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F-rpp semigroups

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Abstract

The investigation of F-rpp semigroups is initiated. After obtaining some important properties of such semigroups, the structure of a class of F-rpp semigroups is established. As an application, we give a new structure of F-regular semigroups and strongly F-abundant semigroups. Thus, the results obtained by Edwards and Guo are extended and enriched.

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1 Introduction

An inverse semigroup S is called *F-inverse* if there exists a group congruence σ on S such that every σ-class of S contains a greatest element with respect to the natural partial order $\leq$ on S. F-inverse semigroups were first studied by McFadden and O'Carroll [10] in 1972. They showed that the notion of F-inverse semigroups is a generalization of residuated inverse semigroups introduced by Blyth [1]. Later on, Edwards [2] has investigated the regular semigroups with the same property of F-inverse semigroups and called them the *F-regular semigroups*. She also defined analogously the F-orthodox semigroups and showed that every F-regular semigroup is in fact an F-orthodox semigroup. She gave a complete characterization of any F-orthodox semigroup in terms of its band of idempotents E, its maximal group homomorphic image, and a particular set of endomorphisms of E.

In generalizing the orthodox semigroups, Fountain [3] considered the starred Green relations $\mathcal{L}^*$ and $\mathcal{R}^*$ instead of weakening the regularity of a semigroup. For elements a, b of a semigroup S, define $a\mathcal{R}^*b$ if and only if for all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$. $\mathcal{L}^*$ is dually defined. Fountain [4] then called a semigroup S *abundant* if each class of $\mathcal{R}^*$ and $\mathcal{L}^*$ contains an idempotent of S. Also, for an abundant semigroup S, define $a \leq b$ if and only if for some $e, f \in E(S)$, $a = eb = bf$. Recently, Guo [7] called an abundant semigroup S *F-abundant* if for each $a \in S$, the congruence class of the minimal cancellative congruence σ containing a has a greatest element m_a with respect to $\leq$. By utilizing a cancellative monoid M and a band B with identity, Guo [7] constructed a semigroup, called an SF-system, on a subset of $M \times B$. His construction of SF-system involved two families of endomorphisms on B. In this way, he gave a generalization of F-regular semigroups in the class of abundant semigroups.

Although, Guo [7] has provided a complete description for strongly F-abundant semigroups by using SF-systems, his method can not be directly applied to construct F-regular semigroups. The main handle is similar to the case, as stated in Edwards [2], that F-inverse semigroups are in general not extendable to F-orthodox semigroups. In order to overcome the difficulties, we introduce more general SF-system, namely, an SFR-system and the concept of F-rpp semigroups. Then, as special cases of SFR-systems, we can re-obtain the methods of construction for F-regular semigroups and strongly F-abundant semigroups.

By an *rpp semigroup*, we mean every principal right ideal of S is projective. Fountain ([5] and [6]) has shown that a semigroup S is rpp if and only if every $\mathcal{L}^*$-class of S contains at least one idempotent of S. This is equivalent to say that a semigroup S is rpp if for all $a \in S$, the left ideal of the form aS^1 of S, regarded as a right S^1-system, is projective (see [5]). Thus, an rpp semigroup is clearly a generalized abundant semigroup. As a generalization of F-abundant semigroups, we call an rpp semigroup S an *F-rpp semigroup* if there exists a left cancellative monoid congruence ρ on S such that each ρ-class contains a greatest element with respect to Lawson partial order $\leq_\ell$ on S.

In this paper, we shall establish a structure theorem for strongly F-rpp semigroups by using SFR-systems. As an application of our result, we consider some special cases and obtain the structure theorems for F-regular semigroups and strongly F-abundant semigroups. Thus, the results obtained by Edwards in [2] and Guo in [7] are extended.

In an abundant semigroup S, we usually denote the idempotent in $L_a^* \bigcap E(S)$ by a^* and the ideampotent in $R_a^* \bigcap E(S)$ by $a^\dagger$. The reader is referred to Howie [8] and Guo [7] for other notations and terminology not given in this paper.

2 Preliminaries

We first cite some useful definitions, results on rpp semigroups and abundant semigroups.

Definition 2.1 *Let x, y be elements of a rpp semigroup S. Define the following Lawson relation $\leq_\ell$ on S by*

> *$x \leq_\ell y$ if and only if $L^*(x) \subseteq L^*(y)$ and there exists $f \in E(S) \bigcap L_x^*$ such that $x = yf$,*

*where $L^*(x)$ is the left *-ideal generated by x (see Fountain [3]).*

By using the same arguments as in Lawson [9] (see Lemma 2.1, Lemma 2.2, Proposition 2.5 and Proposition 2.7 in [9]), we have the following lemmas.

