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A FIXED POINT THEOREM IN MENGER SPACES

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Abstract

In the present work, we prove a fixed point theorem in Menger spaces through weak compatibility.

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1 Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger [10] who was use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [16] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [13].

Sessa [14] introduced weakly commuting maps in metric spaces. Jungck [6] enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces has been introduced by Mishra [11]. Recently, Singh and Jain [15] generalized the results of Mishra [11] using the concept of weak compatibility and compatibility of pair of self maps.

In this paper, using the idea of weak compatibility due to Singh and Jain [15] and the idea of compatibility due to Mishra [11], we prove a common fixed point theorem for six maps under the condition of weak compatibility and compatibility in Menger spaces and give an example to illustrate the theorem.

2 Preliminary Notes

In this section, we recall some definitions and known results in Menger space. For more details we refer the readers to [1-5,7,8,9,12,17].

Definition 2.1 *A triangular norm* $*$ *(shorty t-norm) is a binary operation on the unit interval* $[0,1]$ *such that for all* $a,b,c,d \in [0,1]$ *the following conditions are satisfied:*

(a) $a * 1 = a$;
(b) $a * b = b * a$;
(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(d) $a * (b * c) = (a * b) * c$.
Examples of t-norms are $a * b = \max\{a + b - 1, 0\}$ and $a * b = \min\{a, b\}$.

Definition 2.2 *A distribution function is a function* $F : [-\infty, \infty] \to [0,1]$ *which is left continuous on* $\mathbb{R}$*, non-decreasing and* $F(-\infty) = 0$*,* $F(\infty) = 1$*.*

We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases} .$$

If X is a nonempty set, $F : X \times X \to \Delta$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

Definition 2.3 (Schweizer and Sklar [16]) *The ordered pair* (X, F) *is called a probabilistic metric space (shortly PM-space) if* X *is a nonempty set and* F *is a probabilistic distance satisfying the following conditions: for all* $x, y, z \in X$ *and* $t, s > 0$*,*

(FM-0) $F_{xy}(t) = 1 \Longleftrightarrow x = y$;
(FM-1) $F_{xy}(0) = 0$;
(FM-2) $F_{xy} = F_{yx}$;
(FM-3) $F_{xz}(t) = 1$, $F_{zy}(s) = 1 \Rightarrow F_{xy}(t + s) = 1$.
The ordered triple $(X, F, *)$ is called Menger space if (X, F) is a PM-space, $*$ is a t-norm and the following condition is also satisfies: for all $x, y, z \in X$ and $t, s > 0$,
(FM-4) $F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s)$.

Proposition 2.4 (Sehgal and Bharucha-Reid [13]) *Let* (X, d) *be a metric space. Then the metric* d *induces a distribution function* F *defined by* $F_{xy}(t) = H(t - d(x, y))$ *for all* $x, y \in X$ *and* $t > 0$*. If t-norm* $*$ *is* $a * b = \min\{a, b\}$ *for all* $a, b \in [0,1]$ *then* $(X, F, *)$ *is a Menger space. Further,* $(X, F, *)$ *is a complete Menger space if* (X, d) *is complete.*

Definition 2.5 (Mishra [11]) *Let* $(X, F, *)$ *be a Menger space and* $*$ *be a continuous t-norm.*

(a) A sequence $\{x_n\}$ in X is said to be *converge* to a point x in X (written $x_n \to x$) iff for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be *Cauchy* if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.6 *If $*$ is a continuous t-norm, it follows from (FM-4) that the limit of sequence in Menger space is uniquely determined.*

Definition 2.7 (Singh and Jain [15]) *Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.*

Definition 2.8 (Mishra [11]) *Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_n BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as $n \to \infty$.*

Remark 2.9 *If self maps A and B of a Menger space $(X, F, *)$ are compatible then they are weakly compatible.*

