Taut contact circles on H-contact 3-manifolds

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Abstract

We classify non-Sasakian H-contact 3-manifolds (M, η, g) with (η, η_1) taut contact circle (or, equivalently, Cartan structure), where η_1 is a suitable 1-form orthogonal to η . In particular, a compact 3-manifold admits a taut contact circle if and only if it admits a non-Sasakian H-contact metric structure satisfying $\nabla_\xi \tau = 0$, where ξ is the characteristic vector field and $\tau := L_\xi g$ is the torsion tensor.

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1 Introduction

Recently, many authors have studied the harmonicity of unit vector fields in several geometric situations (see for example [8] for a survey). If (M, g) is a compact and orientable Riemannian manifold, a unit vector field V of M is called *harmonic* if it is a critical point for the energy functional restricted to the set of all unit vector fields of M ([14],[15]).

There are many situations in which a distinguished vector field appears in a natural way, for example the characteristic vector field (also called the Reeb vector field) of a contact metric manifold (see [1]). In this case, it is interesting to study how the criticality of the vector field is related to the geometry of the manifold. On the other hand, the first examples of harmonic vector fields, Hopf unit vector fields, are in fact the characteristic vector fields of the standard Sasakian structure on odd-dimensional spheres. In [12] the present author

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introduced the notion of *H-contact manifold*, namely, a contact metric manifold (M, η, g) for which the characteristic vector field ξ is a harmonic vector field, and proved that a contact metric manifold is H-contact if and only if ξ is an eigenvector of the Ricci operator. The class of H-contact manifolds is very large; it extends the class of Sasakian manifolds.

A pair of contact 1-forms (θ_1, θ_2) on a 3-manifold M is called a *taut contact circle* if $\theta_1 \wedge d\theta_1 = \theta_2 \wedge d\theta_2$ and $\theta_1 \wedge d\theta_2 = -\theta_2 \wedge d\theta_1$. A taut contact circle (θ_1, θ_2) with $\theta_1 \wedge d\theta_2 = 0$, is called a *Cartan structure*. In the main theorem of [6], Geiges and Gonzalo proved that a compact 3-manifold M admits a taut contact circle if and only if M is diffeomorphic to a left quotient of the Lie group G under a discrete subgroup Γ , where G is $SU(2)$, $\tilde{SL}(2, \mathbb{R})$ or $\tilde{E}(2)$.

A Sasakian 3-manifold, in general, does not admit a taut contact circle. In this paper we study the existence of taut contact circles on a non-Sasakian H-contact 3-manifold without the compactness condition. More precisely (see Theorem 3.2), we show that on a non-Sasakian H-contact 3-manifold (M, η, g) , the pair of 1-forms (η, η_1) , where η_1 is a suitable 1-form orthogonal to η , is a taut contact circle (or, equivalently, is a Cartan structure), if and only if (M, η, g) is locally isometric to one of the following Lie groups $SU(2)$, $\tilde{SL}(2, \mathbb{R})$, $\tilde{E}(2)$, equipped with a left-invariant non-Sasakian H-contact metric structure satisfying $\nabla_\xi \tau = 0$, where $\tau := L_\xi g$ is the torsion tensor. In this result, η_1 can be replaced by the 1-form η_2 obtained from η_1 with a rotation of 90° . Moreover (see Theorem 3.4), we show that this class is also characterized as the class of 3-manifolds that admit a non-Sasakian contact metric structure (η, g) with (η, η_i) , $i = 1, 2$, taut contact circles (or, equivalently, Cartan structures). Combining Theorem 3.2 and the above Theorem of [6], we get that a compact 3-manifold admits a taut contact circle if and only if it admits a non-Sasakian H-contact metric structure satisfying $\nabla_\xi \tau = 0$.

Remark 1.1 The condition $\nabla_\xi \tau = 0$ that appears in Theorem 3.2, has been considered for the first time by Chern and Hamilton [4] in their study of compact contact 3-manifolds (M, η) . Then, the present author [9] proved that it is the critical point condition for the functional "integral of the scalar curvature" defined on the set of all metrics associated to the fixed contact form η .

2 Preliminaries on contact metric manifolds

In this section we collect some basic facts about contact metric geometry. All manifolds are supposed to be connected and smooth.