Lemma 2.2 *Let S be an rpp semigroup. For $x, y \in S$ and $e \in E(S)$. Then the following statements holds:*

(i) $\leq_\ell$ is a partial order on S, in particular, $\leq_\ell$ coincide with the usual idempotent order ω on $E(S)$, that is, $e\omega f$ if and only if $e = ef = fe$.

(ii) if $x \leq_\ell e$, then $x^2 = x$ in S.

(iii) if $x \leq_\ell y$ and y is a regular element in S, then x is also a regular element in S.

(iv) Let $y^* \in L_y^* \bigcap E(S)$ and $\omega(y^*) = \{f : f\omega y^*\}$. Then $x \leq_\ell y$ if and only if for all y^*, there exists $f \in \omega(y^*)$ such that $x = yf$.

(v) If $x \leq_\ell y$ and $x\mathcal{L}^*y$, then $x = y$.

Lemma 2.3 [7] *Let S be an abundant semigroup. Then the following statements are equivalent:*

(1) S is idempotent-connected (IC);

(2) for all $a \in S$, two conditions hold:

(i) for some [all] $a^* \in L_a^* \bigcap E(S)$ and for all $e \in \omega(a^*)$, there exists $b \in S$ ($b \in E(S)$) such that $ae = ba$;

(ii) for some [all] $a^\dagger \in R_a^* \bigcap E(S)$ and for all $f \in \omega(a^\dagger)$, there exists $c \in S$ ($c \in E(S)$) such that $fa = ac$.

Definition 2.4 *A congruence ρ on a semigroup S is called a left cancellative monoid congruence if S/ρ is a left cancellative monoid.*

If ρ is a left cancellative monoid congruence, mentioned in the definition of F-rpp semigroup, on an F-rpp semigroup S, then, by the definition, each ρ_x, the ρ-class of S containing x, contains a greatest element, namely, m_x. Now we let $M_S = \{m_x : x \in S\}$. Clearly, $\rho_x = \{y = m_x f : f \in E(S)\}$ and the set is contained in the σ-class of S containing x, where σ is a left cancellative monoid congruence on S. Thus ρ is the smallest left cancellative monoid congruence on S.

Define a multiplication $m_x \star m_y = m_{xy}$ on M_S. Then we can easily to check that $(M_S, \star)$ is indeed isomorphic to S/ρ. (Note : M_S is however not necessarily a left cancellative semigroup under the semigroup multiplication!)

We now study the properties of F-rpp semigroups.

Proposition 2.5 *Let S be an F-rpp semigroup. Then the following properties hold:*

(i) for all $m \in M_S$ and $e \in E(S)$, there exists $f \in E(S)$ such that $em = mf$;

(ii) $E(S)$ is a band with identity;

(iii) for all $m \in M_S$ and m^*, $m^*E(S) \subseteq E(S)m^*$.

Proof: (i) Observe that em and m are in the same ρ-class, we have $em \leq_\ell m$ and hence, by Lemma 2.2, $em = mf$, as required.

(ii) Obviously, $E(S)$ is contained in the same ρ-class. Let m be the greatest element of this ρ-class. If $e \in E(S)$ and $e\mathcal{L}^*m$, then $e \leq_\ell m$ and so, by Lemma 2.2 (v), $e = m$, that is, $m \in E(S)$. Notice that $ef\rho m$ for all $e, f \in E(S)$, we have $ef \leq_\ell m$ and, again by Lemma 2.2, $ef \in E(S)$. Hence $E(S)$ is a band. On the other hand, since $e \leq_\ell m$, we have $e\omega m$ and so $e = em = me$. Hence, m is the identity of $E(S)$.

(iii) For all $e \in E(S)$, we have $me\sigma m$ so that there exists $f \in \omega(m^*)$ such that $me = mf$. This leads to $m^*e = m^*f = f = fm^*$ since $m\mathcal{L}^*m^*$. Thus $m^*E(S) \subseteq E(S)m^*$. □

Proposition 2.6 *Let S be an F-rpp semigroup. Then S is an F-abundant semigroup if and only if S is an idempotent-connected (IC) abundant semigroup.*

