Representations of the group inverse of some 2×2 block matrices

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Abstract

Suppose that P is a given idempotent matrix over the field of complex numbers, P^* is the transpose conjugate of P . Let A, B, C be the matrices from the set $\{P, P^*, PP^*\}$. We can make some 2×2 block matrices of the form $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. For example, M may be $\begin{bmatrix} PP^* & P \\ P & 0 \end{bmatrix}$, $\begin{bmatrix} P & P \\ PP^* & 0 \end{bmatrix}$, $\begin{bmatrix} PP^* & PP^* \\ P & 0 \end{bmatrix}$, $\begin{bmatrix} P & P \\ P^* & 0 \end{bmatrix}$, $\begin{bmatrix} P & PP^* \\ PP^* & 0 \end{bmatrix}$, $\begin{bmatrix} P & PP^* \\ P^* & 0 \end{bmatrix}$ or $\begin{bmatrix} P^* & P \\ P & 0 \end{bmatrix}$. In this paper, we prove that the group inverse of every one of the above seven matrices exists, and give the representations of those group inverse.

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1 Introduction and examples

The research on the generalized inverses of block matrices is important for many applied areas and the computations of generalized inverses. The 2×2 block matrix of the form $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ plays an important role in the problems on optimization solution of differential equations (see[2] and [4]). There is no

known representation for the Drazin inverse of any 2×2 block matrix of the form $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. This is an open research problem proposed by Campbell in [2]. At the present time, peoples only give the representation for the Drazin inverse of such block matrix in some especial cases (see[3]). In this paper, we consider some new special cases on the kind of such block matrix. Suppose that P is a given idempotent matrix over the field of complex number, P^* is the transpose conjugate of P . Let A, B, C be the matrices from the set $\{P, P^*, PP^*\}$. In this paper, we study seven cross-sectional matrices of the form $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. We not only point out their group inverse exists, but the representations for those group inverse are also given.

In this paper, we denote by $\mathbb{C}^{m \times n}$ the linear space of all $m \times n$ matrices over the field of complex numbers $\mathbb{C}$. Let $P_n(\mathbb{C})$ be the set of all idempotent matrices in $\mathbb{C}^{n \times n}$. We denote by I_k the $k \times k$ identity matrix. For $A \in \mathbb{C}^{n \times n}$, the matrix $X \in \mathbb{C}^{n \times n}$ is said to be the group inverse of A if it hold that

$$AX = XA, \quad AXA = A, \quad XAX = X.$$

It is well known that if the group inverse of a complex square matrix exists then it is unique (see [1]). Denote the group inverse of A by $A^\#$.

We state below two lemmas that we will need in the proof of our main results.

Lemma 1.1 (see [1]) *Suppose that $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m}$. If the group inverses of AB and BA exist, then the following conditions hold.*

- (i) $(AB)^\# = A((BA)^\#)^2 B$;
- (ii) $(AB)^\# A = A(BA)^\#, A(BA)^\# B = (AB)^\# AB$;
- (iii) $AA^*(AA^*)^\# A = A, A^*(AA^*)^\# AA^* = A^*$.

Lemma 1.2 *Let $P \in P_n(\mathbb{C})$. Then $(PP^*)^\#$ and $(P - P^*)^\#$ exist and*

- (i) *there is a unitary matrix U such that*

$$P = U \begin{bmatrix} I_r & F \\ 0 & 0 \end{bmatrix} U^*, \quad (1)$$

$$PP^* = U \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad N = I_r + FF^*, \quad (2)$$

$$(P - P^*)^\# = U \begin{bmatrix} 0 & -F(F^*F)^\# \\ (F^*F)^\# F^* & 0 \end{bmatrix} U^*. \quad (3)$$

- (ii) $P(PP^*)^\# = (PP^*)^\# = (PP^*)^\# P^*$.

