A numerical solution of Burger's equation based on modified extended BDF scheme

M. Javidi

Department of Mathematics, Faculty of science,
Razi university, Kermanshah 67149, Iran.

Abstract

In this paper, we present a new method for solving of the Burger's equation by combination of method of lines (MOL) and matrix free modified extended backward differential formula (MF-MEBDF). The method of lines semi discretization approach is used to transform the model partial differential equations (PDEs) in a system of first order ordinary differential equations (ODEs). Firstly theory of application of MOL on Burger's equation presented. Secondly we considers some examples to show goodness of new method. Computational results using a code based on our method are presented for this examples . The numerical results obtained by this way have been compared with the exact solution to show the efficiency of the method.

Mathematics Subject Classification: 35Q53

Keywords: Burger's equation, MF-MEBDF method, method of lines.

1 Introduction

We consider the following initial boundary value problem

$$U_t + UU_x = \mu U_{xx}, \quad 0 < x < 1, \quad 0 < t < T, \quad (1)$$

which is the one dimensional quasilinear parabolic partial differential equation and $\mu > 0$ is the coefficient of the kinematics viscosity of the fluid. This equation was only intended as an approach to the study of turbulence, shock wave and gas dynamics [6, 10, 14]. This equation involves nonlinearity, dissipation and is relatively simple. We consider Eq.(1) with the following initial and boundary conditions:

$$\begin{aligned} U(x, 0) &= \varphi(x), & 0 < x < 1, \\ U(0, t) &= g_1(t), & U(1, t) = g_2(t), & t > 0. \end{aligned} \quad (2)$$

Many researchers have introduced various methods to solve the Burger's equation as a model of turbulence. Darvishi and Javidi [11] studied a numerical solution of Burger's equation by pseudospectral method and Darvishi's preconditioning. Abdou and Soliman [3] used variational iteration method for solving Burger's and coupled Burger's equations. Berzins and Dew [5] a note on C^0 Chebyshev methods for parabolic PDE's. Cole [10] studied the general properties of the Burger's equation. Öziş and Aksan [16] used a finite element approach for solution of Burgers' equation.

Aksan and Özdeş [4] solved the equation by a variational method constructed on the method of discretization in time. Abbasbandy and Darvishi [1] introduced a numerical solution of the Burgers' equation by modified Adomian method. They also [2] solved the problem by time discretization of Adomian's decomposition method. In this paper, Burger's equation is solved by combination of method of lines and MF-MEBDF method. Here, we consider two test problems of [13] and one test problem of [3]. The numerical results are compared with the exact solutions. It was seen that, the numerical solutions were in good agreement with the exact solutions.

Problem 1. Let us consider the Burger's equation [13] with the initial condition

$$\varphi(x) = U(x, 0), \quad 0 \leq x \leq 1 \quad (3)$$

and non-homogeneous boundary conditions

$$g_1(t) = U(0, t), \quad g_1 = U(1, t), \quad 0 \leq t \leq T = 1. \quad (4)$$

The exact solution of the Burger's equation

$$U(x, t) = \frac{0.1e^{-A} + 0.5e^{-B} + e^{-C}}{e^{-A} + e^{-B} + e^{-C}}, \quad (5)$$

$$A = \frac{0.05}{\mu}(x - 0.5 + 4.95t), \quad B = \frac{0.25}{\mu}(x - 0.5 + 0.75t)$$

and

$$C = \frac{0.5}{\mu}(x - 0.5 - 0.375t).$$

Problem 2. Let us consider the Burger's equation [3] with the initial condition

$$\varphi(x) = U(x, 0)$$

and the boundary conditions

$$g_1(t) = U(0, t), \quad g_2(t) = U(1, t), \quad t \geq 0.$$

The exact solution of the Eq.(1) is:

$$U(x, t) = \frac{\alpha + \beta + (\beta - \alpha) \exp(\frac{\alpha}{\mu}(x - \beta t - \lambda))}{1 + \exp(\frac{\alpha}{\mu}(x - \beta t - \lambda))},$$

where the parameters α , β and λ are arbitrary constants and the parameter $\mu > 0$ in Eq.(1), which is so-called viscosity term, plays an important role in determining the behavior of the solution.

2 Semi discrete models for Burger's equation

An approach related to the finite difference method, which has found interesting application the numerical treatment partial differential equation (PDE), is the method of lines, makes use of the finite difference method in the construction of discrete-analytical model of given partial differential equation in which all but one of its independent variables are discretized. This procedure leads to a system of ordinary differential equations (ODEs) in the remaining variable.

We shall consider the problem (1)-(2), to solve it numerically the problem is discretized respect to x -direction by finite difference method and that the variable t remains undiscretized. For this, we define a uniform partition P of the interval $0 \leq x \leq 1$ as:

$$P : 0 = x_0 \leq \dots \leq x_i \leq x_{i+1} \leq \dots \leq x_{n+1} = 1, \quad (6)$$

with a constant spacing $h = x_{i+1} - x_i = \frac{1}{n+1}$, $i = 0, 1, \dots, n$.