Proof: It is clear that an F-abundant semigroup is always an IC quasi-adequate semigroup ([7, Proposition 2.2]). Hence, we only need to prove the converse part. For this purpose, we suppose that S is an IC abundant semigroup and an F-rpp semigroup with a left cancellative monoid congruence ρ on S such that each ρ-class contains a greatest element with respect to $\leq_\ell$. Then $\leq_\ell = \leq$ by [9, Theorem 2.6]. Now let $a, b, c \in S$ and $(ba)\rho = (ca)\rho$. Then there exist $e, f \in \omega(m^*_{ab})$ such that $ba = m_{ba}e$ and $ca = m_{ba}f$. It now follows that S is IC that $ba = gm_{ba}$, for some $g \in E(S)$, since S is IC. Hence, we have

$$(gb)af = gm_{ba}f = (gc)af$$

so that $gb(af)^\dagger = gc(af)^\dagger$ since $af\mathcal{R}^*(af)^\dagger$, where $(af)^\dagger \in R^*_{af} \bigcap E(S)$. Notice that $e\rho$ is the identity of S/ρ, we observe that $(af)^\dagger\rho = g\rho$ is the identity of S/ρ, and hence we have

$$b\rho = ((g\rho)(b\rho)(af)^\dagger\rho) = ((g\rho)(c\rho)(af)^\dagger\rho) = c\rho.$$

This shows that ρ is a right cancellative congruence on S. Therefore ρ is a cancellative congruence. In other words, we have that each ρ-class of S contains a greatest element with respect to $\leq$. This proves that S is an F-abundant semigroup. □

Clearly, a left cancellative monoid M is an F-rpp semigroup but M is in general not an F-abundant semigroup. However, we can see that a cancellative monoid is an F-abundant semigroup but not a F-regular semigroup. Thus, we have the following relations:

Corollary 2.7 $\{$ *F−regular semigroups*$\} \subset \{$ *F−abundant semigroups*$\} \subset \{$ *F − rpp semigroups*$\}$.

Recall from [9] that $\leq_\ell = \leq$ (where $a \leq b \Leftrightarrow a = eb = fb$ for some $e, f \in E(S)$) in an IC abundant semigroup. Since every regular semigroup is an IC abundant semigroup [9], we can easily see that an F-regular semigroup is always an F-rpp semigroup. Also, by [7, p.155], we see that in an F-regular semigroup S, $E(S)m^* = m^*E(S)$ for all $m \in M$ and m^*. However, in general, we do not know whether $E(S)m^*_a = m^*_aE(S)$ in an F-rpp semigroup (F-abundant semigroup). We make the following definition.

Definition 2.8 An F-rpp semigroup S is called *strongly F-rpp* if for all $a \in S$, $E(S)m^*_a = m^*_aE(S)$.

Clearly, F-regular semigroups are strongly F-abundant semigroups (for strongly F-abundant semigroup, see [7]). And it is clear that a left cancellative monoid is strongly F-rpp and a cancellative monoid is strongly F-abundant. Thus, we have the following relations.

Proposition 2.9 *{ F-regular semigroups }* $\subset$ *{ strongly F-abundant semigroups}* $\subset$ *{ strongly F-rpp semigroups }.*

The following result is useful in the sequel.

Lemma 2.10 [7] *Let x be an element of a band B. Then $xB = Bx$ if and only if x is central in B.*

3 Strongly F-rpp semigroups

In this section, we will establish a structure theorem for strongly F-rpp semigroups. To begin with, we give some properties of strongly F-rpp semigroups.

Proposition 3.1 *Let S be a strongly F-rpp semigroup. Then*

(1) $|L^*_m \bigcap E(S)| = 1$, *for all* $m \in M$;

(2) for all $x \in S$, there exists a unique idempotent $f_x \in \omega(m^*_x)$ such that $x = m_x f_x$. In this case, $f_x \mathcal{L}^* x$.

Proof: (1) If $f \in L^*_m \bigcap E(S)$, then $f\mathcal{L}m^*$ so that $f = m^*f$ and $m^* = m^*f$. By Lemma 2.10, m^* is central in $E(S)$ for all $m \in M$ and m^*. Thus $f = m^*f = fm^* = m^*$.

(2) By Lemma 2.2, there exists $f \in \omega(m^*_x)$ such that $x = m_x f$. Now let $p, q \in S^1$ and $xp = xq$. Then $m_x fp = m^* fq$ and $m^*_x fp = m^*_x fq$. It follows that $fp = fq$ since $f \in \omega(m^*_x)$. Together with $x = xf$, we see that $x\mathcal{L}^* f$.

Assume that $g \in \omega(m^*_x)$ such that $x = m_x g$. Then $m_x g = x = m_x f$ and so $m^*_x g = m^*_x f$. Hence $g = f$. Thus (2) holds. This completes the proof. □

Definition 3.2 *Let M be a left cancellative monoid with identity* 1*, and E a band with identity e. Let*

$$\mathcal{C} : M \to E;\ m \mapsto c_m$$

be a mapping of M into E satisfying the following conditions:

(C1) for all $m \in M$, c_m is central in E;

(C2) $c_1 = e$.

Let

$$\mathcal{P} : M \times M \to E; \quad (m,n) \mapsto p_{m,n}$$

be a mapping of $M \times M$ *into* E *such that*

(P1) for all $m \in M$, $p_{1,m} = p_{m,1} = c_m$;

(P2) for all $m, n \in M$, $p_{m,n} = p_{m,n}p_{1,n} \in \omega(c_{mn})$.

