An approximation to the solution of Klein-Gordon equation with initial or boundary value condition

J. Biazar and H. Ebrahimi

Department of Mathematics
Islamic Azad University (Rasht branch)
P. O. Box 41335-3516, Rasht, Iran
biazar@guilan.ac.ir

Abstract

Adomian decomposition method has been applied to solve many functional equations so far. Some authors have used this method for solving Klein-Gordon equation with initial conditions. In this article, Adomian method is applied to solve Klein-Gordon equation with boundary value condition, as well as initial conditions. Three examples are presented to illustrate the method.

Keywords: Adomian decomposition method, Klein-Gordon

1 Introduction

In this work, we will consider the Klein-Gordon equation with initial or boundary conditions and Adomian decomposition method is applied to solve this equation. The Adomian decomposition method has proven to be very effective and results in considerable saving in computation time. Klein-Gordon equation has the following general form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

(1)

Where $c$ is a constant. Let have the following boundary conditions:

$$B_1 u(a_1, y, z, t) = f_1(y, z, t)$$

(2)

$$B_2 u(a_2, y, z, t) = f_2(y, z, t)$$

(3)

Where $B_1$ and $B_2$ are identity or any differentiable operators.
2 The Adomian decomposition method applied to klein-Gordon equation

For solving this equation by Adomian decomposition method, we can pay attention to initial or boundary conditions using operators $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_{yy} = \frac{\partial^2}{\partial y^2}$, and $L_{tt} = \frac{\partial^2}{\partial t^2}$. We use the operator $L_{xx} = \frac{\partial^2}{\partial x^2}$ with the inverse $L_{xx}^{-1} = \int_0^x \int_0^x f(\cdot) dx dy$. Therefore eq.(1) can be written as:

$$L_{xx} u = -\frac{1}{c^2} g + \frac{1}{c^2} L_{tt} u - L_{yy} u - L_{zz} u$$

(4)

By applying the inverse operator $L_{xx}^{-1}$ to both sides of (4), we have:

$$u(x, y, z, t) = u(0, y, z, t) + u_x(0, y, z, t) x - \frac{1}{c^2} L_{xx}^{-1} g + L_{xx}^{-1} \left( \frac{1}{c^2} L_{tt} u - L_{yy} u - L_{zz} u \right)$$

(5)

Let $K_1 = u(0, y, z, t)$ and $K_2 = u_x(0, y, z, t)$. Thus, eq.(5) can be written as:

$$u = K_1 + K_2 x - \frac{1}{c^2} L_{xx}^{-1} g + L_{xx}^{-1} \left( \frac{1}{c^2} L_{tt} u - L_{yy} u - L_{zz} u \right)$$

(6)

To solve this equation by Adomian decomposition method, as usual in this method, the solution $u$ is considered as the sum of the series $u = \sum_{n=0}^{\infty} u_n$ and the integrand on the right side as the sum of a series as:

$$\frac{1}{c^2} L_{tt} u - L_{yy} u - L_{zz} u = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)$$

Where $A_n(u_0, u_1, \ldots, u_n)$ are called Adomian polynomials and should be computed. By using an Alternate Algorithm for computing Adomian polynomial [4], we have:

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{c^2} L_{tt} u_n - L_{yy} u_n - L_{zz} u_n \quad n = 0, 1, 2, \ldots$$

Substituting $u = \sum_{n=0}^{\infty} u_n$ and $A_n(u_0, u_1, \ldots, u_n) in (6)$, we derive:

$$\sum_{n=0}^{\infty} u_n = K_1 + K_2 x - \frac{1}{c^2} L_{xx}^{-1} g + \sum_{n=0}^{\infty} L_{xx}^{-1} \left( \frac{1}{c^2} L_{tt} u_n - L_{yy} u_n - L_{zz} u_n \right)$$

(7)

Therefore from (7) the following procedure can be defined:

$$u_0 = K_1 + K_2 x - \frac{1}{c^2} L_{xx}^{-1} g$$

$$u_{n+1} = L_{xx}^{-1} \left( \frac{1}{c^2} L_{tt} u_n - L_{yy} u_n - L_{zz} u_n \right) \quad n = 0, 1, 2, \ldots$$
For determine $K_1$ and $K_2$, first we consider one-term approximation $\varphi_1$ for the exact solution:

$$\varphi_1 = u_0 = K_1 + K_2 x - \frac{1}{c^2} L_{xx}^{-1} g$$

By using equations (2) and (3), we have:

$$B_1 \varphi_1(a_1, y, z, t) = f_1(y, z, t)$$
$$B_2 \varphi_1(a_2, y, z, t) = f_2(y, z, t)$$

