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Inner invariant means on a locally compact group

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Abstract

For a locally compact group G, $L^1(G)$ is its group algebra and $L^\infty(G)$ is the dual of $L^1(G)$. In this paper, among the other things, we obtain a necessary and sufficient condition for $L^\infty(G)$ to have an inner invariant mean.

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1 Introduction

Let G denote a locally compact group with identity, left Haar measure λ and modular function Δ. Let $L^p(G)$, $1 \leq p \leq \infty$ have the usual meaning and

$$P^p(G) = \{f \in L^p(G);\ f \geq 0,\ \|f\|_p = 1\}.$$

For every $\phi \in L^1(G)$, and $f \in L^p(G)$ the $\star-$ convolution $\phi \star f$,

$$\phi \star f(x) = \int_G \Delta(y)^{\frac{1}{p}} f(y^{-1}xy)\phi(y)\ dy \qquad (x \in G)$$

exists and represents an element of $L^p(G)$ of norm $\|\phi \star f\|_p \leq \|\phi\|_1 \|f\|_p$ (see [10]).

A mean m on $L^\infty(G)$ is a positive linear functional on $L^\infty(G)$ satisfying $m(1) = 1$. For a function $f : G \to C$, we put

$${}_x f_x(y) = f(x^{-1}yx)$$

for any $x, y \in G$. Following Effros [1] and Yuan [10], we say that G is *inner amenable* if there exists a mean m on $L^\infty(G)$ such that $m({}_xf_x) = m(f)$ for all $x \in G$ and $f \in L^\infty(G)$.

Inner amenability is a considerably weaker condition on G than amenability in the usual sense. The free group on two generators is an easily accessible example of a group which is not inner amenable. Amenable locally compact groups and [IN]-groups are inner amenable [4]. We shall follow Hewitt [3] and Yuan [10] for definitions and terminologies not explained here. The literature on inner amenability has grown substantially in recent years (see [5], [6], [7] and [12]).

2 Main Results

Now we state the following characterization Theorem of inner amenable groups.

Theorem 2.1. For a locally compact group G, the following conditions are equivalent:

(1) G is an inner amenable group.

(2) There exists a net (f_α) in $P^2(G) \bigcap C_C(G)$ such that for every compact subset K of G,

$$\lim_\alpha |1 - f_\alpha \star f_\alpha(x)| = 0$$

uniformly on K.

(3) For every compact subset K of G and $\delta,\ \epsilon > 0$, there exist compact subsets U and N in G such that $|U| > 0$, $|N| < \delta$, and

$$\frac{|xUx^{-1}\Delta U|}{|U|} < \epsilon$$

for all $x \in K \setminus N$.

Proof. Assuming that G is inner amenable. Let a compact subset K of G and $\epsilon \in (0, 1)$ be given. By Theorem 1 in [10], there exists $h \in P^2(G)$ such that $\|\Delta(y)^{\frac{1}{2}}\ {}_yh_y - h\|_2 < \frac{\epsilon}{6}$ for all $y \in K$. By density of $C_C(G)$, we may determine $\phi \in C_C(G)$ such that $\|\phi - h\|_2 < \frac{\epsilon}{6}$. For every $y \in K$,

$$\begin{aligned} \|\Delta(y)^{\frac{1}{2}}\ {}_y\phi_y - \phi\|_2 &\leq \|\Delta(y)^{\frac{1}{2}}\ {}_y\phi_y - \Delta(y)^{\frac{1}{2}}\ {}_yh_y\|_2 + \|\Delta(y)^{\frac{1}{2}}\ {}_yh_y - h\|_2 + \|\phi - h\|_2 \\ &= \|\phi - h\|_2 + \|\Delta(y)^{\frac{1}{2}}\ {}_yh_y - h\|_2 + \|\phi - h\|_2 < \frac{\epsilon}{2}. \end{aligned}$$

