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Construction of singular surfaces over a finite field

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Abstract. Fix a prime power q and integers $m \geq 2$, $\sigma > 0$, $x > 0$, $g \geq 0$, $d \geq m\sigma + 1$ such that $q \geq (\delta - 1)\delta^3$, where $\delta := d^3 + 3dx + 4x + 2g - 2$. Let $C \subset \mathbf{P}^3$ be a smooth degree x curve defined over $\mathbb{F}_q$ such that $h^1(\mathbf{P}^3, \mathcal{I}_C(\sigma - 1)) = h^1(C, \mathcal{O}_C(\sigma - 2)) = 0$ and $p_a(C) = g$. Here we prove the existence of a degree d surface $X \subseteq \mathbf{P}^3$ defined over $\mathbb{F}_q$, such that $\mathrm{Sing}(X) = C$ and X has ordinary multiplicity m along C, i.e. for every $P \in C(\bar{F}_q)$ the tangent cone of X at P is reduced and it is the union of m distinct planes containing the tangent line of C at P.

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1. Singular surfaces over a finite field

Many existence theorems are easy over an algebraically closed base field K. When $K = \bar{F}_p$ there is a large p-power q such that a solution may be defined over $\mathbb{F}_q$ with q not too large. Here we prove the existence of certain degree d surfaces $X \subset \mathbf{P}^3$ with as singular locus a prescribed smooth curve and with a prescribed multiplicity at each point of it and defined over $\mathbb{F}_q$. More precisely, we prove the following result.

Theorem 1. *Fix a prime power q and integers $m \geq 2$, $\sigma > 0$, $x > 0$, $g \geq 0$, $d \geq m\sigma + 1$ such that $q \geq (\delta - 1)\delta^3$, where $\delta := d^3 + 3dx + 4x + 2g - 2$. Let $C \subset \mathbf{P}^3$ be a smooth degree x curve defined over $\mathbb{F}_q$ such that $h^1(\mathbf{P}^3, \mathcal{I}_C(\sigma - 1)) = h^1(C, \mathcal{O}_C(\sigma - 2)) = 0$ and $p_a(C) = g$. Then there exists a degree d surface $X \subseteq \mathbf{P}^3$ defined over $\mathbb{F}_q$, such that $Sing(X) = C$ and X has ordinary multiplicity m along C, i.e. for every $P \in C(\bar{F}_q)$ the tangent cone of X at P is reduced and it is the union of m distinct planes containing the tangent line of C at P.*

By Castelnuovo-Mumford's lemma the homogeneous ideal of C is generated by forms of degree at most σ. In the statement of Theorem 1 we do not assume that C is connected. When C is connected we may take $\sigma = x - 1$, and all cases in which we cannot take $\sigma = x - 2$ are classified ([3]).

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Lemma 1. *Let $C \subset \mathbf{P}^3$ be an integral curve such that*

$$h^1(\mathbf{P}^3, \mathcal{I}_C(\sigma-1)) = h^1(C, \mathcal{O}_C(\sigma-2)) = 0.$$

Fix an integer $t \geq \sigma+1$, $P \in C_{reg}$ and a tangent vector τ of $\mathbf{P}^3$ at P. Then $h^1(\mathbf{P}^3, \mathcal{I}_{C\cup\tau}(z)) = h^2(\mathbf{P}^3, \mathcal{I}_{C\cup\tau}(z-1)) = 0$ for all $z \geq \sigma$, the homogeneous ideal of $C\cup\tau$ is generated by forms of degree at most $\sigma+1$ and there exists a degree t surface containing $C\cup\tau$, but smooth at P.

Proof. If τ is tangent to C, then $\tau \subset C$ and hence there is nothing to prove. Thus we may assume that τ is not tangent to C at P. By Castelnuovo-Mumford's lemma it is sufficient to prove $h^1(\mathbf{P}^3, \mathcal{I}_{C\cup\tau}(\sigma)) = h^2(\mathbf{P}^3, \mathcal{I}_{C\cup\tau}(\sigma-1)) = 0$; indeed, note that since the tangent space of $C\cup\tau$ is two-dimensional and $C\cup\tau$ is scheme-theoretically cut out by degree t surfaces, at least one degree t surface containing $C\cup\tau$ must be smooth at P. Since $\dim(\tau) = 0$, we have $h^2(\mathbf{P}^3, \mathcal{I}_{C\cup\tau}(\sigma-1)) = h^1(C\cup\tau, \mathcal{O}_{C\cup\tau}(\sigma-1)) = h^1(C, \mathcal{O}_C(\sigma-1)) = 0$. Let $H \subset \mathbf{P}^3$ be a plane containing τ and transversal to C at P. Since C is the residual scheme of $C\cup\tau$ with respect to H, there is the following exact sequence:

$$0 \to \mathcal{I}_C(\sigma-1) \to \mathcal{I}_{C\cup\tau}(\sigma) \to \mathcal{I}_{(C\cap\tau)\cap H}(\sigma) \to 0 \tag{1}$$