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

Example 2.10 *Let (X, d) be a metric space where $X = [0, 2]$ and $(X, F, *)$ be the induced Menger space with $F_{xy}(t) = H(t - d(x, y))$, $\forall x, y \in X$ and $\forall t > 0$. Define self maps A and B as follows:*

$$Ax = \begin{cases} 2 - x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2, \end{cases} \quad \text{and} \quad Bx = \begin{cases} x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Take $x_n = 1 - 1/n$. Then $F_{Ax_n 1}(t) = H(t - (1/n))$ and $\lim_{n\to\infty} F_{Ax_n 1}(t) = H(t) = 1$. Hence $Ax_n \to \infty$ as $n \to \infty$. Similarly, $Bx_n \to \infty$ as $n \to \infty$. Also $F_{ABx_n BAx_n}(t) = H(t - (1 - 1/n))$ and $\lim_{n\to\infty} F_{ABx_n BAx_n}(t) = H(t - 1) \neq 1$, $\forall t > 0$. Hence the pair (A, B) is not compatible. Set of coincidence points of A and B is $[1, 2]$. Now for any $x \in [1, 2]$, $Ax = Bx = 2$, and $AB(x) = A(2) = 2 = S(2) = SA(x)$. Thus A and B are weakly compatible but not compatible.

Lemma 2.11 (Singh and Jain [15]) *Let $\{x_n\}$ be a sequence in a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x_n x_{n+1}}(kt) \geq F_{x_{n-1} x_n}(t)$$

for all $t > 0$ and $n = 1, 2...$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.12 (Singh and Jain [15]) *Let $(X, F, *)$ be a Menger space. If there exists $k \in (0,1)$ such that*

$$F_{xy}(kt) \geq F_{xy}(t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3 Main Results

Theorem 3.1 *Let A, B, S, T, L and M be self maps on a complete Menger space $(X, F, *)$ with $t * t \geq t$ for all $t \in [0,1]$, satisfying:*

(a) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(b) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} & F^2_{LxMy}(kt) * [F_{ABxLx}(kt).F_{STyMy}(kt)] \\ \geq \; & [pF_{ABxLx}(t) + qF_{ABxSTy}(t)].F_{ABxMy}(2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$;

(c) $AB = BA, ST = TS, LB = BL, MT = TM$;

(d) either AB or L is continuous;

(e) the pair (L, AB) is compatible and (M, ST) is weakly compatible.

Then A, B, S, T, L and M have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. By (a), there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_1 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, ...$

Step1. By taking $x = x_{2n}$ and $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} & F^2_{Lx_{2n}Mx_{2n+1}}(kt) * \left[F_{ABx_{2n}Lx_{2n}}(kt) F_{STx_{2n+1}Mx_{2n+1}}(kt)\right] \\ \geq \; & [pF_{ABx_{2n}Lx_{2n}}(t) + qF_{ABx_{2n}STx_{2n+1}}(t)]F_{ABx_{2n}Mx_{2n+1}}(2kt), \end{aligned}$$

$$\begin{aligned} & F^2_{y_{2n}y_{2n+1}}(kt) * \left[F_{y_{2n-1}y_{2n}}(kt) F_{y_{2n}y_{2n+1}}(kt)\right] \\ \geq \; & [pF_{y_{2n}y_{2n-1}}(t) + qF_{y_{2n-1}y_{2n}}(t)]F_{y_{2n-1}y_{2n+1}}(2kt), \end{aligned}$$

$$\begin{aligned} & F_{y_{2n}y_{2n+1}}(kt) \left[F_{y_{2n-1}y_{2n}}(kt) * F_{y_{2n}y_{2n+1}}(kt)\right] \\ \geq \; & (p+q)\, F_{y_{2n}y_{2n-1}}(t) F_{y_{2n-1}y_{2n+1}}(2kt), \end{aligned}$$

$$F_{y_{2n}y_{2n+1}}(kt)F_{y_{2n-1}y_{2n+1}}(2kt) \geq F_{y_{2n-1}y_{2n}}(t)F_{y_{2n-1}y_{2n+1}}(2kt).$$

Hence, we have

$$F_{y_{2n}y_{2n+1}}(kt) \geq F_{y_{2n-1}y_{2n}}(t).$$

Similarly, we also have

$$F_{y_{2n+1}y_{2n+2}}(kt) \geq F_{y_{2n}y_{2n+1}}(t).$$

In general, for all n even or odd, we have

$$F_{y_n y_{n+1}}(kt) \geq F_{y_{n-1}y_n}(t)$$

for $k \in (0,1)$ and all $t > 0$. Thus, by Lemma 2.11, $\{y_n\}$ is a Cauchy sequence in X. Since $(X, F, *)$ is complete, it converges to a point z in X. Also its subsequences converge as follows: $\{Lx_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.