Let $f = \frac{\phi}{\|\phi\|_2}$. It is easy to see that

$$\|\Delta(y)^{\frac{1}{2}}\ {}_yf_y - f\|_2 < \epsilon$$

whenever $y \in K$. For every $y \in K$, we have

$$\begin{aligned}
\left|1 - \int_G \Delta(y)^{\frac{1}{2}} f(y^{-1}xy)f(x)\ dx\right| &= \left|\int_G f(x)f(x) - \Delta(y)^{\frac{1}{2}} f(y^{-1}xy)f(x)\ dx\right| \\
&\le \int_G |f(x)| \left|f(x) - \Delta(y)^{\frac{1}{2}} f(y^{-1}xy)\right|\ dx \\
&\le \|f\|_2 \left(\int_G \left|f(x) - \Delta(y)^{\frac{1}{2}} f(y^{-1}xy)\right|^2\ dx\right)^{\frac{1}{2}} \\
&= \|f - \Delta(y)^{\frac{1}{2}}\ {}_yf_y\|_2 < \epsilon.
\end{aligned}$$

This shows that $|1 - f \star f(y)| < \epsilon$ for all $y \in K$. Consequently for each pair (K, ϵ), where $K \subseteq G$ is compact and $\epsilon > 0$, there is a $f_{(K,\epsilon)} \in P^2(G) \bigcap C_C(G)$ such that

$$|1 - f_{(K,\epsilon)} \star f_{(K,\epsilon)}(y)| < \epsilon$$

for $y \in K$. Then we define the partial ordering on the index set as $(K, \epsilon) \le (K', \epsilon')$ if $K \subseteq K'$ and $\epsilon \ge \epsilon'$. Clearly $f_{(K,\epsilon)} \star f_{(K,\epsilon)}$ converges to 1 in the uniform topology on compacta. So, (1) implies (2).

For the converse, if $\epsilon > 0$ and compact set $K \subseteq G$ are specified, there is some $f \in P^2(G) \bigcap C_C(G)$ such that

$$|1 - f \star f(y)| < \frac{\epsilon^2}{2}.$$

for all $y \in K$. On the other hand, for every $y \in K$, $\|f\|_2 = \|\Delta(y)^{\frac{1}{2}}\ {}_yf_y\|_2 = 1$. Hence,

$$\begin{aligned}
\left\|f - \Delta(y)^{\frac{1}{2}}\ {}_yf_y\right\|_2^2 &= \int_G \left|f(x) - \Delta(y)^{\frac{1}{2}}\ {}_yf_y(x)\right|^2\ dx \\
&= \|f\|_2^2 + \|\Delta(y)^{\frac{1}{2}}\ {}_yf_y\|_2^2 - 2\int_G \Delta(y)^{\frac{1}{2}}\ {}_yf_y(x)f(x)\ dx \\
&= 2 - 2\int_G \Delta(y)^{\frac{1}{2}}\ {}_yf_y(x)f(x)\ dx \\
&= 2 - 2f \star f(y) < \epsilon^2.
\end{aligned}$$

This shows that $\left\|f - \Delta(y)^{\frac{1}{2}}\ {}_yf_y\right\|_2 < \epsilon$ for all $y \in K$. By Theorem 1 in [10], G is inner amenable.

(1) implies (3). Let a compact subset K of G and $\delta,\ \epsilon > 0$ be given. By Theorem 2 in [11] there exists a compact subset U of G such that $0 < |U|$ and

$$\frac{|aUa^{-1}\Delta U|}{|U|} < \epsilon$$

whenever $a \in K$. Thus (1) implies (3).

Finally it suffices to show that (3) implies (1). Let $K \subseteq G$ be compact with left Haar measure $|K| > 0$ and let $W = KK$. We may assume the unite in G is in K. Let $M = \max\{\Delta(x^{-1});\ x \in W\}$ and $\epsilon > 0$. Now apply (3) to $\delta = \frac{|K|}{2}, \frac{\epsilon}{2M}$ and compact set W. There exist compact subsets U and N in G such that $0 < |U|, |N| < \delta$ and

$$\frac{|aUa^{-1}\Delta U|}{|U|} < \frac{\epsilon}{2M}$$

whenever $a \in W \setminus N$. By Theorem 7.3 in [9],

$$(W \setminus N) \bigcap a(W \setminus N) \neq \emptyset$$

for all $a \in K$. For every $a \in K$, there exists $x, y \in W \setminus N$ such that $a = yx^{-1}$. Thus,

$$\begin{aligned}
\frac{|aUa^{-1}\Delta U|}{|U|} &= \frac{|yx^{-1}Uxy^{-1}\Delta U|}{|U|} = \frac{|x^{-1}Uxy^{-1}\Delta y^{-1}U|}{|U|} \\
&= \Delta(y^{-1})\frac{|x^{-1}Ux\Delta y^{-1}Uy|}{|U|} \\
&\leq M\left(\frac{|x^{-1}Ux\Delta U|}{|U|} + \frac{|y^{-1}Uy\Delta U|}{|U|}\right) < \epsilon.
\end{aligned}$$

By Theorem 2 in [11], G is inner amenable.