A *contact manifold* is a $(2n + 1)$ -dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M . It has an underlying almost contact structure (η, φ, ξ) where ξ is a global vector field (called the

characteristic vector field, or the Reeb vector field) and φ a global tensor of type (1.1) such that

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi.$$

In particular, $J = \varphi|_{\text{Ker}\eta}$ satisfies $J^2 = -I$. A Riemannian metric g can be found such that

$$\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \varphi\cdot), \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot).$$

We refer to (M, η, g) or to $(M, \eta, g, \xi, \varphi)$ as a contact metric (or Riemannian) manifold.

In what follows, we shall denote by ∇ the Levi-Civita connection of M and by R the corresponding Riemannian curvature tensor given by

$$R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

The Ricci tensor of type (0,2) and the corresponding endomorphism field are respectively indicated by ϱ and Q . Moreover, if L denotes the Lie derivative, the tensors $h = \frac{1}{2}L_\xi\varphi$ and $\tau = L_\xi g$ are symmetric and satisfy

$$\tau = 2g(\varphi\cdot, h\cdot), \quad h\varphi = -\varphi h, \quad h\xi = 0, \quad \nabla_\xi\tau = 2g(\varphi, \nabla_\xi h).$$

A *K-contact* manifold is a contact metric manifold such that ξ is a Killing vector field with respect to g . Clearly, M is *K-contact* if and only if $\tau = 0$ (or, equivalently, $h = 0$). Moreover, M is *K-contact* if and only if

$$Q\xi = 2n\xi.$$

A contact metric manifold (M, η, g) is a *Sasakian manifold* if its curvature tensor satisfies

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

for all vector fields X and Y . Any Sasakian manifold is *K-contact* and the converse also holds for 3-dimensional spaces. We refer to [1] for more information about contact Riemannian geometry.

A contact metric manifold (M, η, g) is said to be a *H-contact manifold* if ξ is a harmonic vector field. The following characterization was proved in [12].

Theorem 2.1 *A contact metric manifold (M, η, g) is H-contact if and only if ξ is an eigenvector of the Ricci operator Q and hence $Q\xi = (2n - \text{tr } h^2)\xi$.*

It should be noted that the class of H -contact metric manifolds is very large. In particular, K -contact spaces (and hence, Sasakian manifolds), (k, μ) -spaces, (strongly) locally φ -symmetric spaces are all examples of H -contact manifolds. We refer to [11], [12],[13] for more details on H -contact manifolds.

Next, let $(M, \eta, g, \xi, \varphi)$ be a 3-dimensional contact metric manifold and m a point of M . Let U be the open subset of M where $h \neq 0$ and V the open subset of points $m \in M$ such that $h = 0$ in a neighborhood of m . Then, $U \cup V$ is an open dense subset of M . For every point $m \in U \cup V$ there exists a local orthonormal basis $\{\xi, e_1, e_2 = Je_1\}$ of smooth eigenvectors of h in a neighborhood of m . On U we put $he_1 = \lambda e_1$, where λ is a non-vanishing smooth function which we suppose to be positive. From $h\varphi = -\varphi h$, we have $he_2 = -\lambda e_2$. We recall the following

Lemma 2.2 [3] *On U we have*

$$\begin{aligned}\nabla_{\xi}e_1 &= -ae_2, & \nabla_{\xi}e_2 &= ae_1, \\ \nabla_{e_1}\xi &= -(\lambda + 1)e_2, & \nabla_{e_2}\xi &= -(\lambda - 1)e_1, \\ \nabla_{e_1}e_1 &= \frac{1}{2\lambda}\{(e_2)(\lambda) + A_1\}e_2, & \nabla_{e_2}e_2 &= \frac{1}{2\lambda}\{e_1(\lambda) + A_2\}e_1, \\ \nabla_{e_1}e_2 &= -\frac{1}{2\lambda}\{(e_2)(\lambda) + A_1\}e_1 + (\lambda + 1)\xi, \\ \nabla_{e_2}e_1 &= -\frac{1}{2\lambda}\{e_1(\lambda) + A_2\}e_2 + (\lambda - 1)\xi,\end{aligned}$$

where a is a smooth function, $A_1 = \varrho(\xi, e_1)$ and $A_2 = \varrho(\xi, e_2)$.

We recall that a contact metric manifold $(M, \eta, g, \xi, \varphi)$ is said to be *homogeneous* if there exists a connected Lie group of isometries acting transitively on M and leaving η invariant. It is said to be *locally homogeneous* if the pseudogroup of local isometries acts transitively on M and leaves η invariant. Note that a 3-dimensional locally homogeneous contact metric manifold is locally isometric to a homogeneous one. We now examine these structures when the contact structure is non-Sasakian, H -contact and the additional condition $\nabla_{\xi}\tau = 0$ is satisfied. Such a study will also be useful in the next sections.

