On ideals in some Banach algebras associated with a foundation semigroup

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Abstract

In the present paper, among the other things, for a large family of topological semigroups namely, foundation semigroup S , we study the question of whether a closed subspace of $LUC(S)^*$ ($M_a(S)$, respectively), invariant under L_x^{**} (L_x , respectively), $x \in S$, is a left ideal.

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1 Introduction

Throughout this paper S will denote a topological semigroup, i.e., S is a semigroup with a topology such that the map $(x, y) \mapsto xy$ from $S \times S$ into S is continuous, when $S \times S$ has the product topology.

Recall that on a locally compact Hausdorff topological semigroup S , $M_a(S)$ denotes the space of all measures $\mu \in M(S)$ (the space of all bounded complex Radon measures on S) for which the mappings $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ from S into $M(S)$ are weakly continuous. A topological semigroup S is called a foundation semigroup if $\bigcup \{ \text{supp } \mu; \mu \in M_a(S) \}$ is dense in S .

Recall that, for any Banach algebra A we denote by A^* and A^{**} its first dual and second dual, respectively. The first Arens multiplication on A^{**} is defined in three steps as follows.

For a, b in A , f in A^* and F, G in A^{**} , the elements fa , Ff of A^* and GF of A^{**} are defined by

$$\langle fa, b \rangle = \langle f, ab \rangle, \quad \langle Ff, a \rangle = \langle F, fa \rangle, \quad \langle GF, f \rangle = \langle G, Ff \rangle.$$

Let $LUC(S)$ denote the closed subspace of bounded left uniformly continuous functions on S , we have shown in [8] that $M_a(S)^*M_a(S) = LUC(S)$. $LUC(S)^*$ is a Banach algebra with the Arens multiplication. Further results on foundation semigroup may be found in ([1], [3], [5] and [12]).

2 Main Results

Let L_x (R_x , respectively), $x \in S$ denote the translation operator in $M_a(S)$, that is, for $\mu \in M_a(S)$, $L_x(\mu) = \delta_x * \mu$ ($R_x(\mu) = \mu * \delta_x$, respectively). In the following Theorem we prove that a closed subspace in $M_a(S)$ is a left ideal if and only if it is invariant under each L_x , $x \in S$. The object of this section is to discuss the similarities in this type of behavior where one considers translations in $M_a(S)^{**}$ and $LUC(S)^*$.

Theorem 2.1 *Let S be a foundation Hausdorff locally compact topological semigroup with identity. Then a closed subspace I in $M_a(S)$ is a left ideal if and only if it is invariant under each L_x , $x \in S$.*

Proof. Let I be a closed subspace in $M_a(S)$. Suppose first that I is a left ideal and $\mu \in I$. Let $x \in S$, and let $(e_\alpha)_{\alpha \in D}$ be a bounded approximate identity in $M_a(S)$ [9]. Since $M_a(S)$ is an ideal in $M(S)$ [5],

$$L_x(e_\alpha * \mu) = \delta_x * e_\alpha * \mu \in I,$$

whenever $\alpha \in D$. On the other hand, $\delta_x * e_\alpha * \mu \rightarrow \delta_x * \mu$ in the norm topology, and so $\delta_x * \mu \in I$. This shows that I is invariant under each L_x , $x \in S$.

To prove the converse, let $\mu \in I$ and let ν be a probability measure in $M_a(S)$. We will show that $\nu * \mu \in I$. We assume to the contrary that $\nu * \mu$ is not in the closed subspace I . Part (b) of the separation Theorem 3.4 in [13] shows that there exist $f \in M_a(S)^*$ and $\gamma \in \mathbb{R}$ such that,

$$\operatorname{Re}\langle f, \delta_x * \mu \rangle < \gamma < \operatorname{Re}\langle f, \nu * \mu \rangle,$$

whenever $x \in S$. By Theorem 3.27 in [13], we have

$$\begin{aligned} \operatorname{Re}\langle f, \nu * \mu \rangle &= \operatorname{Re} \int_S \langle f, \delta_x * \mu \rangle d\nu(x) \\ &= \int_S \operatorname{Re}\langle f, \delta_x * \mu \rangle d\nu(x) \\ &\leq \gamma < \operatorname{Re}\langle f, \nu * \mu \rangle. \end{aligned}$$

This is a contradiction. The proof is now complete.