We denote by Ω the family of all nonvoid compact subsets in G. Let also

$$\Omega_0 = \{K \in \Omega;\ |K| > 0\}.$$

Let $\mu \in M(G)$ and $f \in L^p(G)$, we consider

$$L_\mu : f \mapsto \mu * f \qquad (\mu * f(x) = \int_G f(y^{-1}x)\ d\mu(y))$$

We know that G is amenable if and only if $\|L_\mu\| = \|\mu\|$ for any $\mu \in M(G)$ (for details see [8], [9]).

Let $\psi \in L^1(G)$. Define $T_\psi : L^p(G) \to L^p(G)$ given by $T_\psi(f) = \psi \star f$. In the following Theorem, we give sufficient conditions about G such that $\|T_\psi\| = 1$ for all $\psi \in P^1(G)$.

Theorem 2.2. Let p a real number such that $1 \leq p < \infty$, and let $p' = \frac{p}{p-1}$ ($1' = \infty$). Let $\psi \in P^1(G)$. Then $\|T_\psi\| = 1$ if any one of the following conditions hold:

(1) G is an inner amenable group.

(2) $\sup\{inf\{\sup\{\frac{|U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}}{|aUa^{-1}\bigcap V|}\Delta(a)^{\frac{-1}{p}};\ a\in K\};\ U,V\in\Omega_0\};\ K\in\Omega\}=1.$

Proof. Assume G to be inner amenable. By Theorem 2.2 in [2], $\|T_\psi\|=1$ for each $\psi\in P^1(G)$.

Now assume that (2) holds. Let $\phi\in P^1(G)\bigcap C_C(G)$ with $K=supp\ \phi$. Given $\epsilon>0$, by assumption there exist $U,V\in\Omega_0$ such that

$$\sup\{\frac{|U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}}{|aUa^{-1}\bigcap V|}\Delta(a)^{\frac{-1}{p}};\ a\in K\}<1+\epsilon.$$

Let $f=(\frac{1}{|U|^{\frac{1}{p}}})1_U\in P^p(G)$ and $g=(\frac{1}{|V|^{\frac{1}{p'}}})1_V\in P^{p'}(G)$. We have

$$\begin{aligned}
\inf\{\Delta(a)^{\frac{1}{p}}|aUa^{-1}\textstyle\bigcap V|;\ a\in K\} &\leq \int_K \Delta(x)^{\frac{1}{p}}|xUx^{-1}\textstyle\bigcap V|\ \phi(x)\ dx\\
&= |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}\int_G \Delta(x)^{\frac{1}{p}}\frac{|xUx^{-1}\bigcap V|}{|U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}}\ \phi(x)\ dx\\
&= |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}\int_G\phi(x)\int_G\Delta(x)^{\frac{1}{p}}f(x^{-1}yx)g(y)\ dydx\\
&= |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}\int_G\phi\star f(y)g(y)\ dy\\
&= |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}(\phi\star f,g)\\
&\leq |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}\|\phi\star f\|_p\|g\|_{p'}\\
&\leq |U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}\|T_\phi\|.
\end{aligned}$$

Hence

$$1\leq\sup\{\frac{|U|^{\frac{1}{p}}|V|^{\frac{1}{p'}}}{|aUa^{-1}\bigcap V|}\Delta(a)^{\frac{-1}{p}};\ a\in K\}\ \|T_\phi\|<(1+\epsilon)\|T_\phi\|.$$

As $\epsilon>0$ may be chosen arbitrary, we obtain $\|T_\phi\|=1$. It is easy to see that $\|T_\psi\|=1$ for any $\psi\in P^1(G)$.

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