Denote by $End(E)$ *the set of endomorphisms of* E *into itself. Define*

$$\Phi : M \to End(E) \quad by \quad m \mapsto \phi_m,$$

where

$$\phi_m : E \to E \quad is \quad defined \quad by \quad x \mapsto x\phi_m$$

satisfying the conditions below:

(PH1) ϕ_1 *is the identity mapping on* E;

(PH2) for all $m \in M$ *and* $x \in E$, $x\phi_m \in \omega(c_m)$.

Then, we call the above quadruple $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ *an SFR-system if the following conditions are satisfied:*

(SFR1) for all $x \in E$ *and* $m, n \in M$, $p_{m,n}(x\phi_m\phi_n) = (x\phi_{mn})p_{m,n}$;

(SFR2) for all $m, n \in M$ *and* $x \in \omega(c_m), y, z \in \omega(c_n)$

$$p_{m,n}(x\phi_n)y = p_{m,n}(x\phi_n)z \Rightarrow (x\phi_n)y = (x\phi_n)z;$$

(SFR3) for all $m, n, t \in M$, $p_{m,nt}p_{n,t} = p_{mn,t}(p_{m,n}\phi_t)$.

Now, for a given SFR-system $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$, we put

$$SFR = SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi) = \{(m,x) \in M \times E : x \in \omega(c_m)\}.$$

Then, we define a multiplication in the SFR-system by

$$(m,x) \circ (n,y) = (mn, p_{m,n}(x\phi_n)y).$$

Since $p_{m,n} \in \omega(c_{mn})$ and c_{mn} is central in E, we observe that $p_{m,n}(x\phi_n)y \in \omega(c_{mn})$ and so $(m,x) \circ (n,y) \in SFR$. Thus $\circ$ is well-defined.

We now have the following lemmas.

Lemma 3.3 $(SFR, \circ)$ *is a semigroup.*

Proof: Let $(m,x),(n,y),(t,z)\in SFR$. Then, by computing, we have

$$\begin{aligned}[(m,x)\circ(n,y)]\circ(t,z) &= (mn,p_{m,n}(x\phi_n)y)\circ(t,z)\\ &= (mnt,p_{mn,t}(p_{m,n}(x\phi_n)y)\phi_t)z)\\ &= (mnt,p_{mn,t}(p_{m,n}\phi_t)(x\phi_n\phi_t)(y\phi_t)z)\\ &= (mnt,p_{m,nt}p_{n,t}(x\phi_n\phi_t)(y\phi_t)z)\end{aligned}$$

and

$$\begin{aligned}(m,x)\circ[(n,y)\circ(t,z)] &= (m,x)\circ(nt,p_{n,t}(y\phi_t)z)\\ &= (mnt,p_{m,nt}(x\phi_{nt})p_{n,t}(y\phi_t)z)\\ &= (mnt,p_{m,nt}p_{n,t}(x\phi_n\phi_t)(y\phi_t)z).\end{aligned}$$

This proves that $[(m,x)\circ(n,y)]\circ(t,z)=(m,x)\circ[(n,y)\circ(t,z)]$. Thus $(SFR,\circ)$ is a semigroup. □

Lemma 3.4 *$E(SFR)=\{(1,x)\in SFR: x\in E\}$ which is isomorphic to E. Moreover, $(1,e)$ is the identity element of $E(SFR)$.*

Proof: Let $(m,x)\in E(SFR)$. Then we have

$$(m^2,p_{m,m}(x\phi_m)x)=(m,x)^2=(m,x)$$

and so $m^2=m$. Hence $m=1$ since M is a left cancellative monoid. On the other hand, we have

$$(1,x)^2=(1,p_{1,1}(x\phi_1)x)=(1,x).$$

This leads to $E(SFR)=\{(1,x)\in SFR: x\in E\}$. It is easy to see that

$$\theta: E(SFR)\to E; (1,x)\mapsto x$$

is an isomorphism. □

Lemma 3.5 *In the semigroup SFR, $(m,x)\mathcal{L}^*(1,x)$, and hence SFR is an rpp semigroup.*

