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Frames in Finitely or Countably Generated Hilbert C^*-Modules

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Abstract

In this paper we study the relations between frames in Hilbert C^*-modules over a unital C^*-algebra. We also study the behavior of Besel sequences and frames under operators.

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1. Introduction

Frames in Hilbert spaces introduced by Gabor and they serve as a replacement for bases, but with more flexibility. Later frames in Hilbert C^*-modules were investigated in [2, 3, 4, 7, 8]. These concepts are generalizations of some results in [6]. The notion of frames in Hilbert C^*-modules are useful in wavelet theory, in C^*-algebras and in Hilbert bundle theory. In [3] it has been shown that every finitely or countably generated Hilbert C^*-module has a standard frame. In this paper firstly we recall some basic properties of frames in Hilbert C^*-modules, secondly by using adjointable module homomorphism on Hilbert C^*-modules and on $\ell_2(I, A)$ we construct some frames and finally we present a relation between standard frames in Hilbert A-modules. Our references for Hilbert spaces are [1] and [6].

In this paper $\mathbb{N}$ will denote the set of natural numbers and I will be a finite or countable subset of $\mathbb{N}$.

2. Preliminaries

Let A be a unital C^*-algebra, let H and K be finitely or countably generated Hilbert A-modules. For every $x \in H$, we define $|x| = \langle x, x\rangle^{1/2}$ and for $a \in A$, $|a| = (a^*a)^{1/2}$.

A sequence $\{x_i : i \in I\}$ of H is a frame if there exist real constants $C, D > 0$ such that for every $x \in H$,

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_j\rangle\langle x_j, x\rangle \leq D\langle x, x\rangle. \tag{1}$$

If for every $x \in H$, the series in the middle of the inequality in (1) is convergent in norm, we say that the frame is standard. The optimal constants for C and D are called the frame bounds. It is λ-tight if $C = D = \lambda$ and it is a Parseval frame if $C = D = 1$. The sequence $\{x_i : i \in I\}$ is called a *Bessel sequence* if there is a real $D > 0$ such that for every $x \in H$,

$$\sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle.$$

The sequence $\{x_i : i \in I\}$satisfies the lower frame bound if there exists a positive constant $C > 0$

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_j\rangle\langle x_j, x\rangle. \tag{2}$$

Let

$$\ell_2(I, A) = \{a = (a_i)_{i\in I} :\ a_i \in A \ \text{ for each } \ i \in I \ \text{ and } \sum_{i\in I} a_i a_i^* \text{ is convergent in } \|.\|_A\}.$$

We note that, see [7], $\ell_2(I, A)$ is a Hilbert A-module with A-valued inner product $\langle (a_i)_{i\in I}, (b_i)_{i\in I}\rangle = \sum_{i\in I} a_i b_i^*$ and the norm defined by $\|a\| = \|\sum_{i\in I} a_i a_i^*\|_A^{1/2}$ for all $a = (a_i)_{i\in I}$ in $\ell_2(I, A)$.

Now suppose that $\{x_j :\ j \in I\}$ is a standard frame for H. Then the frame transform $\theta :\ H \to \ell_2(I, A)$, defined by

$$\theta(x) = (\langle x, x_i\rangle)_{i\in I} \qquad (x \in H), \tag{3}$$

is an adjointable map with adjoint $\theta^* : \ell_2(I, A) \to H$ and therefore θ is A-linear and bounded, see [8, P. 234] or [7]. Moreover for every $x \in H$,

$$|\theta(x)|^2 = \langle \theta(x), \theta(x)\rangle = \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle. \tag{4}$$

Therefore θ is one-to-one, with a closed range which is complemented in $\ell_2(I,A)$, $\ell_2(I,A) = \theta(H) \oplus Ker\ \theta^*$. We note that $\theta^*(e_j) = x_j$, where $\{e_j : j \in I\}$ is the standard basis of $\ell_2(I,A)$. We also note that $\theta^*|\theta(H)$ is an invertible operator and the frame operator $S = (\theta^*\theta)^{-1}$ is a positive invertible bounded operator on H such that for every $x \in H$,

$$x = \sum_{j\in I}\langle x, Sx_j\rangle x_j = \sum_{j\in I}\langle x, x_j\rangle Sx_j. \tag{5}$$

The sequence $\{Sx_j : j \in I\}$ is a frame for H and is called the canonical dual frame of $\{x_j : j \in I\}$. Throughout this paper, we denote by $\hat{L}(K,H)$, the set of all adjointable maps from K to H and as usual we abbreviate $\hat{L}(H,H)$ by $\hat{L}(H)$.