Hence it is sufficient to prove $h^1(H, \mathcal{I}_{(C\cap\tau)\cap H}(\sigma)) = 0$. We know that $h^1(H, \mathcal{I}_{C\cap H,H}(\sigma-1)) = 0$. Let $R \subset H$ be the line spanned by τ. Since the homogeneous ideal of $C\cap H$ in H is generated by forms of degree at most σ, we have length$(C\cap R) \leq \sigma$. Thus length$((C\cup\tau)\cap R)) \leq \sigma+1$. Since the residual scheme of $(C\cap\tau)\cap H$ with respect to R is $(C\cap H)\backslash(C\cap R)$ we have an exact sequence

$$0 \to \mathcal{I}_{(C\cap H)\backslash(C\cap R),H}(\sigma-1) \to \mathcal{I}_{(C\cup\tau)\cap H,H}(\sigma) \to \mathcal{I}_{C\cap R,R}(\sigma) \to 0 \tag{2}$$

We have $h^1(R, \mathcal{I}_{C\cap R,R}(\sigma)) = 0$, because length$((C\cup\tau)\cap R)) \leq \sigma+1$. We have $H^1(H, \mathcal{I}_{(C\cap H)\backslash(C\cap R),H}(\sigma-1)) \leq h^1(H, \mathcal{I}_{C\cap H,H}(\sigma-1)) = 0$. □

Remark 1. Let X be an integral projective variety and $L, M \in \mathrm{Pic}(W)$. If L is very ample and M is spanned, then $L\otimes M$ is very ample.

Proof of Theorem 1. Let N_C denote the normal sheaf of C in $\mathbf{P}^3$. Since C is smooth, N_C is a rank 2 vector bundle with degree $2g-2+4x$. Let $w : W \to \mathbf{P}^3$ be the blowing-up of C. Set $E := w^{-1}(C)$ and $\mathcal{O}_W(1) := w^*(\mathcal{O}_{\mathbf{P}^3}(1))$. Hence E and $\mathcal{O}_W(1)$ freely generate $\mathrm{Pic}(W)$. We have $\mathcal{O}_W\cdot E\cdot E = -x$, $\mathcal{O}_W(1)\cdot\mathcal{O}_W(1)\cdot E = 0$, $\mathcal{O}_W(1)^3 = 1$ and $E^3 = -4x+2-2g$ ([2], Prop. 6.7). For all integers t, c set $\mathcal{L}_{t,c} := \mathcal{O}_W(t)(-cE)$. Notice that $\mathcal{L}^3_{d,m} = d^3+3dx+4x+2g-2 = \delta$.

(a) Here we will check that $\mathcal{L}_{t,1}$ is spanned for all $t \geq \sigma$. Fix $Q \in W$. First assume $Q \notin E$. Thus $w(Q) \notin C$. Since $t \geq \sigma$, there is a degree t surface $A \subset \mathbf{P}^3$ containg C and with $w(Q) \notin C$. The strict transform A' of A in W is an element of $|\mathcal{L}_{t,c}|$ for some integer $c \geq 1$. Hence $A'' := A' + (c-1)(E) \in |\mathcal{L}_{t,1}|$ and $Q \notin A''$. Now assume $Q \in E$. Thus Q represents a tangent vector τ of $\mathbf{P}^3$ at $w(Q)$ not in the tangent line to C at P. Since C is scheme-theoretically cut out inside $\mathbf{P}^3$ by all degree t hypersurfaces containing C, there is one such surface B whose tangent plane at $w(Q)$ does not contain τ. Use the strict transform of B to show that Q is not in the base locus of $|\mathcal{L}_{t,1}|$.

(b) Here we will check that $\mathcal{L}_{t,1}$ is very ample for all $t \geq \sigma + 1$. It is sufficient to check that $h^0(W, \mathcal{L}_{t,1}(-Z)) = h^0(W, \mathcal{L}_{t,1}) - 2$ for all length two zero-dimensional schemes $Z \subset W$. We need to distinguish 6 cases:

(i) Z is reduced, say $Z = \{Q, Q'\}$, with $Q \notin E$ and $Q' \notin E$;
(ii) Z is reduced, say $Z = \{Q, Q'\}$, with $Q \notin E$ and $Q' \in E$;
(iii) Z is reduced and $Z \subset E$;
(iv) Z is not reduced and $Q := Z_{red} \notin E$.
(v) Z is not reduced, $Q := Z_{red} \in E$ and Z is not in the tangent plane to E at Q;
(vi) Z is not reduced and $Z \subset E$.

We will only check case (vi). By part (a) it is sufficient to check that for all $P \in C$ and all tangent vectors τ of $\mathbf{P}^3$ at P there is a degree t surface $U \subset \mathbf{P}^3$ such that $C \cup \tau \subset U$ and U is smooth at P. This is proved in Lemma 1. By Remark 1 our assumption on the integer d implies that $\mathcal{L}_{d,m}$ is very ample. By [1], Th. 1, our assumption on q and the equality $\delta = \mathcal{L}_{d,m}^3$ implies the existence of a smooth $\Sigma \in |\mathcal{L}_{d,m}|$ defined over $\mathbb{F}_q$. The surface $w(\Sigma)$ is a solution of Theorem 1. □

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