Case I. AB is continuous. Since AB is continuous, $AB(AB)x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$. Since (L, AB) is compatible, $L(AB)x_{2n} \to ABz$.

Step 2. By taking $x = ABx_{2n}$ and $y = x_{2n+1}$ in (b), we have

$$F^2_{L(AB)x_{2n}Mx_{2n+1}}(kt) * \left[F_{AB(AB)x_{2n}L(AB)x_{2n}}(kt)F_{STx_{2n+1}Mx_{2n+1}}(kt)\right]$$

$$\geq [pF_{AB(AB)x_{2n}L(AB)x_{2n}}(t) + qF_{AB(AB)x_{2n}STx_{2n+1}}(t)]F_{AB(AB)x_{2n}Mx_{2n+1}}(2kt)$$

This implies that, as $n \to \infty$

$$\begin{aligned} F^2_{zABz}(kt) * [F_{ABzABz}(kt)F_{zz}(kt)] &\geq [pF_{ABzABz}(t) + qF_{zABz}(t)]F_{zABz}(2kt) \\ &\geq [p + qF_{zABz}(t)]F_{zABz}(kt), \end{aligned}$$

$$\begin{aligned} F_{zABz}(kt) &\geq p + qF_{zABz}(t) \\ &\geq p + qF_{zABz}(kt), \end{aligned}$$

$$F_{zABz}(kt) \geq \frac{p}{1-q} = 1$$

for $k \in (0,1)$ and all $t > 0$. Thus, we have $z = ABz$.

Step 3. By taking $x = z$ and $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} &F^2_{LzMx_{2n+1}}(kt) * \left[F_{ABzLz}(kt)F_{STx_{2n+1}Mx_{2n+1}}(kt)\right] \\ \geq\ &[pF_{ABzLz}(t) + qF_{ABzSTx_{2n+1}}(t)]F_{ABzMx_{2n+1}}(2kt). \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} F^2_{zLz}(kt) * [F_{zLz}(kt)F_{zz}(kt)] &\geq [pF_{zLz}(t) + qF_{zz}(t)]F_{zz}(2kt) \\ F^2_{zLz}(kt) * F_{zLz}(kt) &\geq pF_{zLz}(t) + q. \end{aligned}$$

Noting that $F^2_{zLz}(kt) \leq 1$ and using (c) in Definition 2.1, we have

$$\begin{aligned} F_{zLz}(kt) &\geq pF_{zLz}(t) + q \\ &\geq pF_{zLz}(kt) + q, \end{aligned}$$

$$F_{zLz}(kt) \geq \frac{q}{1-p} = 1$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have $z = Lz = ABz$.

Step 4. By taking $x = Bz$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} &F^2_{L(Bz)Mx_{2n+1}}(kt) * \left[F_{AB(Bz)L(Bz)}(kt)F_{STx_{2n+1}Mx_{2n+1}}(kt)\right] \\ \geq\ &[pF_{AB(Bz)L(Bz)}(t) + qF_{AB(Bz)STx_{2n+1}}(t)]F_{AB(Bz)Mx_{2n+1}}(2kt). \end{aligned}$$

Since $AB = BA$ and $BL = LB$, we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$. Letting $n \to \infty$, we have

$$F^2_{zBz}(kt) * [F_{BzBz}(kt)F_{zz}(kt)] \geq [pF_{BzBz}(t) + qF_{zBz}(t)]F_{zBz}(2kt),$$

$$\begin{aligned} F^2_{zBz}(kt) &\geq [p + qF_{zBz}(t)]F_{zBz}(2kt) \\ &\geq [p + qF_{zBz}(t)]F_{zBz}(kt), \end{aligned}$$

$$\begin{aligned} F_{zBz}(kt) &\geq p + qF_{zBz}(t) \\ &\geq p + qF_{zBz}(kt), \end{aligned}$$

$$F_{zBz}(kt) \geq \frac{p}{1-q} = 1$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have $z = Bz$. Since $z = ABz$, we also have $z = Az$. Therefore, $z = Az = Bz = Lz$.