Proof: Let $(n,y),(t,z)\in SFR^1$ and $(m,x)(n,y)=(m,x)(t,z)$. Then we have

$$(mn,p_{m,n}(x\phi_n)y)=(mt,p_{m,t}(x\phi_t)z)$$

and so $mn=mt, p_{m,n}(x\phi_n)y=p_{m,t}(x\phi_t)z$. Thereby, we deduce that $n=t$ since M is a left cancellative monoid. From (SFR2), the latter formula yields that $(x\phi_n)y=(x\phi_t)z$, that is, $p_{1,n}(x\phi_n)y=p_{1,t}(x\phi_t)z$ since $y,z\in\omega(c_n)$. Thus, we have

$$(1,x)(n,y)=(n,p_{1,n}(x\phi_n)y)=(t,p_{1,t}(x\phi_t)z)=(1,x)(t,z).$$

This equality, together with the fact that

$$(m,x)(1,x)=(m,p_{m,1}(x\phi_1)x)=(m,c_mxx)=(m,x),$$

yield that $(m,x)\mathcal{L}^*(1,x)$, as required. □

Lemma 3.6 *Let $(m,x),(n,y)\in SFR$. Then $(m,x)\leq_\ell (n,y)$ if and only if $m=n$ and $x\omega y$.*

Proof: If $(m,x)\leq_\ell (n,y)$, then, by Lemma 2.2, we have $(1,z)\omega(1,y)$ such that $(m,x)=(n,y)(1,z)$. Hence we have

$$(m,x)=(n,p_{n,1}(y\phi_1)z)=(n,c_nyz)=(n,yz)$$

and so $m=n, x=yz$. On the other hand, we have $(1,z)\omega(1,y)$ if and only if $(1,z)=(1,z)(1,y)=(1,y)(1,z)$. This leads to $z=zy=yz$, that is, $z\omega y$. But since $x=yz$, we have $x=z\omega y$.

Conversely, assume that $m=n$ and $x\omega y$. Then, by the above proof, we see that $(1,x)\omega(1,y)$. Since $(m,x)=(n,y)(1,x)$, by Lemmas 2.2 and 3.5, we obtain that $(m,x)\leq_\ell (n,y)$. This completes the proof. □

Lemma 3.7 *The relation ρ defined by*

$$(m,x)\rho(n,y)\ if\ and\ only\ if\ m=n$$

is a left cancellative monoid congruence on the semigroup SFR.

Proof: It can be easily checked that ρ is a congruence on SFR. We still need to verify that SFR/ρ is left cancellative. However, this follows from the fact that the semigroup SFR/ρ is isomorphic to M. □

Proposition 3.8 *The semigroup SFR is a strongly F-rpp semigroup.*

Proof: It is clear that the ρ-class $\rho_{(m,x)}$ containing (m,x) is the set $\{(m,y)\in SFR : y\in E\}$. Hence, it follows from Lemma 3.6 that (m,c_m) is the greatest element in $\rho_{(m,x)}$ with respect to $\leq_\ell$. Thus SFR is an F-rpp semigroup.

Let $(m,x)\in SFR$. Then, by using the above proof, we have $m_{(m,x)}=(m,c_m)$ and, by Lemma 3.5, $m^*_{(m,x)}=(1,c_m)$. On the other hand, $(1,c_m)$ is central in $E(SFR)$ since c_m is central in E. Hence, it follows from Lemma 2.10 that $m^*_{(m,x)}E(SFR)=E(SFR)m^*_{(m,x)}$. Thus SFR is a strongly F-rpp semigroup. □

In the rest of this section, we will prove that any strongly F-rpp semigroup is isomorphic to some $SFR(M,E;\mathcal{C},\mathcal{P};\Phi)$. For the sake of simplicity, we always assume that S is a strongly F-rpp semigroup with a left cancellative monoid congruence ρ on S such that each ρ-class has a greatest element, and E is the idempotent band of S. We use $(M,\star)=(M_S,\star)$ to denote the left cancellative monoid with identity 1 consisting of the greatest elements in all ρ-classes of S (in the sense of Section 2). We also use $e=1$ to denote the identity of E.

By Proposition 3.1, we see that m^* is central in E for all $m \in M$. We now see that

$$\mathcal{F}: \ M \to E; \quad m \mapsto d_m = m^*$$

is a mapping of M into E which satisfies conditions (C1) and (C2).

Let $m, n \in M$. Since $mn \ \rho \ m \star n$, by Proposition 3.1, there exists a unique idempotent $f_{mn} \in \omega(d_{m\star n})$ such that $mn = m \star n f_{mn}$ and $f_{mn}\mathcal{L}^* mn$. Define

$$\mathcal{Q} : M \times M \to E; \quad (m, n) \mapsto q_{m,n} = f_{mn}.$$

Then, the mapping $\mathcal{Q}$ has the following properties:

(i) Since $m \star 1 = m$ and $m = mm^*$, we have $m = (m \star 1)m^*$ and, by the uniqueness of $q_{m,n}$, we have $q_{m,1} = m^* = d_m$. Similarly, we have $q_{1,m} = d_m$. This shows that $\mathcal{Q}$ satisfies (P1);

(ii) Observe that $mn = mnn^* = (m \star n)(q_{m,n} d_n)$ and $q_{m,n} \in \omega((m \star n)^*)$, we have that $q_{m,n} d_n \in \omega((m \star n)^*)$. Hence $q_{m,n} = q_{m,n} d_n = q_{m,n} q_{1,n}$ by the uniqueness of $q_{m,n}$. This shows that $\mathcal{Q}$ satisfies (P2).