Thus

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} B_1^{-1} f_1 + \frac{1}{c^2} L_{xx}^{-1} g \bigg|_{x=a_1} \\ B_2^{-1} f_2 + \frac{1}{c^2} L_{xx}^{-1} g \bigg|_{x=a_2} \end{bmatrix}$$

By solving eq.(9), we obtain approximate values $K_1$ and $K_2$ and by using Adomian procedure we obtain $u_1$, and consider two-terms approximated values $\varphi_2 = u_0 + u_1$ for the exact solution and using equations (2) and (3), we have:

$$\begin{bmatrix} 1 & a_1 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} B_1^{-1} f_1 + \frac{1}{c^2} L_{xx}^{-1} g \bigg|_{x=a_1} - u_1 \\ B_2^{-1} f_2 + \frac{1}{c^2} L_{xx}^{-1} g \bigg|_{x=a_2} - u_1 \end{bmatrix}$$

Therefore approximation values $K_1$ and $K_2$ from two-terms approximation $\varphi_2$ will be obtained. We can determine the components $u_n$ as far as we like to enhance the accuracy of the approximation and similarity we can be obtain approximation values $K_1$ and $K_2$ in $(n+1)$-terms approximation $\varphi_{n+1} = \sum_{i=0}^{n} u_i$. Also, we have $\lim_{n \to \infty} \varphi_{n+1} = u$.

### 3 Numerical results

**Example 1**: Consider the Klein-Gordon equation with the following boundary conditions.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x^2 - t^2$$

$$u(0, t) = 0$$

$$u(1, t) = \frac{t^2}{2}$$

Regarding boundary conditions we use the operator $\frac{\partial^2}{\partial x^2}$. Therefore, we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + t^2 - x^2$$

Applying the inverse operator $L_{xx}^{-1} = \int_{0}^{x} \int_{0}^{x} \cdot dx$ to both sides of (13),we get:

$$u(x, t) = u(0, t) + \frac{\partial u(0, t)}{\partial x} x + \int_{0}^{x} \int_{0}^{x} (t^2 - x^2) dx dx + \int_{0}^{x} \int_{0}^{x} \frac{\partial^2 u}{\partial t^2} dx dx$$
Let $K_1 = u(0, t)$ and $K_2 = \frac{\partial u(0, t)}{\partial x}$. Therefore, the Adomian scheme would be as follows:

$$u_0 = K_1 + K_2 x + \frac{x^2 t^2}{2} - \frac{x^4}{12}$$

$$u_{n+1} = \int_0^x \int_0^x \frac{\partial^2 u_n}{\partial t^2} dx dx \quad n = 0, 1, 2, \ldots \quad (14)$$

By using (11) we have $K_1 = 0$. To find $K_2$, we consider the following one-term approximation:

$$\varphi_1 = u_0 = K_2 x + \frac{x^2 t^2}{2} - \frac{x^4}{12}$$

By using (12) we have:

$$\varphi_1(1, t) = \frac{t^2}{2} \Rightarrow K_2 = \frac{1}{12}$$

Therefore, $u_0 = \frac{x}{12} + \frac{x^2 t^2}{2} - \frac{x^4}{12}$. By using (14) $u_1$ would be derived as:

$$u_1 = \int_0^x \int_0^x \frac{\partial^2 u_0}{\partial t^2} dx dx = \int_0^x \int_0^x x^2 dx dx = \frac{x^4}{12}$$

To improve the value of $K_2$, let us consider two-terms approximation

$$\varphi_2 = u_0 + u_2 = K_2 x + \frac{x^2 t^2}{2}$$

for solution $u$. Regarding (12), we obtain $K_2 = 0$. Therefore $\varphi_2 = \frac{x^2 t^2}{2}$. Again by using (14), we get:

$$u_2 = \int_0^x \int_0^x \frac{\partial^2 u_1}{\partial t^2} dx dx = \int_0^x \int_0^x (0) dx dx = 0$$

$$u_2 = \int_0^x \int_0^x \frac{\partial^2 u_2}{\partial t^2} dx dx = \int_0^x \int_0^x (0) dx dx = 0$$

$$\vdots$$

$$u_n = 0$$

$$\vdots$$

Therefore $n$-terms approximation is $\varphi_n = \frac{x^2 t^2}{2}$ and solution is:

$$u(x, t) = \lim_{n \to \infty} \varphi_n = \lim_{n \to \infty} \frac{x^2 t^2}{2} = \frac{x^2 t^2}{2}$$

This solution is the exact solution.

