Inner invariant means on a locally compact group

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Abstract

For a locally compact group \( G \), \( L^1(G) \) is its group algebra and \( L^\infty(G) \) is the dual of \( L^1(G) \). In this paper, among the other things, we obtain a necessary and sufficient condition for \( L^\infty(G) \) to have an inner invariant mean.

Mathematics Subject Classification: 43A60

Keywords: Banach algebras, locally compact group, inner invariant mean, unimodular group

1 Introduction

Let \( G \) denote a locally compact group with identity, left Haar measure \( \lambda \) and modular function \( \Delta \). Let \( L^p(G) \), \( 1 \leq p \leq \infty \) have the usual meaning and

\[
P^p(G) = \{ f \in L^p(G); f \geq 0, \|f\|_p = 1 \}.
\]

For every \( \phi \in L^1(G) \), and \( f \in L^p(G) \) the \( \ast\ast \) convolution \( \phi \ast f \),

\[
\phi \ast f(x) = \int_G \Delta(y)^{\frac{1}{p}} f(y^{-1}xy)\phi(y) \, dy \quad (x \in G)
\]

exists and represents an element of \( L^p(G) \) of norm \( \|\phi \ast f\|_p \leq \|\phi\|_1 \|f\|_p \) (see [10]).

A mean \( m \) on \( L^\infty(G) \) is a positive linear functional on \( L^\infty(G) \) satisfying \( m(1) = 1 \). For a function \( f : G \rightarrow C \), we put

\[
x f(x)(y) = f(x^{-1}yx)
\]
for any \( x, y \in G \). Following Effros [1] and Yuan [10], we say that \( G \) is \textit{inner amenable} if there exists a mean \( m \) on \( L^\infty(G) \) such that \( m(xf_x) = m(f) \) for all \( x \in G \) and \( f \in L^\infty(G) \).

Inner amenability is a considerably weaker condition on \( G \) than amenability in the usual sense. The free group on two generators is an easily accessible example of a group which is not inner amenable. Amenable locally compact groups and \([IN]\)-groups are inner amenable [4]. We shall follow Hewitt [3] and Yuan [10] for definitions and terminologies not explained here. The literature on inner amenability has grown substantially in recent years (see [5], [6], [7] and [12]).

\section{Main Results}

Now we state the following characterization Theorem of inner amenable groups.

\textbf{Theorem 2.1.} For a locally compact group \( G \), the following conditions are equivalent:

(1) \( G \) is an inner amenable group.

(2) There exists a net \((f_\alpha)\) in \( P^2(G) \cap C_C(G) \) such that for every compact subset \( K \) of \( G \),

\[
\lim_{\alpha} |1 - f_\alpha \ast f_\alpha(x)| = 0
\]

uniformly on \( K \).

(3) For every compact subset \( K \) of \( G \) and \( \delta, \epsilon > 0 \), there exist compact subsets \( U \) and \( N \) in \( G \) such that \( |U| > 0, |N| < \delta \), and

\[
\frac{|xUx^{-1}\Delta U|}{|U|} < \epsilon
\]

for all \( x \in K \setminus N \).

\textbf{Proof.} Assuming that \( G \) is inner amenable. Let a compact subset \( K \) of \( G \) and \( \epsilon \in (0,1) \) be given. By Theorem 1 in [10], there exists \( h \in P^2(G) \) such that

\[
\|\Delta(y)^{\frac{1}{2}} y_{\phi y} - h\|_2 < \frac{\epsilon}{6}
\]

for all \( y \in K \). By density of \( C_C(G) \), we may determine \( \phi \in C_C(G) \) such that \( \|\phi - h\|_2 < \frac{\epsilon}{6} \). For every \( y \in K \),

\[
\|\Delta(y)^{\frac{1}{2}} y_{\phi y} - \phi\|_2 \leq \|\Delta(y)^{\frac{1}{2}} y_{\phi y} - \Delta(y)^{\frac{1}{2}} y_{h y}\|_2 + \|\Delta(y)^{\frac{1}{2}} y_{h y} - h\|_2 + \|\phi - h\|_2
\]

