Setting the Clock Back to Zero Property of a Family of Bivariate Life Distributions

B. Raja Rao¹, Jasem M. Alhumoud and C.V. Damaraju

¹ Posthumous
Civil Engineering Department, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
Tel: (+965) 967-4825, e-mail: jasem@kuc01.kuniv.edu.kw

Abstract

In the present paper, the Lack of Memory Property (L.M.P), uniquely enjoyed by the exponential distribution, has been extended and it is shown the several other families of life distributions have this extended property. This extended property is called the 'setting the clock back to zero property'. Its analogy for bivariate and multivariate families of life distributions is discuses. As a simple application of this result, it is proved that the life expectancy vector of an organism under the influence of multiple competing risks has a simple expression if its life distribution has the setting the clock back to zero property. Several univariate and bivariate life distributions are used to illustrate the use the result.

Keywords: Force of mortality; survival function; bivariate and multivariate exponential distributions, Gumbel's bivariate exponen distribution; bivariate Gompertz distribution.

1 Introduction

The present paper discusses a health situation in which the individuals in a biomedical or an epidemiological cohort study are exposed simultaneously to several risks of death, such as cancer, heart disease, stroke, renal failure, emphysema, etc. Let us consider a population of individuals in which k causes 0 risks of death indexed by R₁, R₂,...,Rₖ are "competing" for an individual's life. An individual in this population is exposed to the risk of dying of anyone of these causes. Methods of the analysis of mortality experience in such a population are known as competing risk analysis. Let X₁, X₂,..., Xₖ denote an individual's hypothetical survival times or life times due to these risks. That is, if he were allowed to continue, he would die of risk Rᵢ at time Xᵢ. In problems associated with reliability of an engineering system, X₁, X₂, ..., Xₖ would be the theoretic a failure times of k components of the system.
It is supposed that each individual, presumably at birth, is endowed with a set of such times \( X_1, X_2, \ldots, X_k \) one for each risk. Such risks could be cancer, heart disease, kidney failure, emphysema, etc. The actual time of the death is the minimum of \( X_1, X_2, \ldots, X_k \). Most models proposed for human or animal populations postulate the existence of a number of independently operating causes or risks of death. This is an essential component of many of the procedures developed. This postulate assumes that the risk of death from one cause is independent of, and is unaffected by, changes in the risk of death from other causes. The falsity of this assumption, when used in models for human populations, is recognized by many research workers, but in theoretical or mathematical investigations, this assumption is nevertheless made to simplify the discussion. The joint survival function of these life times is defined by equation

\[
S(x_1, x_2, \ldots, x_k) = P(X_1 \geq x_1, X_2 \geq x_2, \ldots, X_k \geq x_k). 
\] (1-1)

We cannot simultaneously observe the failure times. What can be observed is \( U = \min(X_1, X_2, \ldots, X_k) \), whose survival function is

\[
S_U(t) = P(U \geq t) = P\left( \min_{i=1}^{n} X_i \geq t \right) = S(t, t, \ldots, t). 
\] (1-2)

The marginal survival functions of (1-1) are

\[
S_i(x) = S(O, O, \ldots, x, 0, 0, \ldots 0). \quad i=1,2,\ldots,k 
\] (1-3)

and the hazard rates of the marginal distributions are defined by

\[
\lambda_i(x) = -\frac{\partial}{\partial x_i} \log S_i(x). \quad i = 1, 2, \ldots, k. 
\] (1-4)

The crude hazard rate or death rate due to risk \( R_i \) evaluated at \( x_i = x \) for all \( i \) is

\[
h_i(x) = -\frac{\partial}{\partial x_i} \log S(x_1, x_2, \ldots, x_k) \big|_{x_1=x} \quad i = 1, 1, \ldots, k. 
\] (1-5)

As a particular case, if the risks act independently, the crude hazard rates are the same as the hazard rates of the marginal distributions. The multivariate hazard vector is the one whose \( i \)th element evaluated at \( x_i = x \) for all \( i \) is given by equation (1-5). These quantities are useful in actuarial science to prepare life tables for biological populations.