**Example 2**: Consider the following Klein-Gordon

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + xy \quad (15)$$
Klein-Gordon equation

\[ u(x, 0, t) = 0 \quad (16) \]
\[ u(x, 1, t) = \frac{xt^2}{2} \quad (17) \]
\[ u(0, y, t) = 0 \quad (18) \]
\[ u(1, y, t) = \frac{yt^2}{2} \quad (19) \]

We can use one of the operators \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \) for obtain the solution. By using operator \( \frac{\partial^2}{\partial y^2} \) we have:

\[ u(x, y, t) = u(x, 0, t) + \frac{\partial u(x, 0, t)}{\partial y} \cdot y - \int_0^y \int_0^y xy \, dy \, dy + \int_0^y \int_0^y \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \right) \, dy \, dy \]

Consider \( K_1 = u(x, 0, t) \) and \( K_2 = \frac{\partial u(x, 0, t)}{\partial y} \). Then the Adomian scheme would be as follows:

\[ u_0 = K_1 + K_2 y - \frac{xy^3}{6} \]
\[ u_{n+1} = \int_0^y \int_0^y \left( \frac{\partial^2 u_n}{\partial t^2} - \frac{\partial^2 u_n}{\partial x^2} \right) \, dy \, dy \quad n = 0, 1, 2, \ldots \quad (20) \]

By using (16), we have \( K_1 = 0 \) and to obtain \( K_2 \) we consider the following one-term approximation:

\[ \varphi_1 = u_0 = K_2 y - \frac{xy^3}{6} \]

By Considering (17) we have \( K_2 = \frac{xt^2}{2} + \frac{x}{6} \). Therefore \( u_0 = \frac{xyt^2}{2} + \frac{xy}{6} - \frac{xy^3}{6} \).

Also \( u_1 \) would be as

\[ u_1 = \int_0^y \int_0^y \left( \frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} \right) \, dy \, dy = \frac{xy^3}{6} \]

Similarity by using two-terms approximation \( \varphi_2 = u_0 + u_1 \) and (17), we obtain \( k_2 = \frac{xt^2}{2} \).

Therefore \( \varphi_2 = \frac{xyt^2}{2} \) and from (20) we have:

\[ u_2 = \int_0^y \int_0^y \left( \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} \right) \, dy \, dy = 0 \]
\[ u_3 = 0 \]
\[ \vdots \]
\[ u_n = 0 \]

Then \( n \)-terms approximation is \( \varphi_n = \frac{x y t^2}{2} \) and the exact solution is:

\[ u(x, y, t) = \lim_{n \to \infty} \varphi_n = \lim_{n \to \infty} \frac{x y t^2}{2} = \frac{x y t^2}{2} \]

Also by using operator \( \frac{\partial^2}{\partial x^2} \) derive the same solution.

**Example 3:** Consider the following Klein-Gordon equation with the initial conditions:

\[
\frac{\partial^2 u}{\partial t^2} = 4 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + t^2 x^2
\]

\[ u(x, y, z, 0) = \sin y \]

\[ \frac{\partial u(x, y, z, 0)}{\partial t} = z^2 \]

By using operator \( \frac{\partial^2}{\partial t^2} \) Adomian procedure would be as follows:

\[ u_0 = \frac{1}{12} x^2 t^4 + z^2 t + \sin y \]

\[ u_{n+1} = 4 \int_0^t \int_0^t \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} + \frac{\partial^2 u_n}{\partial z^2} \right) dt \quad n = 0, 1, 2, \ldots \]

First few terms would be as follows:

\[ u_0 = \frac{1}{12} x^2 t^4 + z^2 t + \sin y \]

\[ u_1 = \frac{1}{45} t^6 + \frac{4}{3} t^3 - \sin y \frac{(2t)^2}{2!} \]

\[ u_2 = \sin y \frac{(2t)^4}{4!} \]

\[ u_3 = -\sin y \frac{(2t)^6}{6!} \]

\[ \vdots \]

Therefore, the general term would be as:

\[ u_n = (-1)^n \sin y \frac{(2t)^{2n}}{(2n)!} \quad n = 2, 3, 4 \ldots \]

Then, the solution is:

\[ u(x, y, z, t) = \sum_{n=0}^{\infty} u_n = \frac{1}{45} t^6 + \frac{1}{12} x^2 t^4 + \frac{4}{3} t^3 + z^2 t + \sin y \left( 1 - \frac{(2t)^2}{2!} + \frac{(2t)^4}{4!} - \ldots \right) \]

\[ = \frac{1}{45} t^6 + \frac{1}{12} x^2 t^4 + \frac{4}{3} t^3 + z^2 t + \sin y \cos 2t \]
4 Conclusions and Discussion

The Adomian decomposition method is a powerful method, which has provided an efficient potential for the solution of physical applications modeled by linear and nonlinear differential equations [1,2,3]. The main goal of this work has been to derive an approximation for solution of Klein-Gordon equation. We have achieved this goal by applying Adomian decomposition method. We can be understood from the Examples to solve the equation we have different choices, in Example 2 the choices are \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \), and using each of them leads to the same solution. For computations we used the package Maple 9.

5 References


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