\[
= \|\phi - h\|_2 + \|\Delta(y)^{\frac{1}{2}} y_{h y} - h\|_2 + \|\phi - h\|_2 < \frac{\epsilon}{2}
\]
Let $f = \frac{\phi}{\|\phi\|_2}$. It is easy to see that
\[
\|\Delta(y)^{1/2} y f_y - f\|_2 < \epsilon
\]
whenever $y \in K$. For every $y \in K$, we have
\[
1 - \int_G \Delta(y)^{1/2} f(y^{-1} x y) f(x) \, dx = \int_G [f(x) f(x) - \Delta(y)^{1/2} f(y^{-1} x y) f(x)] \, dx
\]
\[
\leq \int_G |f(x)| |f(x) - \Delta(y)^{1/2} f(y^{-1} x y)| \, dx
\]
\[
\leq \|f\|_2 \left( \int_G |f(x) - \Delta(y)^{1/2} f(y^{-1} x y)|^2 \, dx \right)^{1/2}
\]
\[
= \|f - \Delta(y)^{1/2} y f_y\|_2 < \epsilon.
\]
This shows that $|1 - f * f(y)| < \epsilon$ for all $y \in K$. Consequently for each pair $(K, \epsilon)$, where $K \subseteq G$ is compact and $\epsilon > 0$, there is a $f_{(K,\epsilon)} \in P^2(G) \cap C_c(G)$ such that
\[
|1 - f_{(K,\epsilon)} * f_{(K,\epsilon)}(y)| < \epsilon
\]
for $y \in K$. Then we define the partial ordering on the index set as $(K, \epsilon) \leq (K', \epsilon')$ if $K \subseteq K'$ and $\epsilon \geq \epsilon'$. Clearly $f_{(K,\epsilon)} * f_{(K,\epsilon)}$ converges to 1 in the uniform topology on compacta. So, (1) implies (2).

For the converse, if $\epsilon > 0$ and compact set $K \subseteq G$ are specified, there is some $f \in P^2(G) \cap C_c(G)$ such that
\[
|1 - f * f(y)| < \frac{\epsilon^2}{2},
\]
for all $y \in K$. On the other hand, for every $y \in K$, $\|f\|_2 = \|\Delta(y)^{1/2} y f_y\|_2 = 1$. Hence,
\[
\|f - \Delta(y)^{1/2} y f_y\|_2^2 = \int_G |f(x) - \Delta(y)^{1/2} y f_y(x)|^2 \, dx
\]
\[
= \|f\|_2^2 + \|\Delta(y)^{1/2} y f_y\|_2^2 - 2 \int_G \Delta(y)^{1/2} y f_y(x) f(x) \, dx
\]
\[
= 2 - 2 \int_G \Delta(y)^{1/2} y f_y(x) f(x) \, dx
\]
\[
= 2 - 2 f * f(y) < \epsilon^2.
\]
This shows that $\|f - \Delta(y)^{1/2} y f_y\|_2 < \epsilon$ for all $y \in K$. By Theorem 1 in [10], $G$ is inner amenable.

(1) implies (3). Let a compact subset $K$ of $G$ and $\delta$, $\epsilon > 0$ be given. By Theorem 2 in [11] there exists a compact subset $U$ of $G$ such that $0 < |U|$ and
\[
\frac{|aUa^{-1} \Delta U|}{|U|} < \epsilon
\]
whenever $a \in K$. Thus (1) implies (3).

Finally it suffices to show that (3) implies (1). Let $K \subseteq G$ be compact with left Haar measure $|K| > 0$ and let $W = KK$. We may assume the unite in $G$ is in $K$. Let $M = \max\{\Delta(x^{-1}); \ x \in W\}$ and $\epsilon > 0$. Now apply (3) to $\delta = \frac{|K|}{2}, \ \frac{\epsilon}{2M}$ and compact set $W$. There exist compact subsets $U$ and $N$ in $G$ such that $0 < |U|, |N| < \delta$ and

$$\frac{|aUa^{-1}\Delta U|}{|U|} < \frac{\epsilon}{2M}$$

whenever $a \in W \setminus N$. By Theorem 7.3 in [9],

$$(W \setminus N) \cap a(W \setminus N) \neq \emptyset$$

for all $a \in K$. For every $a \in K$, there exists $x, y \in W \setminus N$ such that $a = yx^{-1}$.