2 Setting the Clock Back to Zero Property of a Family of Univariate Life Distributions

Let the non-negative continuous random variable \( X \) have some life distribution with the survival function
Setting the Clock Back to Zero Property

\[ S(x, \beta) = P(X \geq x), \ x \geq 0, \quad (2-1) \]

where \( \beta \) is a parameter or a vector of parameters. When \( \beta \in \) a parameter space \( \Omega \) this defines a family of life distributions \( \{S(x, \beta), x \geq 0, \beta \in \Omega\} \). Only the exponential distribution has the Lack of Memory Property (L M.P.):

\[ P(X \geq x + x_0 \mid X \geq x_0) = P(X \geq x). \quad (2-2) \]

By a simple extension of this property, we can show that a number of families of other life distributions have the extended property:

\[ P(X \geq x + x_0 I X \geq x_0) = P(X^* \geq x) \quad (2-3) \]

where the random variable \( X^* \) has the same distribution as that of \( X \), except that the vector \( \beta \) of parameters is replaced by \( \beta^* \in \Omega \). This property is called the 'setting the clock back to zero' property. We now define this formally as follows:

**Definition (2.1)** A family of life distribution \( \{S(x, \beta), x \geq 0, \beta \in \Omega\} \) is said to have the setting the clock back to zero or is said to be "invariant" property (Rao and Talwalker, 1990) if the survival function satisfies the following condition for each \( \beta \in \Omega \) and \( x_0 > 0 \):

\[ \frac{S(x + x_0, \beta)}{S(x_0, \beta)} = S(x, \beta^*), \text{ where } \beta^* = \beta^*(x_0) \in \Omega. \quad (2-4) \]

In words, this property simply means that the conditional distribution of additional survival time of an organism due to any one risk, given that it has survived \( x_0 \) time units, remains in the family. It is clear that this concept generalizes the Lack of Memory Property of the univariate exponential distribution in the sense that for the exponential distribution, the conditional distribution of additional survival time due to any on risk is exactly the same as the original distribution.

It is easily seen that the family of linear hazard exponential distribution has the 'setting the clock back to zero' property, while the Weibull and Rayleigh distributions do not. The Gompertz distribution has been shown to have this property (Rao and Talwalker 1989; Rao, 1990) since

\[ \frac{S(x + x_0, \beta)}{S(x_0, \beta)} = \exp \left[ -\frac{k'}{\alpha} (e^{\alpha x} - 1) \right] \]

\[ \frac{S(x + x_0, \beta)}{S(x_0, \beta)} = S(x, \beta^*) \quad (2-5) \]
Here $\beta=(k,\alpha)$ is the vector of parameters and the parametric space is $\Omega=\{(k,a): 0 \leq k \leq \infty, -\infty \leq \alpha \leq \infty\}$. The new vector of parameters is $\beta^*=(k',\alpha)$ where $k'=k\exp(\alpha x_0)$. Clearly $\beta^* \in \Omega$. In this operation of setting the clock back to zero, the parameter $\alpha$ remains unchanged and the parameter $k$ becomes $k'=k\exp(\alpha x_0)$. In view of this property truncating a Gompertz distribution at time $x_0$ and setting the origin at $x_0$ (i.e., in terms of time, "Setting the clock back to zero") leaves the form of the distribution unaltered except for the value of $k$, which changes from $k$ to $k'$.

The Gompertz model for $\gamma(x, \beta)$ has been used by Garg et al. (1970), who have studied its properties and obtained maximum likelihood estimates of its parameters. Laird (1969 and 1965) have utilized this property the Gompertzian growth process and have presented several applications in the dynamics of normal, embryonic and tumor growth in animals and mammals. The parameter, which does not undergo any change under this operation has been called a normalizing constant.

Another family of univariate life distributions that has the setting the clock back to zero property is the General Krane family (Krane, 1963; Rao, 1990). The hazard rate for this family is a polynomial of degree $m$:

$$\gamma(x, \beta) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m.$$  \hspace{1cm} (2-6)

Some of its special cases are
i- $m=0$ gives an exponential distribution.
ii- $m=1$ gives a linear hazard exponential distribution.
iii- $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ gives a Weibull distribution.