3. Construction of Standard Frames of H

Lemma 3.1. *Let H and K be Hilbert A-modules.*
(a) If $\{x_i : i \in I\}$ is a Bessel sequence in H with bound D and $T \in \hat{L}(H,K)$, then $\{Tx_i : i \in I\}$ is a Bessel sequence in K with bound $D\|T\|^2$,
(b) If $\{x_i : i \in I\}$ satisfies the lower frame condition and there exists a positive constant K such that for every $y \in \overline{T(H)}$, $K\langle y, y\rangle \leq \langle T^*y, T^*y\rangle$, then $\{Tx_i : i \in I\}$ satisfies the lower frame condition in $\overline{T(H)}$.
Proof. (a) By Proposition 1.2 of [7] for every y in K we have

$$\sum_{i\in I}|\langle y, Tx_i\rangle|^2 = \sum_{i\in I}|\langle T^*y, x_i\rangle|^2 \leq D\langle T^*y, T^*y\rangle \leq |D|\|T\|^2\langle y, y\rangle.$$

(b) For every $y \in \overline{T(H)}$ we have

$$CK\langle y, y\rangle \leq C\langle T^*y, T^*y\rangle \leq \sum_i |\langle T^*y, x_i\rangle|^2 = \sum_i |\langle y, Tx_i\rangle|^2.$$

In the following theorem we give a necessary and sufficient condition for $\{y_i = Tx_i : i \in I\}$ to be a standard frame of $\overline{T(H)}$

Theorem 3.2. *Let $\{x_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds $0 < C \leq D$ and T be a module map in $\hat{L}(H,K)$. Then the following conditions are equivalent:*
(i) The sequence $\{Tx_i : i \in I\}$ is a standard frame of $\overline{T(H)}$.
(ii) There exists a positive constant k such that T^*, the adjoint of T, satisfies

$$k\langle y, y\rangle \leq \langle T^*y, T^*y\rangle \tag{6}$$

for every $y \in \overline{T(H)}$.

Proof: Suppose that $\{Tx_i : i \in I\}$ is a standard frame for $\overline{T(H)}$ with lower bound C'. Then for every $y \in \overline{T(H)}$,

$$C'\langle y, y\rangle \leq \sum_{i \in I} |\langle y, Tx_i\rangle|^2 = \sum_{i \in I} |\langle T^* y, x_j\rangle|^2 \leq D\langle T^* y, T^* y\rangle. \tag{7}$$

From which, condition (6) follows with $k = C'/D$.

The converse follows from the above lemma. Moreover since $\{x_i : i \in I\}$ is a standard frame $\sum |\langle T^* y, x_j >|^2$ is convergent in norm, so $\sum |\langle y, Tx_i\rangle|^2$ is convergent in norm for every $y \in \overline{T(H)}$. □

In previous theorem if $\{Tx_i : i \in I\}$ is a standard frame for K, then by the reconstruction formula (5), $T(H)$ is dense in K, so for $\{Tx_i : i \in I\}$ to be a standard frame of K it is necessary that $T(H)$ be dense in K and consequently the assumption $\overline{T(H)} = K$ yields the following result.

Corollary 3.3. *Let* $\{x_j : j \in \mathbb{J}\}$ *be a standard frame of* H *with bounds* $0 < C \leq D$ *and* T *be a map in* $\hat{L}(H, K)$ *such that* $\overline{T(H)} = K$. *Then the following conditions are equivalent:*

(i) The sequence $\{Tx_i : i \in I\}$ *is a standard frame of* K,

(ii) There exists a positive constant k *such that* T^*, *the adjoint of* T, *satisfies*

$$k\langle y, y\rangle \leq \langle T^* y, T^* y\rangle \tag{8}$$

for every $y \in K$.