Now let $m \in M$ and $e \in E$. Then, by Proposition 3.1, there exists a unique idempotent $f_{em} \in \omega(m^*)$ such that $em = m f_{em}$. Define a mapping

$$\theta_m : \ E \to E \quad by \quad e \mapsto f_{em}.$$

If $g \in E$, then we have

$$m f_{egm} = (egm) = em f_{gm} = m f_{em} f_{gm}$$

and so, by Proposition 3.1, we have $f_{egm} = f_{em} f_{gm}$ since $f_{em} f_{gm} \in \omega(m^*)$. Thus $(eg)\theta_m = (e\theta_m)(g\theta_m)$ and θ_m is a homomorphism of E into itself.

Finally, we define a mapping Θ of M into $End(E)$:

$$\Theta : M \to End(E) \quad by \quad m \mapsto \theta_m.$$

Then, clearly Θ satisfies (PH1) and (PH2).

We now prove the following crucial lemma.

Lemma 3.9 $(M, E; \mathcal{F}, \mathcal{Q}; \Theta)$ *is an SFR-system.*

Proof: Let $m, n \in M$ and $x \in E$. Since

$$\begin{aligned} (m \star n) q_{m,n} (x\phi_m \phi_n) &= mn(x\phi_m\phi_n) \\ &= xmn = x(m \star n) q_{m,n} = (m \star n)(x\phi_{m\star n}) q_{m,n} \end{aligned}$$

and $q_{m,n}(x\phi_m\phi_n), (x\phi_{m\star n})q_{m,n} \in \omega((m \star n)^*)$, by Proposition 3.1, we have $q_{m,n}(x\phi_m\phi_n) = (x\phi_{m\star n})q_{m,n}$. That is, (SFR1) holds.

For $m, n \in M$ and $x \in \omega(d_m), y, z \in \omega(d_n)$, if $q_{m,n}(x\phi_n)y = q_{m,n}(x\phi_n)z$, then we have

$$\begin{aligned} mxny &= mn(x\phi_n)y = (m \star n)q_{m,n}(x\phi_n)y \\ &= (m \star n)q_{m,n}(x\phi_n)z = mn(x\phi_n)z = mxnz \end{aligned}$$

Since $m\mathcal{L}^*d_m = m^*$, by the above formula, we have $m^*xny = m^*xny$ and so $n(x\phi_n)y = n(x\phi_n)z$. This shows that $n^*(x\phi_n)y = n^*(x\phi_n)z$ since $n\mathcal{L}^*n^*$. Consequently, $(x\phi_n)y = (x\phi_n)z$, and hence (SFR2) holds.

Finally, let $m, n, t \in M$. Since

$$\begin{aligned} (m \star n \star t)q_{m,n\star t}q_{n,t} &= m(n \star t)q_{n,t} = mnt \\ &= (m \star n)q_{m,n}t = (m \star n)t(q_{m,n}\phi_t) \\ &= (m \star n \star t)q_{m\star n,t}(q_{m,n}\phi_t) \end{aligned}$$

and $m \star n \star t\mathcal{L}^*(m \star n \star t)^*$, we have

$$(m \star n \star t)^* q_{m,n\star t}q_{n,t} = (m \star n \star t)^* q_{m\star n,t}(q_{m,n}\phi_t). \quad (1)$$

Since $q_{m\star n,t}, q_{m,n\star t} \in \omega((m \star n \star t)^*)$, we have $q_{m,n\star t}q_{n,t}, q_{m\star n,t}(q_{m,n}\phi_t) \in \omega((m \star n \star t)^*)$ and, by the above equality (1), we obtain that $q_{m,n\star t}q_{n,t} = q_{m\star n,t}(q_{m,n}\phi_t)$. That is, (SFR3) also holds. This completes the proof. □

Theorem 3.10 $S \cong SFR = SFR(M, E; \mathcal{F}, \mathcal{Q}; \Theta)$.

Proof: We only need to prove that the mapping

$$\varphi : S \to SFR = SFR(M, E; \mathcal{F}, \mathcal{Q}; \Theta);\ s \mapsto (m_s, f_s),$$

where m_s and f_s have the same meanings as Proposition 3.1, is an isomorphism.

By Proposition 3.1, φ is well-defined and for $(m, x) \in SFR$, we have $(mx)\varphi = (m, x)$. Hence φ is surjective. That φ is injective follows from Proposition 3.1.