Of course, the parameter $m$ for a Weibull distribution need not be an integer.

Rao (1990) has proved that the family of simple time to tumor distributions recently introduced by Chiang and Conforti (1989) has the setting the clock back to zero property. In their stochastic model, the hazard rate is a function of the accumulated effect of an individual's continuous exposure to toxic material in the environment (absorbing coefficient) and his biological reaction to the toxin absorbed (discharging coefficient). The life expectancy of an individual of age $x_0$ plays an important role in biometry and actuarial statistics, under the name biometric function. In general any function arising in probability modelling of lifetimes may be called; biometric function. Other examples are the force of mortality or hazard function and the conditional probability $q(x,y)$ of dying in the interval $(x,y)$ given survival up to age $x_0$, (Chiang, 1968).

The life expectancy $e_{x_0}$ of an organism at age $x_0$ is the expected remaining life of the organism. In other words, $e_{x_0}$ shows how long an organism of age $x_0$ would survive, on the average. In an application of time to tumor incidence, this would mean that if no tumor has been found until time $X_0$, the organism can have on the average an additional $e_{x_0}$ time units of tumor-free life. This is given by
Setting the Clock Back to Zero Property

\[ e_{x_0} = E(X \mid X > x_0) - x_0 \]
\[ e_{x_0} = \frac{1}{S(x_0, \beta)} \int_{x_0}^{\infty} S(x, \beta) \, dx \]
\[ e_{x_0} = \int_{0}^{\infty} \frac{S(x + x_0, \beta)}{S(x_0, \beta)} \, dx \quad (2-7) \]

If the family of survival distributions has the setting the clock back to zero property, equation (2-7) becomes

\[ e_{x_0} = \int_{0}^{\infty} S(x, \beta^*) \, dx = E(X^*) \quad (2-8) \]

For such a family of life distributions, the life expectancy is the expected life length of the organism with \( \beta^* \) being the vector of parameters.

3 Families of Bivariate Life Distributions and an Extension of the Lack of Memory Property

In what follows, we will treat the case of an organism which is expose, simultaneously to two risks \( R_1 \) and \( R_2 \) which are competing for the life of the organism. The case when more than two risks act on an organism follows on similar lines. The additional difficulty is in writing down the various equation only. Once this is done, the rest of the argument is the same.

Let us suppose that the random variables \( X \) and \( Y \) represent the hypothetical life times of the organism due to risks \( R_1 \) and \( R_2 \), respectively. Their joint survival function is defined by the equation

\[ S(x, y, \beta) = P(X \geq x, Y \geq y), \quad 0 \leq x \leq \infty, \quad 0 \leq y \leq \infty \quad (3-1) \]

where \( \beta \) is a parameter or a vector of parameters. When \( \beta \in \Omega \) is a parameter space \( \Omega \), this defines a family of bivariate life distributions \{\( S(x, y, \beta) \), \( x \geq 0, \ y \geq 0, \ \beta \in \Omega \}\}. Suppose that an organism has survived up to age \( x_0 \) for some \( x_0 > 0 \). This means that its hypothetical lifetimes satisfy \( X \geq x_0 \) and \( Y \geq x_0 \). The proportion of such organisms is given by

\[ S(x_0, x_0, \beta) = P(X \geq x_0, Y \geq x_0) \quad (3-2) \]
In cancer research problems, the function represents the proportion of organism that have enjoyed a tumor-free life of \(x_0\) time units. The bivariate form of the L.M.P. states that

\[
P(X \geq x + x_0, Y \geq y + x_0 \mid X \geq x_0, Y \geq y_0) = P(X \geq x, Y \geq y) \quad \text{(3-3)}
\]

By a simple extension of this property, we can describe the bivariate form of the setting the clock back to zero property as follows. The conditional distribution of the additional time of survival of the living organism due to risk \(R_1\) given that the organism has survived both the risks for a time of \(x_0\) units is

\[
P(Y \geq y + x_0 \mid X \geq x_0, Y \geq x_0) = \frac{S(x + x_0, x_0, \beta)}{S(x_0, x_0, \beta)} \quad \text{(3-4)}
\]

