ON JONES POLYNOMIALS OF GRAPHS OF
TORUS KNOTS $K(2, q)$

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Abstract

Let $K(2, q)$ be torus knot. Let $K_1 \# K_2 \# \dots \# K_n$ be connected sums of torus knots $K(2, q)$. Algorithms are given for finding the Jones polynomials of graphs of torus knot $K(2, q)$ and graphs of connected sums $K_1 \# K_2 \# \dots \# K_n$

Keywords: knot, graph, torus knot, knot graph, Jones polynomials.

1. Introduction.

In 1984, V. F. R. Jones defined a new polynomial invariant for knots or links. This dramatic discovery opened a new era in knot theory. The invariant was unexpectedly defined through operator algebra, but its combinatorial description indicated that through graph theory knot theory could most benefit from this invariant. In particular, Murasugi has been applied to knot graphs this polynomial [1],[2],[3].

A *graph* is a pair $G(V, E)$ of sets satisfying $E \subseteq [V]^2$; thus, the elements of E are 2-element subsets of V . We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$ respectively. Throughout this paper, every graph is assumed to be finite, simple and connected (See [4] for the basic terminology of graph theory). A *knot* is a simple closed curve in a space S^3 . The torus knot $K(p, q)$ of type p, q is the knot which wraps around the standard solid torus K in the longitudinal direction p times and in the meridional direction q times.

Where integers p, q are relatively prime. Thus the trefoil is $K(2, 3)$. (See [5],[6] for many standard terminologies in knot theory).

2. Knot graphs

A projection of a knot or a link on a 2-dimensional plane divides the plane into several domains. It is a frequently used method to separate these domains into two classes, white domains and black domains in the study of knot theory. Using this method, C. Bankwitz [7] introduced the notion of knot graph. Let D be a regular projection of a knot on a 2-dimensional sphere S . If D has n double points $D_1, D_2, \dots, D_n$, then it divides S into $n+2$ domains, each of which is

homeomorphic to an open disk. Now, separate these domains into two classes α and β (Fig. 1(a)).

Figure 1.

Starting with the outermost domain, we can color the domains either black or white (or inner and outer). Now, we shall color the outermost domain black (or white). We can color the domains so that neighboring domains are never the same color. Let $W_1, W_2, \dots, W_v$ be the domains of class α . Take points $c_i \in W_i$ ($i = 1, 2, \dots, v$) and connect these points by n non-intersecting arcs $d_1, d_2, \dots, d_n$ in such a way that each d_r corresponds to D_r ($r = 1, 2, \dots, n$) and c_i and c_j are connected by d_r , if and only if, W_i and W_j have a common double point D_r on their boundaries. The vertices of graph are the centers of the white domains (Figure 1(b)). The domains of class α in all can be considered as a projection of a surface spanning K which is twisted 180° at each double point of D . However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the under - and over - crossing at a crossing. To this end, in Figure 2 is shown a way of assigning to each edge of *graph* either the sign + or - .

A + sign is assigned to an edge e if the domains are colored in the manner of Figure 2(a), and - sign if they are as in Figure 2(b). A signed plane graph that has been formed by means of the above process is said to be the *graph of K* and denote it by $G(K)$. From the same -consideration about the class β , we get another graph $G'(K)$. We call this the *dual graph of $G(K)$* .

Figure 2.

3. Jones polynomial of a graph

Let K be a knot and G be its graph. We fix some notations, before we define Jones polynomial $J_G(x, y, z)$ for any finite signed graph G . Throughout this talk, a graph frequently represents the geometric realization of a finite 1-dim CW-complex in $S^3 = R^3 \cup \infty$. A vertex correspond to a 0-simplex and an edge correspond to a 1-simplex. Thus, denote $\beta_i(G)$ the i^{th} Betti number of G . $|G|$ denotes the cardinality of a set G . A graph G is said to be *signed* if either +1 or -1 is assigned to each edge. Let H be a subgraph of G . Denote $p(H)$ and $n(H)$, respectively, the number of positive and negative edges in H . A subgraph H of G has induced sign function $f_H = f_G|E(H)$. A subgraph H is a spanning subgraph if $V(H) = V(G)$. Let $S_G(r, s)$ be the set of all spanning subgraphs H such that $\beta_0(H) = r + 1$ and $\beta_1(H) = s$. Therefore is the set of all spanning trees in G .

