gamma + \delta)}, \quad (18)$$

then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{C}(\alpha)$.

Putting $\Phi(z) = z/(1 - z)^2$ and $\Psi(z) = z$ in Theorem 1, we have

Corollary 3. *Under the constraints (9), if the function $f(z)$ defined by (1) satisfies*

$$\sum_{n=2}^{\infty} n a_n \leq \frac{(1 - \alpha)(2 - \gamma)(2 + \beta + \delta)}{2(2 - \gamma + \delta)}, \quad (19)$$

then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{P}(\alpha)$.

Putting $\Phi(z) = (z + (1 - 2\beta)z^2)/(1 - z)^{3-2\alpha}$ and $\Psi(z) = z/(1 - z)^{2-2\alpha}$ in Theorem 1, we have

Corollary 4. *Under the constraints (9), if the function $f(z)$ defined by (1) satisfies*

$$\sum_{n=2}^{\infty} C(\alpha, n)(n - \alpha) a_n \leq \frac{(1 - \alpha)(2 - \gamma)(2 + \beta + \delta)}{2(2 - \gamma + \delta)}, \quad (20)$$

then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{R}(\alpha)$.

Next we prove

Theorem 2. *Let*

$$h(z) = z + \sum_{n=2}^{\infty} \frac{(2 - \gamma + \delta)_{n-1}(1)_n}{(2 - \gamma)_{n-1}(2 + \beta + \delta)_{n-1}} z^n \quad (21)$$

be in the class $\mathcal{E}(zf'(z), f(z); \alpha)$, then $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{S}^*(\alpha)$.

Proof. Notice from (14) and (21) that

$$J_{0,z}^{\beta,\gamma,\delta} f(z) = z + \sum_{n=2}^{\infty} \frac{(2-\gamma+\delta)_{n-1}(1)_n}{(2-\gamma)_{n-1}(2+\beta+\delta)_{n-1}} a_n z^n = (h * f)(z), \quad (22)$$

which readily yields

$$\frac{z(J_{0,z}^{\beta,\gamma,\delta} f(z))'}{J_{0,z}^{\beta,\gamma,\delta} f(z)} = \frac{z(h * f)'(z)}{(h * f)(z)} = \frac{(h * (zf'))(z)}{(h * f)(z)}. \quad (23)$$

Setting $\Phi(z) = z(f(z))$ and $\Psi(z) = f(z)$ in (23), we find that

$$\operatorname{Re} \left(\frac{z(J_{0,z}^{\beta,\gamma,\delta} f(z))'}{J_{0,z}^{\beta,\gamma,\delta} f(z)} \right) > \alpha \quad (24)$$

which implies that $J_{0,z}^{\beta,\gamma,\delta} f(z) \in \mathcal{S}^*(\alpha)$.

References

- [1] O.P. Juneja, T.R. Reddy and M.L. Mogra, *A convolution approach for analytic functions with negative coefficients*, Soochow J. Math. 11 (1985), 69-81.
- [2] H.M. Srivastava, M. Saigo and S. Owa, *A class of distortion theorems involving certain operators of fractional calculus*, J. Math. Anal. Appl. 131 (1988), 412-420.
- [3] S. Ruscheweyh, *Linear operators between classes of prestarlike functions*, Comment. Math. Helv. 52 (1977), 497- 509.

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