Similarly, the conditional distribution of the additional time of survival of the organism due to risk \(R_2\) given that the organism has survived both the risks for \(x_0\) units is

\[
P(Y \geq y + x_0 \mid X \geq x_0, Y \geq x_0) = \frac{S(x_0, y + x_0, \beta)}{S(x_0, x_0, \beta)}
\]

We now state the following definition: Definition (3.1): A family of bivariate life distributions, \(\{S(x,y,\beta), x \geq 0, y \geq 0, \beta \in \Omega\}\) is said to have the setting the clock back to zero property or is Said to be invariant if, for each \(\beta \in \Omega\) and \(x_0 > 0\), the survival function satisfies the two equations

\[
\frac{S(x + x_0, x_0, \beta)}{S(x_0, x_0, \beta)} = S(x, x_0, \beta^*) \quad \text{(3-6)}
\]

\[
\frac{S(x + x_0, x_0, \beta)}{S(x_0, x_0, \beta)} = S(x, x_0, \beta^{**})
\]

where \(\beta^* = \beta^*(x_0) \beta \in \Omega\) and \(\beta^{**} (x_0) \beta \in \Omega\). Here \(\beta^*\) and \(\beta^{**}\) are functions of the age \(x_0\). The two equations in (3.6) may be alternatively written as

\[
P(X \geq x + x_0 \mid X \geq x_0, Y \geq x_0) = P(X^* \geq x, Y \geq x_0)
\]
where the random variables \( X^* \) and \( Y \) have exactly the same bivariate distribution as the lifetimes \( X \) and \( Y \) (see equation 3-1), except that the parameter vector \( \beta \) is replaced by the vector \( \beta^* \). Similarly, the random variables \( X \) and \( Y^* \) have exactly the same bivariate distribution as the lifetimes \( X \) and \( Y \) (see equation 3-1), except that the parameter vector \( \beta \) is replaced by the vector \( \beta^{**} \).

This shows that the conditional distribution of the additional time of survival of the living organism due to risk \( R_1 \) (assuming that the risk \( R_2 \) has no killed it first) given that the organism has survived both the risks for \( x_0 \) time unit remains in the family. Similarly, the conditional distribution of the additional time of survival of the living organism due to risk \( R_2 \) (assuming that the risk \( R \) has not killed it first) given that the organism has survived both the risks for \( x_0 \) time units, remains in the family.

It will be shown in the next section that the life expectancy vector of an organism takes a simple form if its life distribution under the influence of competing risks has the setting the clock back to zero property.

### 4 The Life Expectancy Vector of an Organism

The life expectancy of an organism of age \( x_0 \) which is exposed to two competing risks \( R_1 \) and \( R_2 \) is represented by the vector

\[
e_{x_0} = \begin{pmatrix} e^{(1)}_{x_0} \\ e^{(2)}_{x_0} \end{pmatrix}
\]

where the components \( e^{(1)}_{x_0} \) and \( e^{(2)}_{x_0} \) are defined as follows. The component \( e^{(1)}_{x_0} \) is the expected remaining life of an organism of age \( x_0 \) due to risk \( R_1 \) that is.

\[
e^{(1)}_{x_0} = E(X - x_0 \mid X \geq x_0, Y \geq x_0)
\]

\[
e^{(1)}_{x_0} = \frac{1}{S(x_0, x_0, \beta)} \int_0^\infty \int_0^\infty f(x, y, \beta) \, dx \, dy - x_0,
\]

where \( f(x, y, \beta) \) is the density corresponding to the joint survival function \( S(x, y, \beta) \) in equation (3-1). Equation (4-2) may be simplified by noting that
This gives the first component of the life expectancy vector

\[ e^{(1)}_{x_0} = \frac{1}{S(x_0, x_0, \beta)} \int_{x_0}^{\infty} S(x, x_0, \beta) \, dx \]

\[ e^{(1)}_{y_0} = \int_{0}^{\infty} S(x + x_0, y_0, \beta) \frac{dy}{S(x_0, x_0, \beta)}. \]  (4-4)