Step 5. Since $L(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Lz = STv$. By taking $x = x_{2n}$, $y = v$ in (b), we have

$$\begin{aligned} &F^2_{Lx_{2n}Mv}(kt) * [F_{ABx_{2n}Lx_{2n}}(kt)F_{STvMv}(kt)] \\ \geq\ &[pF_{ABx_{2n}Lx_{2n}}(t) + qF_{ABx_{2n}STv}(t)]F_{ABx_{2n}Mv}(2kt) \end{aligned}$$

which implies that, as $n \to \infty$

$$F^2_{zMv}(kt) * [F_{zz}(kt)F_{zMv}(kt)] \geq [pF_{zz}(t) + qF_{zz}(t)]F_{zMv}(2kt),$$

$$F^2_{zMv}(kt) * F_{zMv}(kt) \geq (p+q)\, F_{zMv}(2kt).$$

Noting that $F^2_{zMv}(kt) \leq 1$ and using (c) in Definition 2.1, we have

$$\begin{aligned} F_{zMv}(kt) &\geq F_{zMv}(2kt) \\ &\geq F_{zMv}(t) \end{aligned}$$

Thus, by Lemma 2.12, we have $z = Mv$ and so $z = Mv = STv$. Since (M, ST) is weakly compatible, we have $STMv = MSTv$. Thus, $STz = Mz$.

Step 6. By taking $x = x_{2n}$, $y = z$ in (b) and using Step 5, we have

$$\begin{aligned} & F^2_{Lx_{2n}Mz}(kt) * [F_{ABx_{2n}Lx_{2n}}(kt)F_{STzMz}(kt)] \\ \geq\ & [pF_{ABx_{2n}Lx_{2n}}(t) + qF_{ABx_{2n}STz}(t)]F_{ABx_{2n}Mz}(2kt) \end{aligned}$$

which implies that, as $n \to \infty$

$$F^2_{zMz}(kt) * [F_{zz}(kt)F_{MzMz}(kt)] \geq [pF_{zz}(t) + qF_{zMz}(t)]F_{zMz}(2kt),$$

$$\begin{aligned} F^2_{zMz}(kt) &\geq [p + qF_{zMz}(t)]F_{zMz}(2kt) \\ &\geq [p + qF_{zMz}(t)]F_{zMz}(kt), \end{aligned}$$

$$\begin{aligned} F_{zMz}(kt) &\geq p + qF_{zMz}(t) \\ &\geq p + qF_{zMz}(kt), \end{aligned}$$

$$F_{zMz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = Mz$ and therefore $z = Az = Bz = Lz = Mz = STz$.

Step 7. By taking $x = x_{2n}$, $y = Tz$ in (b), we have

$$\begin{aligned} & F^2_{Lx_{2n}M(Tz)}(kt) * \left[F_{ABx_{2n}Lx_{2n}}(kt)F_{ST(Tz)M(Tz)}(kt)\right] \\ \geq\ & [pF_{ABx_{2n}Lx_{2n}}(t) + qF_{ABx_{2n}ST(Tz)}(t)]F_{ABx_{2n}M(Tz)}(2kt). \end{aligned}$$

Since $MT = TM$ and $ST = TS$, we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$. Letting $n \to \infty$, we have

$$F^2_{zTz}(kt) * [F_{zz}(kt)F_{TzTz}(kt)] \geq [pF_{zz}(t) + qF_{zTz}(t)]F_{zTz}(2kt),$$

$$F^2_{zTz}(kt) \geq [p + qF_{zTz}(t)]F_{zTz}(kt),$$

$$\begin{aligned} F_{zTz}(kt) &\geq p + qF_{zTz}(t) \\ &\geq p + qF_{zTz}(kt), \end{aligned}$$

$$F_{zTz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = Tz$. Since $Tz = STz$, we also have $z = Sz$. Therefore, $z = Az = Bz = Lz = Mz = Sz = Tz$, that is, z is the common fixed point of the six maps.

Case II. L is continuous. Since L is continuous, $LLx_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$. Since (L, AB) is compatible, $(AB)Lx_{2n} \to Lz$.