Finally, let $s, t \in S$. Then we have

$$st = m_s f_s m_t f_t = m_s m_t (f_s \theta_{m_t}) f_t = (m_s \star m_t) q_{m_s,m_t}(f_s \theta_{m_t}) f_t.$$

This implies that $f_{st} = q_{m_s,m_t}(f_s\theta_{m_t})f_t$ by Proposition 2.1. Thus we have

$$(st)\varphi = (m_s \star m_t, q_{m_s,m_t}(f_s\theta_{m_t})f_t) = (m_s, f_s)(m_t, f_t) = (s\varphi)(t\varphi)$$

and so φ is a homomorphism from S onto SFR. Hence φ is an isomorphism, as required. □

Summing up Proposition 3.8 and Theorem 3.10, we obtain the following structure theorem for strongly F-rpp semigroups.

Theorem 3.11 *Let $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ be an SFR-system. Then $SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ is a strongly F-rpp semigroup.*

Conversely, any strongly F-rpp semigroup can be constructed in this manner.

4 Applications

As an applications to the structure theorem of strongly F-rpp semigroups, we will also provide the structure theorem for F-regular semigroups and strongly F-abundant semigroups because these semigroups are special cases of strongly F-rpp semigroups.

We first let E be a band which is expressible as a semilattice of rectangular bands E_α, say $E = \bigcup_{\alpha\in Y} E_\alpha$. If $e \in E_\alpha$ then we denote the rectangular band E_α by $E(e)$. If $E_\alpha E_\beta \subseteq E_\beta$ then we write $E_\beta \leq_b E_\alpha$.

By using Theorem 3.11, we are now able to give the following structure theorem of F-regular semigroups.

Theorem 4.1 *Let $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ be an SFR-system. If M is a group satisfying the following condition:*

(FR) for all $m \in M$, $p_{m^{-1},m} \in E$ and $E(c_m) \leq_b E(p_{m^{-1},m}), E(c_{m^{-1}\phi_m})$,

then $SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ is an F-regular semigroup.

Conversely, any F-regular semigroup can be constructed in this above manner.

Proof: Suppose that M is a group satisfying condition (FR). Since all F-rpp regular semigroups are F-regular semigroups, we need only prove that $SFR = SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ is regular. To see this, let $(m, x) \in SFR$. Then, we can derive the following equalities:

$$\begin{aligned}(m,x)(m^{-1},c_{m^{-1}})(m,x) &= (m,x)(m^{-1}m, p_{m^{-1},m}(c_{m^{-1}}\phi_m)x)) \\ &= (mm^{-1}m, p_{m,1}(x\phi_1)p_{m^{-1},m}(c_{m^{-1}}\phi_m)x) \\ &= (m, c_m x p_{m^{-1},m}(c_{m^{-1}}\phi_m)x) \\ &= (m, x(c_{m^{-1}}\phi_m)x) = (m,x),\end{aligned}$$

Thereby, we have shown that (m, x) is regular and SFR is regular.

Conversely, if S is an F-regular semigroup with a left cancellative monoid congruence ρ on S such that each ρ-class of S contains a greatest element with respect to the Lawson partial order $\leq_\ell$, then S is a strongly F-rpp semigroup. Hence, by Theorem 3.11, $S \cong SFR(M, E; \mathcal{F}, \mathcal{Q}; \Theta)$. By using the arguments in Proposition 2.6, ρ is indeed a cancellative congruence on S. This leads to S/ρ is a cancellative monoid. Since S/ρ is regular, $M \cong S/\rho$ is a group.

Now, we let $m \in M$. Then, by a result of Edwards [2, Theorem 3.2], there exists a unique inverse element m^{-1} of m in M. Clearly, m^{-1} in $(M, \star)$. Since $m^{-1} \star m = 1$ (the identity of M), we have $m^{-1}m \in E(S)$and $q_{m^{-1},m} \in E(S)$. Also, since $mm^{-1}m = m$, we have

$$d_m \mathcal{L} m \mathcal{L} m^{-1}m \mathcal{L}^* q_{m^{-1},m},$$

and consequently, we have $E(d_m) = E(q_{m^{-1},m}) = E(m^{-1}m)$. Observe that

$$m^{-1}m = m^{-1}d_{m^{-1}}m = m^{-1}m(d_{m^{-1}}\theta_m),$$

we can hence deduce that $E(m^{-1}m) \leq_b E(d_{m^{-1}}\theta_m)$. This leads to $E(d_m) \leq_b E(d_{m^{-1}}\theta_m)$. Thus, condition (FR) is satisfied. This completes the proof. □

We now give a construction theorem for strongly F-abundant semigroups. We first describe a structure theorem of abundant strongly F-rpp semigroups.