Similarly, the second component is

\[ e^{(2)}_{y_0} = \int_{0}^{\infty} S(x_0, y + x_0, \beta) \frac{dy}{S(x_0, x_0, \beta)}. \]  (4-5)

Let us suppose now that the family of bivariate life distributions has the setting the clock back to zero property. Then using equations (3-6) the two components of the life expectancy vector become

\[ e^{(1)}_{x_0} = \int_{0}^{\infty} S(x, x_0, \beta^*) \, dx \].

\[ e^{(2)}_{y_0} = \int_{0}^{\infty} S(x_0, y, \beta^{**}) \, dy \].  (4-6)
5 Conclusions and Examples

Example (5.1): Let us consider the family of Gumbel's (1960) bivariate exponential distributions with the joint survival function

$$S(x, y, \beta) = \exp(-\lambda_1 x - \lambda_2 y - \delta xy), \ x \geq 0, \ y \geq 0 \quad (5-1)$$

and \( \beta \) is the parameter vector \( \beta=(\lambda_1, \lambda_2, \delta) \) and the parametric space is

$$\Omega = [(\lambda_1, \lambda_2, \delta) \mid \lambda_1 > 0, \lambda_2 > 0 \text{ and } \delta \geq 0]$$

We now form the ratios (see equations 3-6)

$$\frac{S(x + x_0, x_0, \beta)}{S(x_0, x_0, \beta)} = e^{-(\lambda_1 + \delta \delta_0) x} = S(x, x_0, \beta^*) \quad (5-2)$$

and

$$\frac{S(x_0, y + x_0, \beta)}{S(x_0, x_0, \beta)} = e^{-(\lambda_2 + \delta \delta_0) y} = S(x_0, y, \beta^{**}) \quad (5-3)$$

where the new parameter vectors are

$$\beta^* = (\lambda_1^* , \lambda_2^*, \delta) \in \Omega, \quad \beta^{**} = (\mu_1^*, \mu_2^*, \delta) \in \Omega, \quad \lambda_1^* = \lambda_1 + \delta \delta_0, \quad \lambda_2^* = 0, \quad \lambda_1^* = 0, \quad \text{and} \quad \mu_2^* = \lambda_2 + \delta \delta_0.$$

Thus the family of Gumbel's Bivariate exponential distributions has the setting the clock back to zero property. Then from equations (4-6) the life expectancy vector is given by

$$e_{e_0} = \left( \begin{array}{c} e_{e_0}^{(1)} \\ e_{e_0}^{(2)} \end{array} \right)$$

where
Observe that the components of the life expectancy vector depend on the age $x_0$ of the organism.

**Example (5-2):** Let us consider the family of bivariate exponential distributions (see Johnson and Koltz, 1975; Johnson et al., 1997; Johnson et al., 2000) with the survival function

$$S(x, y, \theta) = e^{-(x+y)}\left[1 + \theta(1 - e^{-x})(1 - e^{-y})\right] x \geq 0, y \geq 0, \theta > 0.$$  \hfill (5-5)

Its marginals are univariate exponential distributions. Let us consider a slightly more general form of the survival function

$$S(x, y, \theta) = e^{-(x+y)}\left[1 + \phi(1 - \alpha e^{-x})(1 - \beta e^{-y})\right] x \geq 0, y \geq 0, \phi > 0, \alpha > 0, \beta > 0,$$  \hfill (5-6)

and

$$S(x + x_0, y, \theta) = e^{-(x+x_0)}\left[1 + \phi(1 - \alpha' e^{-x})(1 - \beta e^{-y})\right]$$  \hfill (5-9)

where $\alpha' = \alpha e^{-x_0}$. We get the ratio
Setting the Clock Back to Zero Property

\[
\frac{S(x + x_0, x_0, \theta)}{S(x_0, x_0, \theta)} = S(x, x_0, \theta^*) \text{ from equation (5-7)}
\]