Step 8. By taking $x = Lx_{2n}$, $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} & F^2_{LLx_{2n}Mx_{2n+1}}(kt) * \left[F_{ABLx_{2n}LLx_{2n}}(kt)F_{STx_{2n+1}Mx_{2n+1}}(kt)\right] \\ \geq\ & [pF_{ABLx_{2n}LLx_{2n}}(t) + qF_{ABLx_{2n}STx_{2n+1}}(t)]F_{ABLx_{2n}Mx_{2n+1}}(2kt). \end{aligned}$$

This implies that, as $n \to \infty$

$$F^2_{zLz}(kt) * [F_{LzLz}(kt)F_{zz}(kt)] \geq [pF_{LzLz}(t) + qF_{zLz}(t)]F_{zLz}(2kt),$$

$$\begin{aligned} F^2_{zLz}(kt) &\geq [p + qF_{zLz}(t)]F_{zLz}(2kt) \\ &\geq [p + qF_{zLz}(t)]F_{zLz}(kt), \end{aligned}$$

$$\begin{aligned} F_{zLz}(kt) &\geq p + qF_{zLz}(t) \\ &\geq p + qF_{zLz}(kt), \end{aligned}$$

$$F_{zLz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = Lz$ and using Steps 5-7, we have $z = Lz = Mz = Sz = Tz$.

Step 9. Since $M(X) \subseteq AB(X)$, there exists $v \in X$ such that $z = Mz = ABv$. By taking $x = v$, $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} & F^2_{LvMx_{2n+1}}(kt) * \left[F_{ABvLv}(kt)F_{STx_{2n+1}Mx_{2n+1}}(kt)\right] \\ \geq\ & [pF_{ABvLv}(t) + qF_{ABvSTx_{2n+1}}(t)]F_{ABvMx_{2n+1}}(2kt) \end{aligned}$$

which implies that, as $n \to \infty$

$$F^2_{zLv}(kt) * [F_{zLv}(kt)F_{zz}(kt)] \geq [pF_{zLv}(t) + qF_{zz}(t)]F_{zz}(2kt),$$

$$\begin{aligned} F^2_{zLv}(kt) * F_{zLv}(kt) &\geq pF_{zLv}(t) + q \\ &\geq pF_{zLv}(kt) + q. \end{aligned}$$

Noting that $F^2_{zLv}(kt) \leq 1$ and using (c) in Definition 2.1, we have

$$F_{zMv}(kt) \geq pF_{zLv}(kt) + q,$$

$$F_{zMv}(kt) \geq \frac{q}{1-p} = 1.$$

Thus, we have $z = Lv = ABv$. Since (L, AB) is weakly compatible, we have $Lz = ABz$ and using Step 4, we also have $z = Bz$. Therefore $z = Az = Bz = Sz = Tz = Lz = Mz$, that is, z is the common fixed point of the six maps in this case also.

Step 10. For uniqueness, let w $(w \neq z)$ be another common fixed point of A, B, S, T, L and M. Taking $x = z$, $y = w$ in (b), we have

$$\begin{aligned} & F^2_{LzMw}(kt) * [F_{ABzLz}(kt)F_{STwMw}(kt)] \\ \geq\ & [pF_{ABzLz}(t) + qF_{ABzSTw}(t)]F_{ABzMw}(2kt) \end{aligned}$$

which implies that

$$\begin{aligned} F^2_{zw}(kt) &\geq [p + qF_{zw}(t)]F_{zw}(2kt) \\ &\geq [p + qF_{zw}(t)]F_{zw}(kt), \end{aligned}$$

$$\begin{aligned} F_{zw}(kt) &\geq p + qF_{zw}(t) \\ &\geq p + qF_{zw}(kt), \end{aligned}$$

$$F_{zw}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = w$. This completes the proof of the theorem.

If we take $B = T = I_X$ (the identity map on X) in the main Theorem, we have the following:

Corollary 3.2 *Let A, S, L and M be self maps on a complete Menger space $(X, F, *)$ with $t * t \geq t$ for all $t \in [0, 1]$, satisfying:*

(a) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;

(b) there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} & F^2_{LxMy}(kt) * [F_{AxLx}(kt).F_{SyMy}(kt)] \\ \geq \ & [pF_{AxLx}(t) + qF_{AxSy}(t)].F_{AxMy}(2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$;

(c) either A or L is continuous;

(d) the pair (L, A) is compatible and (M, S) is weakly compatible.

Then A, S, L and M have a unique common fixed point.