Remark. **(1)** There exists a Hilbert A-module H and a module map $T \in \hat{L}(H)$ such that T^* is injective, but $\overline{T(H)} \neq H$ (cf. [8], Exercise 15.F). For this reason, in Corollary 3.3 we supposed that $\overline{T(H)} = K$. But if $\overline{T(H)}$ is a complemented submodule of H then condition (8) implies that $\overline{T(H)} = H$.

(2) If T is a self adjoint module map in $\hat{L}(H)$ and satisfies condition (8), then T is invertible (cf. [7], Lemma 3.1). In particular T is surjective. Then $T(H) = H$.

(3) Suppose that $T \in \hat{L}(H)$ and $T(H)$ is closed. Then $T(H)$ is a complemented submodule of H and $T(H) \oplus \ker T^* = H$ (cf. [7], Theorem 3.2). If T^* satisfies condition (8), then $\ker T^* = \{0\}$ and $T(H) = H$.

Corollary 3.4. *Let* $\{x_j : j \in \mathbb{J}\}$ *be a standard frame for* H. *If* T *is an adjointable module map from* H *onto* K, *then the following conditions are equivalent:*

(i) The sequence $\{y_j = Tx_j : j \in \mathbb{J}\}$ is a standard frame for K.
(ii) There exists a positive constant k such that

$$k\langle y, y\rangle \leq \langle T^*y, T^*y\rangle$$

for every $y \in K$.

By corollary 3.4, we can construct some standard frames for a closed submodule of H, with a given standard frame.

Now, let $T \in \hat{L}(l_2(A))$ and let $\eta = \{\eta_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds C_η and D_η and frame transform θ_η. We use T to construct the sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$, where

$$\xi_j = \theta_\eta^*(T(e_j)) \quad , \quad (j \in \mathbb{J}) \tag{9}$$

such that

$$T(e_j) = \sum_{i\in\mathbb{J}} \alpha_{ji}e_i \quad , \quad \{\alpha_{ji}\}_{i\in\mathbb{J}} \in l_2(A).$$

Then

$$\theta_\eta^*(T(e_j)) = \sum_{i\in\mathbb{J}} \alpha_{ji}\theta_\eta^*(e_i) = \sum_{i\in\mathbb{J}} \alpha_{ji}\eta_i.$$

But the sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is not always a standard frame for H (e.g. $T = 0$). Now we want to make $\xi = \{\xi_j : j \in \mathbb{J}\}$ a standard frame under an appropriate condition on T.

Theorem 3.5. *Let $\eta = \{\eta_j : j \in \mathbb{J}\}$ be a standard frame of H with bounds D_η and C_η. If $T \in \hat{L}(l_2(A))$ then the following conditions are equivalent:*
(i) The sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is a standard frame of H defined by (9).
(ii) There exists a positive constant k such that

$$k\langle y, y\rangle \leq \langle T^*y, T^*y\rangle \tag{10}$$

for every $y \in \theta_\eta(H)$, where θ_η is the frame transform of $\eta = \{\eta_j : j \in \mathbb{J}\}$.

Proof: Suppose that the sequence $\xi = \{\xi_j : j \in \mathbb{J}\}$ is a standard frame of H defined by (9). Then there are two constants $0 < C_\xi \leq D_\xi$ such that

$$C_\xi\langle x, x\rangle \leq \sum_{j\in\mathbb{J}} \langle x, \xi_j\rangle\langle \xi_j, x\rangle = \langle \theta_\xi(x), \theta_\xi(x)\rangle \leq D_\xi\langle x, x\rangle \tag{11}$$

for every $x \in H$, where θ_ξ is the frame transform of $\xi = \{\xi_j : j \in \mathbb{J}\}$. Also

$$\begin{aligned} \langle \theta_\xi(x), e_n \rangle &= \langle x, \theta_\xi^*(e_n) \rangle \\ &= \langle x, \xi_n \rangle \\ &= \langle x, \theta_\eta^*(T(e_n)) \rangle \\ &= \langle T^*(\theta_\eta(x)), e_n \rangle \end{aligned}$$