where \( \theta^* \) is the new parameter vector, \( \theta^* = (\phi^*, \alpha^*, \beta^*) \in \Omega \). This shows that the family of bivariate exponential distributions given in (5-2-1) has the setting the clock back to zero property. Its life expectancy vector can be found. Its first component from equation (5-7) is

\[
e_{x_0}^{(1)} = \int_{0}^{\infty} S(x, x_0, \theta^*) dx
\]

\[
e_{x_0}^{(1)} = e^{-\theta^* \alpha^*} \left[ \int_{0}^{\infty} e^{-\phi^* x} dx + \phi^* \left( 1 - \beta^* e^{-\theta^* \alpha^*} \right) \right]
\]

\[
e_{x_0}^{(1)} = e^{-\theta^* \alpha^*} \left[ 1 + \phi^* \left( 1 - \beta^* e^{-\theta^* \alpha^*} \right) - \frac{\phi^* \alpha^*}{4} \left( 1 - \beta^* e^{-\theta^* \alpha^*} \right) \right]
\] (5-10)

Similarly we can find its second component. The two components depend on the age \( x_0 \) of the organism.

**Example (5-3):** Let us consider the Marshall and Olkin family (1967; 1997) of bivariate exponential distributions with the survival function

\[
S(x, y, \theta) = \exp \{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)\}
\] (5-11)

\( \lambda_1 > 0, \lambda_2 > 0, \lambda_{12} > 0 \), and \( x \geq 0, y \geq 0 \). This gives

\[
S(x, x_0, \theta) = \exp \{-\lambda_1 x - \lambda_2 x_0 - \lambda_{12} \max(x, x_0)\}
\]

\[
S(x, x_0, \theta) = \exp \{-\lambda_1 x - \lambda_2 x_0 - \lambda_{12} x\}, \text{ since } x \geq x_0.
\] (5-12)

and

\[
S(x+x_0, x_0, \theta) = \exp \{-\lambda_1 (x+x_0) - \lambda_2 x_0 - \lambda_{12} x - \lambda_{12} x_0\}
\] (5-13)

also

\[
S(x+x_0, x_0, \theta) = \exp \{-\lambda_1 x - \lambda_2 x_0 - \lambda_{12} x_0\}
\] (5-14)
So that the ratio
\[
\frac{S(x + x_0, x_0, \theta)}{S(x_0, x_0, \theta)} = \exp\{- (\lambda_1 + \lambda_{12}) x\} = S(x, x_0, \theta^*) \tag{5-15}
\]

where \(\theta^* = (\lambda_1^*, \lambda_2^*, \lambda_{12}^*)\), and where \(\lambda_1^* = \lambda_1 + \lambda_{12}\), \(\lambda_2^* = 0\), \(\lambda_{12}^* = 0\). Similarly the ratio

\[
\frac{S(x_0, y + x_0, \theta)}{S(x_0, x_0, \theta)} = \exp\{- (\lambda_1 + \lambda_{12}) y\} = S(x_0, y, \theta^{**}) \tag{5-16}
\]

where \(\theta^{**} = (\mu_1^*, \mu_2^*, \mu_{12}^*) \in \Omega\) and \(\mu_1^* = 0\), \(\mu_2^* = \lambda_2 + \lambda_{12}\), \(\mu_{12}^* = 0\). Thus the Marshall and Olkin family of bivariate exponential distribution has the setting the clock back to zero property. Then from equation (4-6), the life expectancy vector is given by

\[
e_{x_0} = \begin{pmatrix}
e_{x_0}^{(1)} \\
e_{x_0}^{(2)}
\end{pmatrix}
\]

where the first component

\[
e_{x_0}^{(1)} = \int_0^\infty S(x, x_0, \theta) dx = (\lambda_1 + \lambda_{12})^{-1}
\]

and

\[
e_{x_0}^{(2)} = \int_0^\infty S(x_0, y, \theta^{**}) dx = (\lambda_2 + \lambda_{12})^{-1}
\]

Observe that the two components of the life expectancy vector are independent of the age \(x_0\) of the organism. This is of course, due to the bivariate lack of memory property of the family.