In recent years there has been shown considerable interest by harmonic analysts in the ideal problem of $L^1(G)$, $LUC(S)^*$ and $L^1(G)^{**}$ of a locally compact group G (see [2], [6], [7] and [9]). It seems to the author that the ideal problem of corresponding algebras of topological semigroups has not been touched so far. The following Theorem is indeed a main result of this paper.

Theorem 2.2 *Let S be a foundation Hausdorff locally compact topological semigroup with identity. If I is a $\sigma(LUC(S)^*, LUC(S))$ -closed subspace of $LUC(S)^*$ invariant under all L_x^{**} , then I is a left ideal.*

Proof. Suppose that I is a $\sigma(LUC(S)^*, LUC(S))$ -closed subspace of $LUC(S)^*$ invariant under all L_x^{**} , and let $F \in I$, $\mu \in M_a(S)$. We first prove that $\mu F \in I$. If μF is not in the closed subspace I , then there exists $f \in LUC(S)$ such that $\langle \mu F, f \rangle = 1$ but $\langle L_x^{**}(F), f \rangle = 0$ for every $x \in S$ (see Theorem 3.5 in [13]). Consequently, $\langle F, L_x^*(f) \rangle = 0$ for every $x \in S$. Since $f \in LUC(S) = M_a(S)^* M_a(S)$ (see Lemma 2.2 in [8]), so $f = g\nu$ for some $g \in M_a(S)^*$, $\nu \in M_a(S)$. For any $\eta \in M_a(S)$, and any $x \in S$,

$$\begin{aligned} \langle L_x^*(f), \eta \rangle &= \langle f, L_x(\eta) \rangle = \langle g\nu, \delta_x * \eta \rangle \\ &= \langle g, \nu * \delta_x * \eta \rangle = \langle g\nu * \delta_x, \eta \rangle. \end{aligned}$$

This implies that $L_x^*(f) = g\nu * \delta_x$ for all $x \in S$. On the other hand,

$$\begin{aligned} \langle \mu F, f \rangle &= \langle F, f\mu \rangle = \langle F, (g\nu)\mu \rangle = \langle F, g\nu * \mu \rangle \\ &= \langle Fg, \nu * \mu \rangle = \int_S \langle Fg, \nu * \delta_x \rangle d\mu(x) \\ &= \int_S \langle F, g\nu * \delta_x \rangle d\mu(x) = \int_S \langle F, L_x^*(f) \rangle d\mu(x) = 0. \end{aligned}$$

This is a contradiction. Thus $\mu F \in I$ for any $\mu \in M_a(S)$.

Let $G \in LUC(S)^*$. Since $M_a(S)$ is $\sigma(LUC(S)^*, LUC(S))$ -dense in $LUC(S)^*$, there is a net $(\nu_\alpha) \subseteq M_a(S)$ such that $\nu_\alpha \rightarrow G$ in the $\sigma(LUC(S)^*, LUC(S))$ -topology. Clearly, $\nu_\alpha F \rightarrow GF$ in the $\sigma(LUC(S)^*, LUC(S))$ -topology. By assumption, we have $GF \in I$. So I is a left ideal in $LUC(S)^*$.

Theorem 2.3 *If I is a right ideal in $M_a(S)^{**}$, then I is invariant under R_x^{**} , $x \in S$.*

Proof. If I is a right ideal in $M_a(S)^{**}$ and $x \in S$, then $R_x^{**}(F) = R_x^{**}(FE)$, where E is a right identity for $M_a(S)^{**}$. For any $f \in M_a(S)^*$, and any $\mu, \nu \in M_a(S)$,

$$\begin{aligned} \langle R_x^*(f\mu), \nu \rangle &= \langle f\mu, R_x(\nu) \rangle = \langle f\mu, \nu * \delta_x \rangle \\ &= \langle f, \mu * \nu * \delta_x \rangle = \langle f, R_x(\mu * \nu) \rangle \\ &= \langle R_x^*(f), \mu * \nu \rangle = \langle R_x^*(f)\mu, \nu \rangle. \end{aligned}$$

This implies that, $R_x^*(f\mu) = R_x^*(f)\mu$.