We use an $o(h^4)$ difference scheme to estimate $\frac{\partial U}{\partial x}(x_i, t)$ and $\frac{\partial^2 U}{\partial x^2}(x_i, t)$ on the internal nodes. Hence we have the following difference scheme(see [12]):

for $x = x_1$ we use

$$\frac{\partial U}{\partial x}(x_1, t) = \frac{1}{24h}(-6y_0(t) - 20y_1(t) + 36y_2(t) - 12y_3(t) + 2y_4(t)) = E_1(t) \quad (7)$$

and for $x = x_i$, $i = 2, \dots, n-1$ we use

$$\frac{\partial U}{\partial x}(x_i, t) = \frac{1}{24h}(2y_{i-2}(t) - 16y_{i-1}(t) + 16y_{i+1}(t) - 2y_{i+2}(t)) = E_i(t), \quad (8)$$

for $x = x_n$ we use

$$\frac{\partial U}{\partial x}(x_n, t) = \frac{1}{24h}(-2y_{n-3}(t) + 12y_{n-2}(t) - 36y_{n-1}(t) + 20y_n(t) + 6y_{n+1}(t)) = E_n(t), \quad (9)$$

Similarly for $x = x_1$ we use

$$\frac{\partial^2 U}{\partial x^2}(x_1, t) = \frac{1}{12h^2}(20g_1(t) - 30y_1(t) - 8y_2(t) + 28y_3(t) - 12y_4(t) + 2y_5(t)) = R_1(t) \quad (10)$$

and for $x = x_i$, $i = 2, \dots, n-1$ we use

$$\frac{\partial^2 U}{\partial x^2}(x_i, t) = \frac{1}{12h^2}(-2y_{i-2}(t) + 32y_{i-1}(t) - 60y_i(t) + 32y_{i+1}(t) - 2y_{i+2}(t)) = R_i(t) \quad (11)$$

for $x = x_n$ we use

$$\frac{\partial^2 U}{\partial x^2}(x_n, t) = \frac{1}{12h^2}(2y_{n-4}(t) - 12y_{n-3}(t) + 28y_{n-2}(t) - 8y_{n-1}(t) - 30y_n(t) + 20g_2(t)) = R_n(t), \quad (12)$$

where $U(x_i, t) = y_i(t)$.

By substituting (7)-(12) into (1) and taking into account that $y_0(t) = g_1(t)$ and $y_{n+1}(t) = g_2(t)$ we obtain the following system of ODEs with the initial condition

$$\begin{aligned} y'_i(t) + y_i(t)E_i(t) &= \mu R_i(t) \\ y_i(0) &= \varphi(x_i), \quad i = 1, \dots, n. \end{aligned} \quad (13)$$

Then the system (13) can be rewritten in the following form:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(0) = f_0, \end{cases} \quad (14)$$

where

$$\begin{aligned} y(t) &= [y_1(t), y_2(t), \dots, y_n(t)]^T, \\ y_0 &= [y_1(0), y_2(0), \dots, y_n(0)]^T, \\ y'(t) &= [y'_1(t), y'_2(t), \dots, y'_n(t)]^T, \\ f(t, y(t)) &= [f_1(t, y(t)), f_2(t, y(t)), \dots, f_n(t, y(t))]^T \end{aligned}$$

and

$$f_i(t, y(t)) = -y_i(t)E_i(t) + \mu R_i(t), \quad i = 1, 2, \dots, n. \quad (15)$$

Consider system (14) and suppose that it is stiff. In the backward differentiation formula (BDF), we have

$$y_n = \sum_{j=1}^k \alpha_j y_{n-j} + \Delta t \beta_k f(t_n, y_n), \quad (16)$$

where k is the order of scheme and the values of α_j and β_k are given in [8]. From (16) we have the following system of nonlinear equations to be solved:

$$F(x_n) = x_n - \Delta t f(t_n, \alpha_n + \beta_k x_n), \quad (17)$$

where $\alpha_n = \sum_{j=1}^k \alpha_j y_{n-j}$ and $x_n = \Delta t f(t_n, y_n)$, see [9]. Usually, to solve a system like (17), a modified Newton method is used. Then a direct method is usually used to solve any resulting system of linear equations. Hence in each step, we need to obtain the jacobian matrix and its related decomposed matrices. Therefore causing more computational cost and running time. In matrix free (MF-BDF) method, the jacobian matrix is not used explicitly, because the exact Newton method is used and then the IOM algorithm, see [17], is applied to solve the resulting system of linear equations. Cash, modified the BDF method, this modification led to the EBDF method [8], and MEBDF method [7]. In [15], authors applied the Cashes modification to the MF-BDF scheme and obtained a new method to solve ODE's and PDE's. They called their method MF-MEBDF.

In this study we want to apply the MF-MEBDF methods to solve system of ODEs.

3 Numerical results

We now obtain numerical solutions of Burger's equation. To show the efficiency of the present method for our problem in comparison with the exact solution we report norm two of error which is defined by

$$\|E\|_2 = \left(\sum_{i=1}^n (\hat{U}(x_i, t) - U(x_i, t))^2 \right)^{\frac{1}{2}}$$

where $\hat{U}(x_i, t)$ is the solution obtained by equation (14) solved by MF-MEBDF method and $U(x_i, t)$ is the exact solution.