**Example (5-4)**: Consider a bivariate life distribution, whose joint survival function is

\[
S(x, y, \theta) = \exp(1 - e^{\alpha x + \beta y}), \ x \geq 0, \ y \geq 0 \tag{5-17}
\]
Setting the Clock Back to Zero Property

where $\theta = (\alpha, \beta) \in \Omega$, and $\Omega = \{ (\alpha, \beta) : \alpha > 0, \beta > 0 \}$. Clearly

$S(x, x_0, \theta) = \exp(1-\exp(\alpha x)) = \exp(1- \beta \cdot \exp(\alpha x))$ \hspace{1cm} (5-18)

$S(x+x_0, x_0, \theta) = \exp(1-\alpha \cdot \beta \cdot e^{\alpha x})$, $\alpha' = e^\alpha$, $\beta' = e^{\beta x}$ \hspace{1cm} (5-19)

and

$S(x_0, x_0, \theta) = \exp(1-\alpha' \beta')$ \hspace{1cm} (5-20)

This gives the

\[
\frac{S(x+x_0, x_0, \theta)}{S(x_0, x_0, \theta)} = \exp\left(\alpha' \beta (1-e^{\alpha x})\right) = S(x, x_0, \theta^*).
\]

where $\theta^* \in \Omega$. Similarly the ratio

\[
\frac{S(x_0, y+x_0, \theta)}{S(x_0, x_0, \theta)} = \exp\left(\alpha' \beta (1-e^{\alpha x})\right) = S(x_0, y, \theta^{**}),
\]

where $\theta^{**} \in \Omega$. We conclude that the family of bivariate life distributions given in equation (5-4) has the setting the clock back to zero property. Its life expectancy vector has the first component

\[
e_x^{(1)} = \int_0^\infty \exp\left(\alpha' \beta (1-e^{\alpha x})\right)dx.
\]

**Example (5-5):** Let us consider a family of Bivariate Gompertz distributions given by Elandt-Johnson (1976) equation (5-11). Its marginals are univariate Gompertz distributions with hazard rates and survival functions

$\lambda_1(x) = \frac{k_1}{a_1} e^{a_1 x}$, $S_1(x) = \exp\left[\frac{k_1}{a_1} \left(1-e^{a_1 x}\right)\right]$

$\lambda_2(x) = \frac{k_2}{a_2} e^{a_2 x}$, $S_2(x) = \exp\left[\frac{k_2}{a_2} \left(1-e^{a_2 x}\right)\right]$
where \( k_1 > 0, K_2 > 0 \) and \(-\infty < a_1 < \infty \) and \(-\infty < a_2 < \infty \). Assuming \( a_1 = a_2 = a \) and letting \( \beta = (k_1, k_2, a, \theta) \in \Omega \), the joint survival function of the Bivariat Gompertz distribution becomes

\[
S(x, y, \beta) = \exp \left\{ \frac{k_1}{a} (1 - e^{ax}) + \frac{k_2}{a} (1 - e^{ay}) - \theta \frac{a}{k_1} \frac{k_1 k_2}{a} \frac{1 - e^{ax}}{1 - e^{ay}} \right\} 
\]

(5-21)

This gives

\[
S(x, y, \beta) = \exp \left\{ \frac{k_1 + k_2}{a} (1 - e^{ax}) - \frac{\theta k_1 k_2}{a(k_1 + k_2)} (1 - e^{ax}) \right\}
\]

\[
S(x, y, \beta) = \exp \left\{ \frac{A}{a} (1 - \alpha) \right\} = e^\lambda
\]

(5-22)

where, \( A = k_1 + k_2 - \frac{\theta k_1 k_2}{k_1 + k_2} \) and \( \alpha = e^{ax} \). Also

\[
S(x, x_0, \beta) = \exp \left\{ \frac{k_1}{a} (1 - e^{ax}) + k_2' - \theta \frac{k_1 k_2}{a} \frac{1 - e^{ax}}{1 - e^{ax}} \right\}
\]