If we take $A = S, L = M$ and $B = T = I_X$ in the main Theorem, we have the following:

Corollary 3.3 *Let $(X, F, *)$ be a complete Menger space with $t*t \geq t$ for all $t \in [0,1]$ and let A and L be compatible maps on X such that $L(X) \subseteq A(X)$. If A is continuous and there exists a constant $k \in (0,1)$ such that*

$$\begin{aligned} & F^2_{LxLy}(kt) * [F_{AxLx}(kt).F_{AyLy}(kt)] \\ \geq \ & [pF_{AxLx}(t) + qF_{AxAy}(t)].F_{AxLy}(2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$, then A and L have a unique fixed point.

Example 3.4 *Let $X = [0,1]$ with the metric d defined by $d(x, y) = |x - y|$ and define $F_{xy}(t) = H(t - d(x, y))$ for all $x, y \in X, t > 0$. Clearly $(X, F, *)$ is a complete Menger space where t-norm $*$ is defined by $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Let A, B, S, T, L and M be maps from X into itself defined as*

$$Ax = x, Bx = \frac{x}{2}, Sx = \frac{x}{5}, Tx = \frac{x}{3}, Lx = 0, Mx = \frac{x}{6}$$

for all $x \in X$. Then

$$L(X) = \{0\} \subset \left[0, \frac{1}{15}\right] = ST(X) \text{ and } M(X) = \left[0, \frac{1}{6}\right] \subset \left[0, \frac{1}{2}\right] = AB(X).$$

Clearly $AB = BA, ST = TS, LB = BL, MT = TM$ and AB, L are continuous. If we take $k = 1/2$ and $t = 1$, we see that the condition (b) of the main Theorem is also satisfied. Moreover, the maps L and AB are compatible if $\lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Lx_n = \lim_{n\to\infty} ABx_n = 0$ for $0 \in X$. The maps M and ST are weakly compatible at 0. Thus, all conditions of the main Theorem are satisfied and 0 is the unique common fixed point of A, B, S, T, L and M.

References

[1] G. Constantin, I. Istratescu, *Elements of Probabilistic Analysis,* Ed. Acad. Bucureşti and Kluwer Acad. Publ., 1989.

[2] O. Hadzic, Common fixed point theorems for families of mapping in complete metric space, *Math. Japon.,* **29** (1984), 127-134.

[3] T. L. Hicks, Fixed point theory in probabilistic metric spaces, *Rev. Res. Novi Sad,* **13** (1983), 63-72.

[4] I. Istratescu, On some fixed point theorems in generalized Menger spaces, *Boll. Un. Mat. Ital.,* **5** (13-A) (1976), 95-100.

[5] I. Istratescu, On generalized complete probabilistic metric spaces, *Rev. Roum. Math. Pures Appl.,* **XXV** (1980), 1243-1247.

[6] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly,* **83** (1976), 261-263.

[7] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Sci.,* **9** (1986), 771-779.

[8] G. Jungck, B. E. Rhoades, Some fixed point theorems for compatible maps, *Internat. J. Math. & Math. Sci.,* **3** (1993), 417-428.

[9] S. Kutukcu, D. Turkoglu, C. Yildiz, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, *Commun. Korean Math. Soc.,* in press.

[10] K. Menger, Statistical metric, *Proc. Nat. Acad.,* **28** (1942), 535-537.

[11] S. N. Mishra, Common fixed points of compatible mappings in PM-spaces,ic spaces, *Math. Japon.,* **36** (1991), 283-289.

[12] E. Pap, O. Hadzic, R. Mesiar, A fixed point theorem in probabilistic metric spaces and an application, *J. Math. Anal. Appl.,* **202** (1996), 433-449.

[13] V. M. Sehgal, A. T. Bharucha-Reid, Fixed point of contraction mapping on PM spaces, *Math. Systems Theory,* **6** (1972), 97-100.

[14] S. Sessa, On a weak commutative condition in fixed point consideration, *Publ. Inst. Math.,* **32** (1982), 146-153.

[15] B. Singh, S. Jain, A fixed point theorem in Menger Space through weak compatibility, *J. Math. Anal. Appl.,* **301** (2005), 439-448.

[16] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces,* North-Holland, Amsterdam, 1983.

[17] R. M. Tardiff, Contraction maps on probabilistic metric spaces, *J. Math. Anal. Appl.,* **165** (1992), 517-523.

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