In [10], the present author studied 3-manifolds admitting a homogeneous contact metric structure and showed that these manifolds are locally isometric to a Lie group G with a left-invariant contact metric structure (η, ξ, g, φ) . Because of the invariance under left translations, it is enough to describe these structures on the associated Lie algebra $\underline{g}$. We shall consider only the case of unimodular Lie groups because in the non-unimodular case the condition " ξ harmonic " is not satisfied. Then, there exists an orthonormal basis $\{e_1 = \xi, e_2, e_3 = Je_2\}$ such that

$$(2.1) \quad [e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = \lambda_2 e_2,$$

where λ_2, λ_3 are constant. Using (2.1) and the first Cartan structural equations, one gets

$$(2.2) \quad (\nabla_{e_i} e_j) = \begin{pmatrix} 0 & \frac{\lambda_2 + \lambda_3 - 2}{2} e_3 & \frac{-\lambda_2 - \lambda_3 + 2}{2} e_2 \\ \frac{\lambda_2 - \lambda_3 - 2}{2} e_3 & 0 & \frac{\lambda_3 - \lambda_2 + 2}{2} e_1 \\ \frac{\lambda_2 - \lambda_3 + 2}{2} e_2 & \frac{\lambda_3 - \lambda_2 - 2}{2} e_1 & 0 \end{pmatrix},$$

moreover the torsion τ is given by the formula

$$(2.3) \quad \tau = (\lambda_2 - \lambda_3)(\theta^2 \otimes \theta^3 + \theta^3 \otimes \theta^2),$$

where $(\theta^1 = \eta, \theta^2, \theta^3)$ are the 1-forms dual to the vector fields (ξ, e_2, e_3) [10]. Since $\nabla_{e_i} \theta_j$ is the 1-form dual to $\nabla_{e_i} e_j$, using (2.2) and (2.3), we find

$$(2.4) \quad \nabla_\xi \tau = (2 - \lambda_2 - \lambda_3) \tau \varphi.$$

Thus,

$$(2.5) \quad \nabla_\xi \tau = 0 \Leftrightarrow \tau = 0 \text{ or } \lambda_2 + \lambda_3 = 2.$$

Using (2.5), Theorem 3.1 of [10] and Theorem 2.1, we then get the following

Theorem 2.3 *Let M be a homogeneous contact metric 3-manifold. Then, M is H -contact and $\nabla_\xi \tau = 0$ if and only if M is locally isometric to one of the following unimodular Lie groups G equipped with a left-invariant contact metric structure:*

- (1) the Heisenberg group H when $\tau = 0$ with $\lambda_2 = \lambda_3 = 0$;
- (2) the 3-sphere $SU(2)$ when $\tau = 0$ or $\tau \neq 0$ with $\lambda_2 > 0$ and $\lambda_3 = 2 - \lambda_2 > 0$;
- (3) the group $\tilde{E}(2)$ (the universal covering of the group of rigid motions of the Euclidean 2-space) when $\tau \neq 0$ and $(\lambda_2, \lambda_3) = (0, 2)$ or $(2, 0)$;
- (4) the group $\tilde{SL}(2, \mathbb{R})$ when $\tau = 0$ or $\tau \neq 0$ with either $\lambda_2 < 0$ and $\lambda_3 = 2 - \lambda_2 > 0$ or $\lambda_3 < 0$ and $\lambda_2 = 2 - \lambda_3 > 0$.

3 Taut contact circles on contact metric 3-manifolds

We recall the following definitions introduced in [6]. A pair of contact 1-forms (θ_1, θ_2) on a 3-manifold M is called a *contact circle* if for any $(a_1, a_2) \in S^1$, the unit circle in $\mathbb{R}^2$, the linear combination $a_1\theta_1 + a_2\theta_2$ is also a contact form. This implies that any non trivial linear combination $a_1\theta_1 + a_2\theta_2$ with constant coefficients $(a_1, a_2) \neq (0, 0)$ is again a contact form. In [7] it was proved that every compact, orientable 3-manifold admits a contact circle. A more restricted class is the class of those 3-manifolds that admit a taut contact circle. A *taut contact circle* is a pair of contact 1-forms (θ_1, θ_2) such that

$$\theta_1 \wedge d\theta_1 = \theta_2 \wedge d\theta_2, \quad \theta_1 \wedge d\theta_2 = -\theta_2 \wedge d\theta_1.$$

Finally, (θ_1, θ_2) is called a *Cartan structure* if it is a taut contact circle and $\theta_1 \wedge d\theta_2 = 0$. The main result of [6] is the following classification theorem.

