On Coefficient Problems of Certain Analytic Functions Involving Hadamard Products

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Abstract. For $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta < 1$, $0 \leq \nu < 1$ and $\beta > 0$, let $MS(\Phi, \Psi; \lambda, \alpha, \delta, \nu, \beta)$ be the class of functions defined in the open unit disk D by

$$\left| \arg \left(\frac{\lambda z^2 f''(z) + z f'(z)}{\lambda z g'(z) + (1 - \lambda)g(z)} - \delta \right) \right| < \frac{\pi \alpha}{2}, \quad z \in D$$

where $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$ is analytic function and satisfies

$$\left| \arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} - \nu \right) \right| < \frac{\pi \beta}{2}, \quad z \in D$$

for some $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ ($n \geq 2$). For $f \in MS(\Phi, \Psi; \lambda, \alpha, \delta, \nu, \beta)$ and given by $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, a sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ when $3\mu\sigma[1 + \delta(2 + \delta)] \geq 2\eta(\eta + 2\gamma_2)(1 + \delta)$ where $\eta = \Upsilon_2 - \gamma_2(1 + \nu)$ and $\sigma = \Upsilon_3 - \gamma_3(1 + \nu)$. Further, for a fixed value of λ , δ and ν we obtained an interesting result.

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1 Introduction.

Let A denote the family of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are in the open unit disc $D = \{z : |z| < 1\}$. Let S denote the class of functions which are univalent in D . Let the function f be defined by (1.1). Also let the function g be defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \quad (1.2)$$

Then the Hadamard product (or convolution) of the functions f and g is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (1.3)$$

Fekete and Szegő[9] obtained sharp upper bound for $|a_3 - \mu a_2^2|$ when μ is real. For various subclass of S , namely the class of starlike functions, the class of convex functions, and the class of close-to-convex functions denoted by S^* , C and K respectively, sharp upper bound for functional $|a_3 - \mu a_2^2|$ has been studied by many different authors including ([1]-[8], [10]-[12], [14]-[19], [21]-[22]). Obviously, the classical ones are from the work done by Fekete and Szegő[9], and Keogh and Merkes[14]. In this paper we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when f belonging to the class of functions defined as follows:

Definition 1.1. Let $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta < 1$, $0 \leq \nu < 1$ and $\beta > 0$, and let $f \in A$. Then $f \in MS(\Phi, \Psi; \lambda, \alpha, \delta, \nu, \beta)$ if and only if

$$\left| \arg \left(\frac{\lambda z^2 f''(z) + z f'(z)}{\lambda z g'(z) + (1 - \lambda)g(z)} - \delta \right) \right| < \frac{\pi \alpha}{2}, \quad (1.4)$$

with $g \in A$ and satisfies

$$\left| \arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} - \nu \right) \right| < \frac{\pi \beta}{2}, \quad z \in D \quad (1.5)$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ ($n \geq 2$).

Note that $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, 0, 0, \beta) = K(\beta)$ the class of close-to-convex functions defined in [3] and $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, 1, 0, 0, 1) = K(1)$ is the class of normalised close-to-convex functions defined by Kaplan [13]. Whereas, $MS(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, \alpha, 0, 0, \beta) = K(\alpha, \beta)$, is the class of normalised close-to-convex functions defined in [7].

2 Main result.

In order to derive our main result, we have to recall here the following lemma[20].

Lemma 2.1. Let $h \in P$ that is, h be analytic in D and be given by $h(z) = 1 + c_1z + c_2z^2 + \dots$, and $\operatorname{Re} h(z) > 0$ for $z \in D$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2} \quad (2.1)$$

Theorem 2.2. Let f be given by (1.1). If $f \in MS(\Phi, \Psi; \lambda, \alpha, \delta, \beta)$; $0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta < 1$, $0 \leq \nu < 1$, $\beta > 0$ and $3\mu\sigma[1 + \delta(2 + \delta)] \geq 2\eta(\eta + 2\gamma_2)(1 + \delta)$ where $\eta = \Upsilon_2 - \gamma_2(1 + \nu)$ and $\sigma = \Upsilon_3 - \gamma_3(1 + \nu)$, then we have the sharp inequality

$$\begin{aligned} |a_3 - \mu a_2^2| \leq & \frac{\beta^2}{3\sigma\eta^2} \{3\mu\sigma[1 + \delta(2 + \delta)] - [2\eta(\eta + 2\gamma_2)(1 + \delta)]\} \\ & + \frac{\alpha([\alpha\eta + 2\beta(1 + \lambda)(1 + \delta)][3\mu(1 + 2\lambda)] - 2[\alpha\eta + 2\beta(1 + \lambda)](1 + \lambda)^2)}{3\eta(1 + \lambda)^2(1 + 2\lambda)} \end{aligned} \quad (2.2)$$