Definition 3.1

$$J_G(x, y, z) = \sum_{r,s} \left\{ \sum_{H \in S_G(r,s)} x^{p(H)-n(H)} \right\} y^r z^s$$

where the first summation runs over all spanning subgraphs H in $S_G(r, s)$. $J_G(x, y, z)$ will be called the *Jones polynomials* of a graph G .

Proposition 3.1[1]

If G is a positive graph, then

$$J_G(x, y, z) = x^{v-1} \sum_{r,s} |S_G(r, s)| \left(\frac{y}{x} \right)^r (xz)^s$$

where $v = |V(G)|$

Example 3.1

Let $K(2,3)$ be torus knot (The trefoil knot, Figure 3(a)). G is its graph (Figure 3(b)).

Figure 3.

The spanning subgraphs of K are:

i. $G_1 =$ $I_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \text{rank} I_0 = 0, \text{rank} I_1 = 0, r = 2, s = 0$

$J_{G_1}(x, y, z) = y^2$

ii. $G_2 =$ $I_0 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}, \text{rank} I_0 = 1, \text{rank} I_1 = 0, r = 1, s = 0$

$J_{G_2}(x, y, z) = 3xy$

iii. $G_3 =$ $I_0 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \text{rank} I_0 = 1, \text{rank} I_1 = 0, r = 1, s = 0$

$J_{G_3}(x, y, z) = 3x^2$

iv. $G_4 =$ $I_0 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \text{rank} I_0 = 2, \text{rank} I_1 = 0, r = 0, s = 1$

$J_{G_4}(x, y, z) = x^3z$

The $J_G(x, y, z) = x^3z + 3x^2 + 3xy + y^2$.

Example 3.2 If G consists of only one vertex, then $J_G(x, y, z) = 1$.

Example 3.3 Jones polynomials of graphs of some $K(2, q)$ torus knots are

$$i) \quad K(2,5) \quad J_G(x, y, z) = x^5z + 5x^4 + 10x^3y + 10x^2y^2 + 5xy^3 + y^4$$

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$$ii) \quad K(2,7) \quad J_G(x, y, z) = \frac{x^7z + 7x^6 + 21x^5y + 35x^4y^2 + 35x^3y^3 + 21x^2y^4 + 7xy^5 + y^6}{126x^4y^4 + 84x^3y^5 + 36x^2y^6 + 9xy^7 + y^8}$$

$$iii) \quad K(2,9) \quad J_G(x, y, z) = \frac{x^9z + 9x^8 + 36x^7y + 84x^6y^2 + 126x^5y^3 + 126x^4y^4 + 84x^3y^5 + 36x^2y^6 + 9xy^7 + y^8}{126x^4y^4 + 84x^3y^5 + 36x^2y^6 + 9xy^7 + y^8}$$

Proposition 3.2 For $q = 1, 3, 5, \dots$, Let $K(2, q)$ be torus knot and G also be its graph. Then,

$$J_G(x, y, z) = x^n z + \sum_{k=1}^n \binom{n}{k} x^{n-k} y^{k-1}$$

where $n = |E(G)|$. [8]

Proof. By Induction.

Proposition 3.3 Let $K_1, K_2, \dots, K_n$ be $K(2, q)$ torus knots and also $K_1 \# K_2 \# \dots \# K_n$ be its connected sums. If the graph of knot $K_1 \# K_2 \# \dots \# K_n$ is G , then

$$J_G(x, y, z) = y^{n-1} \prod_{i=1}^n J_{G_i}(x, y, z)$$

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