Thus,

$$\frac{|aUa^{-1}\Delta U|}{|U|} = \frac{|yx^{-1}Ux^{-1}y^{-1}\Delta U|}{|U|} = \frac{|x^{-1}Uxy^{-1}\Delta y^{-1}U|}{|U|}$$

$$= \Delta(y^{-1}) \frac{|x^{-1}Ux\Delta y^{-1}Uy|}{|U|}$$

$$\leq M \left( \frac{|x^{-1}Ux\Delta U|}{|U|} + \frac{|y^{-1}Uy\Delta U|}{|U|} \right) < \epsilon.$$

By Theorem 2 in [11], $G$ is inner amenable.

We denote by $\Omega$ the family of all nonvoid compact subsets in $G$. Let also

$$\Omega_0 = \{K \in \Omega; \ |K| > 0\}.$$

Let $\mu \in M(G)$ and $f \in L^p(G)$, we consider

$$L_\mu: f \mapsto \mu * f \quad (\mu * f(x) = \int_G f(y^{-1}x) \ d\mu(y))$$

We know that $G$ is amenable if and only if $\|L_\mu\| = \|\mu\|$ for any $\mu \in M(G)$ (for details see [8], [9]).

Let $\psi \in L^1(G)$. Define $T_\psi: L^p(G) \rightarrow L^p(G)$ given by $T_\psi(f) = \psi * f$. In the following Theorem, we give sufficient conditions about $G$ such that $\|T_\psi\| = 1$ for all $\psi \in P^1(G)$.

**Theorem 2.2.** Let $p$ a real number such that $1 \leq p < \infty$, and let $p' = \frac{p}{p-1}$ ($1' = \infty$). Let $\psi \in P^1(G)$. Then $\|T_\psi\| = 1$ if any one of the following conditions hold:
(1) $G$ is an inner amenable group.

(2) $\sup \{ \inf \{ \frac{\| |U|^p |V|^q}{|aUa^{-1}\cap V|} \Delta(a)^{\frac{1}{p}} ; \; a \in K \} ; \; U, V \in \Omega_0 ; \; K \in \Omega \} = 1$.

Proof. Assume $G$ to be inner amenable. By Theorem 2.2 in [2], $\| T_\psi \| = 1$ for each $\psi \in P^1(G)$.

Now assume that (2) holds. Let $\phi \in P^1(G) \cap C_c(G)$ with $K = \text{supp} \phi$. Given $\epsilon > 0$, by assumption there exist $U, V \in \Omega_0$ such that

$$\sup \{ \frac{\| |U|^p |V|^q}{|aUa^{-1}\cap V|} \Delta(a)^{\frac{1}{p}} ; \; a \in K \} < 1 + \epsilon.$$ 

Let $f = (\frac{1}{|V|^p})1_U \in P^p(G)$ and $g = (\frac{1}{|V'|^p})1_V \in P^{p'}(G)$. We have

$$\inf \{ \Delta(a)^{\frac{1}{p}}|aUa^{-1}\cap V| ; \; a \in K \} \leq \int_K \Delta(x)^{\frac{1}{p}}|xU.x^{-1}\cap V| \phi(x) \; dx$$

$$= \| |U|^p |V|^{\frac{1}{p}} \int_G \Delta(x)^{\frac{1}{p}} \frac{|xU.x^{-1}\cap V|}{|U|^p |V|^{\frac{1}{p}}} \phi(x) \; dx$$

$$= \| |U|^p |V|^{\frac{1}{p}} \int_G \phi(x) \int_G \Delta(x)^{\frac{1}{p}} f(x^{-1}y|x)g(y) \; dydx$$

$$= \| |U|^p |V|^{\frac{1}{p}} \int_G \phi \ast f(y)g(y) \; dy$$

$$\leq \| |U|^p |V|^{\frac{1}{p}} \| \phi \ast f \|_p \| g \|_{p'}$$

$$\leq \| |U|^p |V|^{\frac{1}{p}} \| T_\phi \|.$$ 

Hence

$$1 \leq \sup \{ \frac{\| |U|^p |V|^{\frac{1}{p}}}{|aUa^{-1}\cap V|} \Delta(a)^{\frac{1}{p}} ; \; a \in K \} \| T_\phi \| < (1 + \epsilon)\| T_\phi \|.$$ 

As $\epsilon > 0$ may be chosen arbitrary, we obtain $\| T_\phi \| = 1$. It is easy to see that $\| T_\psi \| = 1$ for any $\psi \in P^1(G)$.

References


Received: November 26, 2005