Proof. Let $f \in MS(\Phi, \Psi; \lambda, \alpha, \delta, \nu, \beta)$. It follows from (1.4) that

$$\lambda z^2 f''(z) + z f'(z) = [\lambda z g'(z) + (1 - \lambda)g(z)][q^\alpha(z) + \delta] \quad (2.3)$$

for $z \in D$, with $q \in P$ given by $q(z) = 1 + q_1z + q_2z^2 + q_3z^3 + \dots$. Equation coefficients, we obtain

$$2a_2(1 + \lambda) = b_2(1 + \lambda)(1 + \delta) + \alpha q_1 \quad (2.4)$$

and

$$3a_3(1 + 2\lambda) = \alpha q_2 + \alpha q_1 b_2(1 + \lambda) + \frac{\alpha(\alpha - 1)}{2} q_1^2 + b_3(1 + 2\lambda)(1 + \delta) \quad (2.5)$$

Also, it follows from (1.5) that

$$g(z) * \Phi(z) = (g(z) * \Psi(z))(p^\beta(z) + \nu) \quad (2.6)$$

where $p \in P$ with $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$ for $z \in D$. Thus equating coefficients, we obtain

$$\eta b_2 = \beta p_1 \quad (2.7)$$

and

$$\sigma b_3 = \beta \left(p_2 + \frac{\beta(\eta + 2\gamma_2) - \eta}{2\eta} p_1^2 \right) \quad (2.8)$$

From (2.4), (2.5), (2.7) and (2.8) we have

$$\begin{aligned}
 a_3 - \mu a_2^2 &= \frac{\alpha}{3(1+2\lambda)} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{\beta(1+\delta)}{3\sigma} \left(p_2 - \frac{p_1^2}{2} \right) \\
 &\quad + \frac{\alpha^2 q_1^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\}}{12(1+\lambda)^2(1+2\lambda)} \\
 &\quad + \frac{\alpha\beta p_1 q_1 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)(1+\delta)\}}{6\eta(1+\lambda)(1+2\lambda)} \\
 &\quad + \frac{\beta^2 p_1^2 \{2\eta^2(1+\delta) - 3\mu\sigma[1+\delta(2+\delta)] + 4\eta\gamma_2(1+\delta)\}}{12\sigma\eta^2}
 \end{aligned} \tag{2.9}$$

Assume that $a_3 - \mu a_2^2$ positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$ by applying the same technique done by London[18]. And so from (2.9) and by using lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$, we obtain

$$\begin{aligned}
 \operatorname{Re}(a_3 - \mu a_2^2) &= \frac{\alpha}{3(1+2\lambda)} \operatorname{Re} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{\alpha^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} \operatorname{Re} q_1^2}{12(1+\lambda)^2(1+2\lambda)} \\
 &\quad + \frac{\beta(1+\delta)}{3\sigma} \operatorname{Re} \left(p_2 - \frac{p_1^2}{2} \right) + \frac{\alpha\beta \{2(1+\lambda)^2 - 3\mu(1+2\lambda)(1+\delta)\} \operatorname{Re} p_1 q_1}{6\eta(1+\lambda)(1+2\lambda)} \\
 &\quad + \frac{\beta^2 \{2\eta^2(1+\delta) - 3\mu\sigma[1+\delta(2+\delta)] + 4\eta\gamma_2(1+\delta)\} \operatorname{Re} p_1^2}{12\sigma\eta^2} \\
 &\leq \frac{2\alpha}{3(1+2\lambda)} (1 - R^2) + \frac{\alpha^2 \{2(1+\lambda)^2 - 3\mu(1+2\lambda)\} R^2 \cos 2\phi}{3(1+\lambda)^2(1+2\lambda)} \\
 &\quad + \frac{2\beta(1+\delta)}{3\sigma} (1 - r^2) + \frac{2\alpha\beta \{2(1+\lambda)^2 - 3\mu(1+2\lambda)(1+\delta)\} rR \cos(\theta+\phi)}{3\eta(1+\lambda)(1+2\lambda)} \\
 &\quad + \frac{\beta^2 \{2\eta^2(1+\delta) - 3\mu\sigma[1+\delta(2+\delta)] + 4\eta\gamma_2(1+\delta)\} r^2 \cos 2\theta}{3\sigma\eta^2} \\
 &\leq \frac{2\alpha}{3(1+2\lambda)} (1 - R^2) + \frac{\alpha^2 \{3\mu(1+2\lambda) - 2(1+\lambda)^2\} R^2}{3(1+\lambda)^2(1+2\lambda)} \\
 &\quad + \frac{2\beta(1+\delta)}{3\sigma} (1 - r^2) + \frac{2\alpha\beta \{3\mu(1+2\lambda)(1+\delta) - 2(1+\lambda)^2\} rR}{3\eta(1+\lambda)(1+2\lambda)} \\
 &\quad + \frac{\beta^2 \{3\mu\sigma[1+\delta(2+\delta)] - [2\eta(\eta+2\gamma_2)(1+\delta)]\} r^2}{3\sigma\eta^2} = G(r, R).
 \end{aligned}$$