for every $n \in \mathbb{J}$ and $x \in H$.Since the set of A-linear combinations of $\{e_n\}_{n\in\mathbb{J}}$ is dense in $l_2(A)$, we have

$$\theta_\xi = T^*\theta_\eta. \tag{12}$$

So, by using the left inequality of (11),and (12), we conclude that

$$C_\xi\langle x, x\rangle \leq \langle \theta_\xi(x), \theta_\xi(x)\rangle = \langle T^*\theta_\eta(x), T^*\theta_\eta(x)\rangle \tag{13}$$

for every $x \in H$. Then

$$\frac{C_\xi}{D_\eta}\langle \theta_\eta(x), \theta_\eta(x)\rangle \leq C_\xi\langle x, x\rangle \leq \langle T^*\theta_\eta(x), T^*\theta_\eta(x)\rangle \tag{14}$$

for every $x \in H$. From which, condition (10) follows with $k = \frac{C_\xi}{D_\eta}$.

Conversely, since for every $x \in H$, $\sum |\langle x, \eta_i\rangle|^2$ is convergent in A, by using Proposition 1.2 of [7], we have

$$\begin{aligned} C_\eta k\langle x, x\rangle &\leq k\langle \theta_\eta(x), \theta_\eta(x)\rangle \\ &\leq \langle T^*\theta_\eta(x), T^*\theta_\eta(x)\rangle = \sum_i |\langle x, \xi_i\rangle|^2 \\ &\leq ||T||^2\langle \theta_\eta(x), \theta_\eta(x)\rangle = \|T\|^2 \sum_i |\langle x, \eta_i\rangle|^2 \\ &\leq ||T||^2 D_\eta\langle x, x\rangle. \end{aligned} \tag{15}$$

for every $x \in H$. Therefore $\{\xi_j : j \in I\}$ is a frame with frame transform $\theta_\xi = T^*\theta_\eta$. □

4. Characterization of Standard Frames of H

The aim of this section is to characterize all standard frames of H. In Theorem 4.2, we will show how any two standard frames of H are related with each other.

Definition 4.1. *Frames $\{\eta_i : i \in I\}$ and $\{\xi_i : i \in I\}$ of H and K, respectively are similar if there is an A-linear adjointable, bounded operator $U : H \to K$*

such that for each $i \in I$, $U(\eta_i) = \xi_i$ and U is invertible.

Theorem 4.2. *Let sequences $\eta = \{\eta_j : j \in \mathbb{J}\}$ and $\xi = \{\xi_j : j \in \mathbb{J}\}$ be standard frames of H and K, respectively. Then they are similar. Conversely if $\{\eta_i : \ i \in I\}$ is a standard frame for H and $\{\xi_i : \ i \in I\}$ is a frame for K which is similar to $\{\eta_i : \ i \in I\}$, then $\{\xi_i : \ i \in I\}$ is standard.*

Proof. Since $\{\eta_i : \ i \in I\}$ and $\{\xi_i : \ i \in I\}$ are standard frames of H and K, respectively, then $\theta_\eta(H)$ and $\theta_\xi(K)$ are complemented in $\ell_2(I, A)$. Therefore the orthogonal projections $p : \ell_2(I, A) \to \theta_\eta(H)$ and $q : \ell_2(I, A) \to \theta_\xi(K)$ are adjointable. if we take $U = \theta_\xi^{-1} \circ p \circ \theta_\eta : H \to K$, then U is an A-linear, bounded adjointable operator with $U^* = \theta_\eta^* \circ p \circ \theta_\xi^{-1} : K \to H$ such that for each $i \in I$, $U^*(\xi_i) = \eta_i$ and similarly the map $V = \theta_\eta^{-1} \circ q \circ \theta_\xi : K \to H$ is adjointable with $V^* = \theta_\eta^* \circ q \circ \theta_\eta^{*-1} : H \to K$ such that for each $i \in I$, $V^*(\eta_i) = \xi_i$. Hence $V^*U^* = id_K$ and $U^*V^* = id_H$. Therefore we have the result. The converse is obvious. □

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