Proof (i) Firstly, by Theorem 2.3.1 of [5] we have that there is a unitary matrix U such that

$$P = U \begin{bmatrix} D & F \\ 0 & E \end{bmatrix} U^*,$$

where D is an $r \times r$ invertible upper triangular matrix, E is a upper triangular matrix which all diagonal elements are equal to zero's. It follows from $P^2 = P$ that $D = D^2$ and $E = E^2$. Hence we have $D = I_r$ and $E = 0$. This means that (1) holds. Furthermore, it is easy to get the conclusion (2).

Since PP^* and $P - P^*$ are Hermitian and skew-Hermitian matrices respectively, we see that their group inverse exist. Now let us prove the conclusion (3). In fact, we firstly by (1) have that

$$P - P^* = U \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} U^*.$$

Let $X = \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix}$ and $M = \begin{bmatrix} 0 & -F(F^*F)^\# \\ (F^*F)^\#F^* & 0 \end{bmatrix}$. By Lemma 1.1 and the direct computation, we have

$$MX = XM = \begin{bmatrix} FF^*(FF^*)^\# & 0 \\ 0 & FF^*(FF^*)^\# \end{bmatrix}.$$

It is easy to check that $MXM = M$ and $XM = X$. This implies that (3) holds.

(ii) The conclusion is easily proved. The proof is omitted.

2 Main Results

Theorem 2.1 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} PP^* & P \\ P & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} PP^*(I_n - P) & P \\ (PP^*)^2(P - I_n) + P & -PP^*P \end{bmatrix}. \quad (4)$$

Proof Denote by X the right-hand side matrix in the equation (4). Note that $P^2 = P$, we have

$$MX = XM = \begin{bmatrix} P & 0 \\ PP^*(I_n - P) & P \end{bmatrix}.$$

One can check that $MXM = M$ and $XM = X$. Thus, $M^\# = X$.

Theorem 2.2 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} P & P \\ P^* & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} (PP^*)^\#(P - I_n) & (PP^*)^\#P \\ PP^*(PP^*)^\# - (PP^*)^\#(P - I_n) & -(PP^*)^*P \end{bmatrix}. \quad (5)$$

Proof Let X be the right-hand side matrix in the equation (5). By $P^2 = P$, (iii) of Lemma 1.1 and the definition of group inverse, we have

$$MX = XM = \begin{bmatrix} PP^*(PP^*)^\# & 0 \\ P - PP^*(PP^*)^\# & P \end{bmatrix},$$

Furthermore, we see that $MXM = M$ and $XM = X$. Thus, $M^\# = X$.

Theorem 2.3 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} PP^* & PP^* \\ P & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} -P + PP^*(PP^*)^\# & PP^*(PP^*)^\# \\ P - PP^*(PP^*)^\# + (PP^*)^\#P & -PP^*(PP^*)^\# \end{bmatrix}. \quad (6)$$

Proof Denote by X the right-hand side matrix in the equation (6). By the definition of the group inverse and Lemmas 1.1 and 1.2, we have

$$MX = XM = \begin{bmatrix} P & 0 \\ PP^*(PP^*)^\# - P & PP^*(PP^*)^\# \end{bmatrix}.$$

Meanwhile, by the direct computation we have that $MXM = M$ and $XM = X$. So, (6) holds.

Theorem 2.4 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} P & P \\ P^* & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} (PP^*)^\#(P - I_n) & (PP^*)^\#P \\ P^*(PP^*)^\# + P^*((PP^*)^\#)^2(I_n - P) & -(P^*P)^\# \end{bmatrix}. \quad (7)$$

Proof Let X be the right-hand side matrix in the equation (7). It is not difficult to check by (ii) and (iii) of Lemma 1.1, (ii) of Lemma 1.2 and the fact $P^2 = P$ that

$$MX = XM = \begin{bmatrix} PP^*(PP^*)^\# & 0 \\ P^*(PP^*)^\#(P - I_n) & P^*(PP^*)^\#P \end{bmatrix}.$$

Furthermore, one can conclude that $MXM = M$ and $XM = X$. This means that (7) holds.