Theorem 3.1 *A compact 3-manifold M admits a taut contact circle if and only if M is diffeomorphic to a left quotient of the Lie group G under a discrete subgroup Γ , where G is $SU(2)$, $\tilde{SL}(2, \mathbb{R})$ or $\tilde{E}(2)$.*

On the other hand, the Heisenberg group H^3 admits a Sasakian structure. Consequently, a Sasakian 3-manifold, in general, does not admit a taut contact circle.

Now, let $(M, \eta, g, \xi, \varphi)$ be a non-Sasakian contact metric 3-manifold, namely the torsion $\tau \neq 0$ at any point, and $\{\xi, e_1, e_2 = Je_1\}$ an orthonormal basis of smooth eigenvectors of h with $he_1 = \lambda e_1$ and $\lambda > 0$. Let ω_1, ω_2 be the 1-forms dual to e_1 and e_2 , respectively. Since the three eigenvalues 0, λ and $-\lambda$ of h are everywhere distinct (for $h \neq 0$ at any point), the corresponding line fields are global. The 1-forms $\eta_1 := \omega_2 + \omega_1 = g(e_2 + e_1, \cdot)$ and $\eta_2 = J\eta_1 := g(J(e_2 + e_1), \cdot) = \omega_2 - \omega_1$ are orthogonal to the contact form η . We now prove the following

Theorem 3.2 *Let (M, η, g) be a non-Sasakian H -contact 3-manifold. Then, the following properties are equivalent:*

- (1) (η, η_1) is a taut contact circle;
- (2) (η, η_2) is a taut contact circle;
- (3) (M, η, g) is locally isometric to one of the following Lie groups $SU(2)$, $\tilde{SL}(2, \mathbb{R})$, $\tilde{E}(2)$, equipped with a left-invariant non-Sasakian H -contact metric structure satisfying $\nabla_\xi \tau = 0$.

Moreover, (η, η_i) is a taut contact circle if and only if it is a Cartan structure.

Proof. Since M is H-contact, ξ is an eigenvector of the Ricci operator, namely $A_1 = \varrho(\xi, e_1) = 0$ and $A_2 = \varrho(\xi, e_2) = 0$. Then, by using Lemma 2.2, straightforward computations give

$$\begin{aligned}
 (\eta \wedge d\eta)(\xi, e_1, e_2) &= (d\eta)(e_1, e_2) = g(e_1, Je_2) = -1 \\
 (\eta \wedge d\omega_1)(\xi, e_1, e_2) &= (d\omega_1)(e_1, e_2) = -\frac{1}{2}g(e_1, \nabla_{e_1}e_2) \\
 &= \frac{e_2(\lambda) + A_1}{4\lambda} = \frac{e_2(\lambda)}{4\lambda} \\
 (\eta \wedge d\omega_2)(\xi, e_1, e_2) &= (d\omega_2)(e_1, e_2) = \frac{1}{2}g(e_2, \nabla_{e_2}e_1) \\
 &= -\frac{e_1(\lambda) + A_2}{4\lambda} = -\frac{e_1(\lambda)}{4\lambda} \\
 (\omega_1 \wedge d\eta)(\xi, e_1, e_2) &= (d\eta)(e_2, \xi) = \frac{1}{2}g(\xi, \nabla_\xi e_2) = 0 \\
 (\omega_2 \wedge d\eta)(\xi, e_1, e_2) &= (d\eta)(\xi, e_1) = \frac{1}{2}g(\xi, \nabla_\xi e_1) = 0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (3.1) \quad (\eta \wedge d\eta_1)(\xi, e_1, e_2) &= \frac{e_2(\lambda) - e_1(\lambda)}{4\lambda} \\
 &= (\eta \wedge d\eta_2)(\xi, e_1, e_2) + \frac{e_2(\lambda)}{2\lambda}
 \end{aligned}$$

and

$$(3.2) \quad (\eta_1 \wedge d\eta)(\xi, e_1, e_2) = (\eta_2 \wedge d\eta)(\xi, e_1, e_2) = 0.$$