Theorem 4.2 *Let $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ be an SFR-system. If M is a cancellative monoid, then $SFR = SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ is an abundant semigroup if and only if for all $(m, x) \in SFR$, there exists $[m, x] \in E$ such that*

(AB1) $([m, x]\phi_m)x = x$;

(AB2) for all $n \in M$ and $y, z \in \omega(c_n)$,

$$p_{n,m}(y\phi_m)x = p_{n,m}(z\phi_m)x \Rightarrow y[m, x] = z[m, x].$$

Proof: The proof follows from the following computation:

$(m, x)\mathcal{R}^*(1, [m, x])$ if and only if $(1, [m, x])(m, x) = (m, x)$, and for all $(n, y), (t, z) \in SFR^1$, we have

$$(n, y)(m, x) = (t, z)(m, x) \Rightarrow (n, y)(1, [m, x]) = (t, z)(1, [m, x]);$$

if and only if $([m, x]\phi_m)x = x$ and for all $(n, y), (t, z) \in SFR^1$, we have

$$n = t, p_{n,m}(y\phi_m)x = p_{t,m}(z\phi_m)x \Rightarrow n = t, c_n(y\phi_1)[m, x] = c_t(z\phi_1)[m, x];$$

if and only if $([m, x]\phi_m)x = x$ and for all $n \in M$ and $y, z \in \omega(c_n)$, we have

$$p_{n,m}(y\phi_m)x = p_{n,m}(z\phi_m)x \Rightarrow y[m, x] = z[m, x].$$

□

Theorem 4.3 *Let $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ be an SFR-system. If M is a cancellative monoid and for all $(m, x) \in SFR = SFR(M, E; \mathcal{C}, \mathcal{P}; \Phi)$, there exists $[m, x] \in E$ satisfying the conditions (AB1) and (AB2) in Theorem 4.2 and also the following conditions are satisfied:*

(IC) for some $y \in \omega(x)$, there exists $z \in E$ such that $(z\phi_m)x = y$;

(C) $[m, x]$ is central in $E(S)$,

then SFR is a strongly F-abundant semigroup.

Conversely, any such semigroup can be constructed in the above manner.

Proof: Suppose that the SFR-system $(M, E; \mathcal{C}, \mathcal{P}; \Phi)$ satisfies the conditions in the theorem. Then SFR is a strongly F-rpp semigroup. By (C), $(1, [m, x])$ is central in $E(S)$ and, by Lemma 2.10, $(1, [m, x])E(S) = E(S)(1, [m, x])$. Thus, by Proposition 2.6, it suffices to verify that SFR is an IC abundant semigroup. By Lemma 4.2, SFR is clearly an abundant semigroup.

Now let $(m, x) \in SFR$. Then $(1, x)\mathcal{L}^*(m, x)\mathcal{R}^*(1, [m, x])$. We consider the following cases:

- If $(1, y) \in \omega((1, x))$, then, by Lemma 3.6, $y \in \omega(x)$. Now, by computation and by the condition (IC), we have

$$\begin{aligned}(m, x)(1, y) &= (m, p_{m,1}(x\phi_1)y) = (m, c_m xy) = (m, xy) \\ &= (m, y) = (m, (z\phi_m)x) \quad (z \in E) \\ &= (1, z)(m, x).\end{aligned}$$

  This shows that S satisfies condition (i) in Lemma 2.3.

- By Proposition 3.1, we can also see that S satisfies condition (ii) in Lemma 2.3.

Thus S is an IC abundant semigroup. This proves that S is a strongly F-abundant semigroup.

Conversely, assume that S is a strongly F-abundant semigroup. Then S is an IC abundant semigroup and a strongly F-rpp semigroup with a left cancellative monoid congruence ρ such that each ρ-class contains a greatest element. With the notations in Section 3, $S \cong SFR(M, E; \mathcal{F}, \mathcal{Q}; \Theta)$. By the proof of Proposition 2.6, ρ is a cancellative congruence on S, and so $M \cong S/\rho$ is a cancellative monoid. Conditions (AB1) and (AB2) follows from Theorem 4.2. Since S is a strongly F-abundant semigroup, we have $(1, [m, x])E(S) = E(S)(1, [m, x])$ and, by Lemma 2.10, $(1, [m, x])$ is central in $E(SFR)$. Hence $[m, x]$ is central in $E(S)$. That is, condition (C) holds. On the other hand, we observe that $SFR(M, E; \mathcal{F}, \mathcal{Q}; \Theta)$ is IC. Use Lemma 2.3 (i) and refer to the above proof that S satisfies condition (i) in Lemma 2.3, we can easily check that condition (IC) also holds. This completes the proof. □

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