Theorem 2.5 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} P & PP^* \\ PP^* & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} ((PP^*)^\#)^2(P - I_n) & (PP^*)^\# \\ ((PP^*)^\#)^3(I_n - P) + (PP^*)^\# & -((PP^*)^\#)^2 \end{bmatrix}. \quad (8)$$

Proof Denote by X the right-hand side matrix in the equation (8). Since $P = P^2$, by the definition of group inverse and Lemmas 1.1 and 1.2, one can conclude that

$$MX = XM = \begin{bmatrix} PP^*(PP^*)^\# & 0 \\ (PP^*)^\#(P - I_n) & PP^*(PP^*)^\# \end{bmatrix}.$$

Furthermore, we have by the direct computation that $MXM = M$ and $XXM = X$. The proof is completed.

Theorem 2.6 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} P & PP^* \\ P^* & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} PP^*(P - I_n) & PP^*(PP^*)^\# \\ P^*(PP^*)^\# - (P^*P)^\# + P^*((PP^*)^\#)^2 & -P^*(PP^*)^\# \end{bmatrix}. \quad (9)$$

Proof Let X be the right-hand side matrix in the equation (9). It is easy to check by Lemmas 1 and 2 that

$$MX = XM = \begin{bmatrix} PP^*(PP^*)^\# & 0 \\ P^*(PP^*)^\#(P - I_n) & P^* \end{bmatrix}.$$

Furthermore, we have $MXM = M$ and $XXM = X$. This complete the proof.

Theorem 2.7 Suppose that $P \in P_n(\mathbb{C})$ and $M = \begin{bmatrix} P^* & P \\ P & 0 \end{bmatrix}$. Then $M^\#$ exists and

$$M^\# = \begin{bmatrix} ((P - P^*)^\#)^2(I_n - P) & (P - P^*)^\#P^*P + P \\ PP^*(PP^*)^\#[I_n - ((P - P^*)^\#)^2(I_n - P)] & -P^*(P - P^*)^\#P^*P - P^*P \end{bmatrix}. \quad (10)$$

Proof Denote the right-hand side matrix in the equation (10) by $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_i \in \mathbb{C}^{n \times n}$, $i = 1, 2, 3, 4$. Since $P^2 = P$, we see by lemma

1.2 that there is a unitary matrix U such that (1), (2) and (3) hold. Meanwhile, by Lemmas 1.1 and 1.2, we have

$$X_1 = U \begin{bmatrix} 0 & (F^*F)^\# F \\ 0 & -(F^*F)^\# \end{bmatrix} U^*,$$

$$X_2 = U \begin{bmatrix} I_n - F(F^*F)^\# F^* & 0 \\ (F^*F)^\# F^* & (F^*F)^\# F^* F \end{bmatrix} U^*,$$

$$X_3 = U \begin{bmatrix} I_n & -(F^*F)^\# F \\ 0 & 0 \end{bmatrix} U^*$$

and

$$X_4 = U \begin{bmatrix} I_n - F(F^*F)^\# F^* & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

So, we can rewrite M and X as following:

$$M = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} I_r & 0 & I_r & F \\ F^* & 0 & 0 & 0 \\ I_r & F & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix},$$

$X =$

$$\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & F(F^*F)^\# & I_r - F(F^*F)^\# F^* & 0 \\ 0 & -(F^*F)^\# & (F^*F)^\# F^* & (F^*F)^\# F^* F \\ I & -F(F^*F)^\# & -I_r + F(F^*F)^\# F^* & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix}.$$

This, together with Lemmas 1.1 and 1.2, gives that

$$MX = XM = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & (F^*F)^\# F^* F & 0 & 0 \\ 0 & 0 & I_r & F \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix}.$$

Then, it is easy to check that $MXM = M$ and $XM = X$. The proof is completed.

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