Letting α , δ , ν , β and μ fixed and differentiating $G(r, R)$ partially when $0 < \alpha \leq 1$, $0 \leq \delta < 1$, $0 \leq \nu < 1$, $\beta > 0$ and $3\mu\sigma[1+\delta(2+\delta)] \geq 2\eta(\eta+2\gamma_2)(1+\delta)$

where $\eta = \Upsilon_2 - \gamma_2(1 + \nu)$ and $\sigma = \Upsilon_3 - \gamma_3(1 + \nu)$, we observe that

$$\begin{aligned} & G_{rr}G_{RR} - (G_{rR})^2 \\ &= \frac{16\alpha\beta(1+\delta)}{9\sigma(1+2\lambda)} + \frac{4\alpha^2\beta^2\{3\mu(1+2\lambda)-2(1+\lambda)^2\}\{3\mu\sigma[1+\delta(2+\delta)]-[2\eta(\eta+2\gamma_2)(1+\delta)]\}}{9\sigma\eta^2(1+\lambda)^2(1+2\lambda)} \\ &\quad - \frac{8\beta\alpha^2(1+\delta)\{3\mu(1+2\lambda)-2(1+\lambda)^2\}}{9\sigma(1+\lambda)^2(1+2\lambda)} - \frac{8\alpha\beta^2\{3\mu\sigma[1+\delta(2+\delta)]-[2\eta(\eta+2\gamma_2)(1+\delta)]\}}{9\sigma\eta^2(1+2\lambda)} \\ &\quad - \frac{\alpha^2\beta^2\{3\mu(1+2\lambda)(1+\delta)-2(1+\lambda)^2\}^2}{9\eta^2(1+\lambda)^2(1+2\lambda)^2} < 0 \end{aligned}$$

Therefore, the maximum of $G(r, R)$ occurs on the boundaries. Thus the desired inequality follows by observing that

$$\begin{aligned} G(r, R) \leq G(1, 1) &= \frac{\beta^2}{3\sigma\eta^2}\{3\mu\sigma[1+\delta(2+\delta)]-2\eta(\eta+2\gamma_2)(1+\delta)\} \\ &\quad + \frac{\alpha([\alpha\eta+2\beta(1+\lambda)(1+\delta)][3\mu(1+2\lambda)]-2[\alpha\eta+2\beta(1+\lambda)](1+\lambda)^2)}{3\eta(1+\lambda)^2(1+2\lambda)} \end{aligned} \quad (2.10)$$

The equality is attained when choosing $p_1 = q_1 = 2i$ and $p_2 = q_2 = -2$ in (2.9).

Remark 1. Letting $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{(1-z)}$, $\lambda = 0$, $\delta = 0$ and $\nu = 0$ in Theorem 2.2, we have the result given by Darus and Thomas [7].

Remark 2. Letting $\Phi(z) = \frac{z}{(1-z)^2}$, $\Psi(z) = \frac{z}{(1-z)}$, $\lambda = 0$, $\delta = 0$, $\nu = 0$ and $\alpha = 1$ in Theorem 2.2, we have the result given by Jahangiri [12].

Remark 3. Letting $\lambda = 0$, $\delta = 0$ and $\nu = 0$ in Theorem 2.2, we have the result given by Darus and Hong [17].

Remark 4. Letting $\delta = 0$ and $\nu = 0$ in Theorem 2.2, we have the result given by Darus [8].

Next by letting $\lambda = 1$, $\delta = 0$ and $\nu = 0$ in Definition 1.1, we achieve the following definition:

Definition 2.3 Let $0 < \alpha \leq 1$ and $\beta > 0$, and let $f \in A$. Then $f \in MS(\Phi, \Psi; 1, \alpha, 0, 0, \beta)$ if and only if

$$\left| \arg \left(\frac{(zf'(z))'}{g'(z)} \right) \right| < \frac{\pi\alpha}{2}, \quad (2.11)$$

with $g \in A$ and satisfies

$$\left| \arg \left(\frac{g(z) * \Phi(z)}{g(z) * \Psi(z)} \right) \right| < \frac{\pi\beta}{2}, \quad z \in D \quad (2.12)$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} \Upsilon_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n$ analytic in D such that $g(z) * \Psi(z) \neq 0$, $\Upsilon_n \geq 0$, $\gamma_n \geq 0$ and $\Upsilon_n > \gamma_n$ ($n \geq 2$).

Note also that $MS(\frac{z}{(1-z)^2}; \frac{z}{1-z}; 1, 1, 0, 0, 1) = QC$ is the class of normalised quasi-convex functions defined by Noor [23]. We note that every quasi-convex is close-to-convex and hence univalent in D .

By using the same technique employed in the proof of Theorem 2.2, we can prove the following:

Theorem 2.4. Let f be given by (1.1). If $f \in MS(\Phi, \Psi; 1, \alpha, 0, 0, \beta)$; $0 < \alpha \leq 1$, $\beta > 0$ and $3\mu\sigma \geq 2\eta(\eta + 2\gamma_2)$ where $\eta = \Upsilon_2 - \gamma_2$, and $\sigma = \Upsilon_3 - \gamma_3$, then we have the sharp inequality

$$|a_3 - \mu a_2^2| \leq \frac{\beta^2}{3\sigma\eta^2} \{3\mu\sigma - 2\eta(\eta + 2\gamma_2)\} + \frac{\alpha(9\mu-8)(\alpha\eta+4\beta)}{36\eta} \quad (2.13)$$

We omit the proof here as the details are tedious.

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