Problem 1. In Table 1 we show norm two of error for various values of μ , t and n with $\Delta t = 0.0001$.

Table1. Norm two of error for various values of μ , t and n

μ, n	$\mu = 1, n = 10$	$\mu = 1, n = 20$	$\mu = 0.1, n = 10$	$\mu = 0.1, n = 20$
t				
0.2	$2.4244E - 11$	$1.7892E - 10$	$2.9949E - 5$	$2.7542E - 6$
0.4	$6.4101E - 11$	$1.8886E - 10$	$6.7446E - 5$	$5.1001E - 6$
0.6	$3.7221E - 10$	$5.0883E - 10$	$8.9219E - 5$	$6.9656E - 6$
0.8	$6.0105E - 10$	$6.8384E - 10$	$8.2415E - 5$	$7.3609E - 6$
1	$6.3600E - 10$	$7.2012E - 10$	$5.5745E - 5$	$5.7378E - 6$

Problem 1. In Table 2 we show norm two of error for various values of μ , t and n with $\Delta t = 0.0001$ with $\alpha = 0.4$, $\lambda = 0.125$ and $\beta = 0.6$.

Table2. Norm two of error for various values of μ , t and n

μ, n	$\mu = 1, n = 10$	$\mu = 1, n = 20$	$\mu = 0.1, n = 10$	$\mu = 0.1, n = 20$
t				
0.2	$6.0661E - 10$	$1.6111E - 11$	$6.2557E - 5$	$7.6966E - 6$
0.4	$6.7393E - 10$	$3.0020E - 11$	$8.5129E - 5$	$1.4013E - 5$
0.6	$1.0850E - 9$	$5.1813E - 10$	$2.0394E - 4$	$1.8400E - 5$
0.8	$1.2968E - 9$	$7.9854E - 10$	$2.5610E - 4$	$2.1588E - 5$
1	$1.3264E - 9$	$8.3287E - 10$	$2.3027E - 4$	$2.0974E - 5$

4 Conclusions

In this paper, we have proposed an efficient method for solving Burger's equations, with high convergence and small error. As seen in Table 1-2, errors are very small and they are very better than the results of another papers cited in this article.

References

- [1] S.Abbasbandy and M. T. Darvishi, A numerical solution of Burgers' equation by modified Adomian method, *Appl. Math. Comput.*, **163**(2005) 1265-1272.
- [2] S. Abbasbandy and M. T. Darvishi, A numerical solution of Burgers' equation by time discretization of Adomian method, *Appl. Math. Comput.*, (in press).

- [3] M. A. Abdou, A.A Soliman, Variational iteration method for solving Buger's and coupled Burger's equations, *J. Comput. Applied math.*, (in press).
- [4] E. N. Aksan and A. Özdeş, A numerical solution of Bergurs' equation, *Appl. Math. Comput.*, **156** (2004) 395-402.
- [5] M. Berzins, and P.M. Dew, A note on C^0 Chebyshev methods for parabolic PDE's, *IMA. J. Numer. Anal.*, **17**(1987) 15-37.
- [6] J. M. Burger, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Math.* **1** (1947) 171-179.
- [7] J. R. Cash, The integration of stiff initial value problems in ODE's using modified extended backward formula, *Comp. Math. Applic.*, **9**(1983), 645-657.
- [8] J. R. Cash, On the integration of stiff system of ODE's using extended backward formula. *Numer. Math.* **34**. No. 2, (1980), 235-246.
- [9] J. R. Cash, The numerical solution of ordinary differential equations, *Proc. Comp. Phil. Soc.*, **76** (1974), 443-457.
- [10] J. D. Cole, On a quasilinear parabolic equations occuring in aerodynamics, *Quart. Appl. Math.* **9** (1951) 225-236.
- [11] M. T. Darvishi, and M. Javidi, A numerical solution of Burger's equation by pseudospectral method and Darvishi's preconditioning, *Appl. Math. Comput.*, **173**, no. 1 (2006) 421-429.
- [12] M. T. Darvishi, and M. Javidi, Method of lines for solving system of time dependent partial differential equations, *WSEAS Transactions on Mathematics*, **1**, Issues 1, (2002), 218-222.
- [13] H.M. El-Hawary, E.O. Abdel-Rahman, Numerical solution of the generalized Burger's equation via spectral/spline methods, *Appl. Math. Comput.*, (in press).
- [14] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Commun. Pure Appl. Math.* **3**(1950) 201-230.
- [15] S. M. Hosseini, and G. Hojatti, Matrix free MEBDF method for the solution of stiff systems of ODE's, *Mathematical and computer Modeling*, **29** (1999), 67-77.
- [16] Özış, E. N. Aksan, A finite element approach for solution of Burgers' equation, *Appl. Math. Comput.* **139** (2003) 417-428.
- [17] Y. Saad, Krylove subspace methods for solving large unsymmetric linear systems, *Math. Comp*, **76** (1974), 443-457.

Received: January 9, 2006