(5-23)

where \( k_2' = \frac{k_2}{a} (1 - e^{ax}) = \frac{k_2}{a} (1 - \alpha) \). This gives

\[
S(x + x_0, x_0, \beta) = \exp \left\{ \frac{k_1}{a} (1 - e^{ax}) + k_2' - \theta \frac{k_1 k_2}{a} \frac{1 - e^{ax}}{1 - e^{ax}} \right\}
\]

(5-24)

We now form the ratio
Setting the Clock Back to Zero Property

\[
\frac{S(x+x_0,x_0,\beta)}{S(x_0,x_0,\beta)} = \exp \left[ \frac{k_1}{a} (1 - e^{ax}) + k_2' - \theta \left( \frac{k_1}{a} \frac{(1 - e^{ax})}{1 - e^{ax}} + k_2' \right) \right] = S(x,x_0,\beta^*) \tag{5-25}
\]

where \( \beta^* \) is the vector of the new parameter as shown. Similarly we can show that the ratio

\[
\frac{S(x+x_0,x_0,\beta)}{S(x_0,x_0,\beta)} = S(x_0,y,\beta^{**}). \tag{5-26}
\]

Thus we conclude that the family of the bivariate Gompertz life distributions has the setting the clock back to zero property. Its life expectancy vector can be found.

**Example (5-6):** Let us consider the family of bivariate Pareto distributions having the survival function

\[
S(x,y,\theta) = \left( A + \frac{x}{\sigma_1} + \frac{y}{\sigma_2} \right)^{-\alpha}, \quad x \geq 0, \ y \geq 0, \ \sigma_1 > 0, \ \sigma_2 > 0, \ \alpha > 1, \ A > 0. \tag{5-27}
\]

where \( \theta = (A, \ \sigma_1, \ \sigma_2, \ \alpha) \in \Omega \) is a parameter vector. This gives

\[
S(x_0,x_0,\theta) = P(X \geq x_0, Y \geq x_0) = \left[ A + x_0 \left( \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) \right]^{-\alpha}
\]

\[
S(x,x_0,\theta) = \left( A + \frac{x}{\sigma_1} + \frac{x_0}{\sigma_2} \right)^{-\alpha}
\]

\[
S(x+x_0,x_0,\theta) = \left( A + \frac{x}{\sigma_1} + x_0 \left( \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) \right)^{-\alpha}
\]

We now from the ratio
\[
\frac{S(x + x_0, x_0, \beta)}{S(x_0, x_0, \beta)} = \left( A^* + \frac{x}{\sigma_1^*} + \frac{x_0}{\sigma_2^*} \right) = S(x_0, y, \theta^*).
\]

where \( \theta^* = \left( A^*, \sigma_1^*, \sigma_2^*, \alpha \right) \in \Omega \), and similarly the ratio

\[
\frac{S(x_0, y + x_0, \theta)}{S(x_0, x_0, \beta)} = S(x_0, y, \theta^{**}).
\]

where \( \theta^{**} = \left( A^{**}, \sigma_1^{**}, \sigma_2^{**}, \alpha \right) \in \Omega \). Thus the family has the setting the clock back tot zero property. The first component of the life expectancy vector is

\[
e(1)_{x_0} = \int_0^\infty S(x, x_0, \theta^*) dx
\]

\[
e(1)_{x_0} = \int_0^\infty \left( A^* + \frac{x}{\sigma_1^*} + \frac{x_0}{\sigma_2^*} \right)^{-\alpha} dx = \frac{\sigma_1^*}{(\alpha - 1) \left( A^* + \frac{x_0}{\sigma_2^*} \right)^{\alpha - 1}} \quad \text{(5-28)}
\]

Similarly the second component is

\[
e(2)_{x_0} = \frac{\sigma_1^{**}}{\left( \alpha - 1 \right) \left( A^{**} + \frac{x_0}{\sigma_2^{**}} \right)^{\alpha - 1}}. \quad \text{(5-29)}
\]

References


Setting the Clock Back to Zero Property