On the other hand,

$$\begin{aligned} \langle R_x^{**}(E)f, \mu \rangle &= \langle R_x^{**}(E), f\mu \rangle = \langle E, R_x^*(f\mu) \rangle \\ &= \langle E, R_x^*(f)\mu \rangle = \langle \mu, R_x^*(f) \rangle \\ &= \langle R_x^*(f), \mu \rangle. \end{aligned}$$

Consequently, $R_x^{**}(E)f = R_x^*(f)$. We have

$$\begin{aligned}\langle R_x^{**}(F), f \rangle &= \langle R_x^{**}(FE), f \rangle = \langle FE, R_x^*(f) \rangle \\ &= \langle F, R_x^*(f) \rangle = \langle F, R_x^{**}(E)f \rangle \\ &= \langle FR_x^{**}(E), f \rangle.\end{aligned}$$

Thus $R_x^{**}(F) = FR_x^{**}(E) \in I$ as asserted.

A topological semigroup S is called compactly cancellative if CD^{-1} and $C^{-1}D$ are compact subsets of S for every two compact subsets C and D of S , where $CD^{-1} = \{z \in S; zd \in C \text{ for some } d \in D\}$, $C^{-1}D = \{z \in S; cz \in D \text{ for some } c \in C\}$. In [4] Civin proved that, if G is not discrete, there exists a closed left ideal in $L^1(G)^{**}$ which is not invariant under L_x^{**} , $x \in G$. In the following Theorem we will show that $LUC(S)^*$ has a closed subspace which is invariant under R_x^{**} for all $x \in S$ and which is neither a left nor a right ideal.

Theorem 2.4 *Let S be a foundation Hausdorff compactly cancellative topological semigroup with identity. If S is not discrete, there exists a closed subspace of $LUC(S)^*$ which is invariant under R_x^{**} for all $x \in S$ and which is neither a left nor a right ideal.*

Proof. Let S be a non compact semigroup. $M_a(S)$ is clearly invariant under R_x^{**} for each $x \in S$ and which is neither a left nor a right ideal by Corollary 4 in [11]. Now, we may suppose that S is compact. We have $LUC(S)^* = M(S)$. Clearly $M(S)$ has a closed subspace which is neither a left nor a right ideal. The proof is now complete.

References

- [1] M. Amini and A. R. Medghalchi, Fourier algebras on topological foundation *-semigroup, *Semigroup Forum*, **68** (2004), 322-334.
- [2] J. W. Baker and M. Filali, On minimal ideals in some Banach algebras associated with a locally compact group, *J. London Math. Soc.*, **63** (2001), 83 - 98.
- [3] J. F. Berglund and H. D. Junghenn and P. Milnes, *Analysis on semigroups, function spaces, compactifications, representations*, New York, 1989.
- [4] P. Civin, Ideals in the second conjugate algebra of a group algebra, *Math. Scand.*, **11** (1962), 161-174.
- [5] H. A. M. Dzinotyiweyi, *The analogue of the group algebra for topological semigroup*, Pitman, Boston, London, 1984.

- [6] M. M. Filali, Finite dimensional left ideals in some Banach algebras associated with a locally compact group, *Proc. Amer. Math. Soc.*, **127** (1999), 2325-2333.
- [7] M. M. Filali, Finite dimensional right ideals in some Banach algebras associated with a locally compact group, *Proc. Amer. Math. Soc.*, **127** (1999), 1729-1734.
- [8] A. Ghaffari, Convolution operators on semigroup algebras, *Southeast Asian Bulletin of Mathematics*, **27** (2004), 1025-1036.
- [9] A. Ghaffari and A. R. Medghalchi, The Socle and finite dimensionality of some Banach algebras, *Proc. Indian Acad. Sci.*, **115** (2005), 327-330.
- [10] M. Lashkarizadeh Bami, Function algebras on weighted topological semigroups, *Math. Japon.*, **47** (1998), 217-227.
- [11] M. Lashkarizadeh Bami, Ideals of $M(S)$ as ideals of $LUC(S)^*$ of a compactly cancellative topological semigroup S , *Math. Japon.*, **48** (1998), 363-366.
- [12] M. Lashkarizadeh Bami, The topological centers of $LUC(S)^*$ and $M_a(S)^*$ of certain foundation semigroups, *Glasg. Math. J.*, **42** (2000), 335-343.
- [13] W. Rudin, *Functional analysis*, McGraw Hill, New York, 1991.

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