Moreover, we find

$$(\omega_1 \wedge d\omega_1)(\xi, e_1, e_2) = \frac{\lambda - 1 + a}{2}, \quad (\omega_2 \wedge d\omega_2)(\xi, e_1, e_2) = \frac{a - \lambda - 1}{2},$$

$$(\omega_1 \wedge d\omega_2)(\xi, e_1, e_2) = 0 = (\omega_2 \wedge d\omega_1)(\xi, e_1, e_2),$$

from which we get

$$\begin{aligned}
 (3.3) \quad (\eta_1 \wedge d\eta_1)(\xi, e_1, e_2) &= (\eta_2 \wedge d\eta_2)(\xi, e_1, e_2) \\
 &= (\eta \wedge d\eta)(\xi, e_1, e_2) + a.
 \end{aligned}$$

(1) $\Rightarrow$ (3) From (3.1)-(3.3), we obtain that (η, η_1) is a taut contact circle, equivalently, a Cartan structure, if and only if

$$(3.4) \quad \begin{cases} e_2(\lambda) - e_1(\lambda) = 0, \\ a = 0. \end{cases}$$

Next, from Lemma 2.2, taking into account (3.4), we have

$$\begin{aligned} R(\xi, e_1)e_1 &= -\nabla_\xi \nabla_{e_1} e_1 + \nabla_{e_1} \nabla_\xi e_1 + \nabla_{[\xi, e_1]} e_1 \\ &= -\nabla_\xi \frac{e_2(\lambda)}{2\lambda} e_2 + (\lambda + 1) \nabla_{e_2} e_1 \\ &= \left\{ -\xi \left(\frac{e_2(\lambda)}{2\lambda} \right) - \frac{(\lambda + 1)}{2\lambda} e_1(\lambda) \right\} e_2 + (\lambda^2 - 1) \xi \end{aligned}$$

and

$$\begin{aligned} R(\xi, e_2)e_2 &= -\nabla_\xi \nabla_{e_2} e_2 + \nabla_{e_2} \nabla_\xi e_2 + \nabla_{[\xi, e_2]} e_2 \\ &= -\nabla_\xi \frac{e_1(\lambda)}{2\lambda} e_1 + (\lambda - 1) \nabla_{e_1} e_2 \\ &= \left\{ -\xi \left(\frac{e_1(\lambda)}{2\lambda} \right) - \frac{(\lambda - 1)}{2\lambda} e_2(\lambda) \right\} e_1 + (\lambda^2 - 1) \xi. \end{aligned}$$

Then,

$$g(R(\xi, e_1)e_2, e_1) = \varrho(\xi, e_2) = 0 \quad \text{and} \quad g(R(\xi, e_2)e_1, e_2) = \varrho(\xi, e_1) = 0$$

imply

$$(3.5) \quad \xi \left(\frac{e_2(\lambda)}{2\lambda} \right) + \frac{(\lambda + 1)}{2\lambda} e_1(\lambda) = 0$$

and

$$(3.6) \quad \xi \left(\frac{e_1(\lambda)}{2\lambda} \right) + \frac{(\lambda - 1)}{2\lambda} e_2(\lambda) = 0.$$

From (3.4)-(3.6) it follows

$$e_1(\lambda) = e_2(\lambda) = 0.$$

Moreover, $2\xi(\lambda) = [e_1, e_2](\lambda) = 0$. Therefore, λ is locally constant. Since λ is continuous and M is connected, λ is globally constant on M . Then,

$$(3.7) \quad [\xi, e_1] = (\lambda + 1)e_2, \quad [e_2, \xi] = (1 - \lambda)e_1, \quad [e_1, e_2] = 2\xi,$$

where λ is constant, $\lambda \neq 0$.

Then, (3.7) implies that M is locally isometric to a Lie group equipped with a left-invariant contact metric structure (see [10]). Moreover, by (2.4), $\nabla_\xi \tau = 0$. Thus, Theorem 2.3 yields that M is locally isometric to one of the

following Lie groups $SU(2)$, $\tilde{SL}(2, \mathbb{R})$, $\tilde{E}(2)$, equipped with a left-invariant non-Sasakian H-contact metric structure satisfying $\nabla_\xi \tau = 0$.

(2) $\Rightarrow$ (3) From (3.1)-(3.3) we obtain that (η, η_2) is a taut contact circle, equivalently, a Cartan structure, if and only if

$$(3.8) \quad \begin{cases} e_2(\lambda) + e_1(\lambda) = 0, \\ a = 0. \end{cases}$$

Then, as in the proof of (1) $\Rightarrow$ (3), taking into account (3.8), we get (3.7) and hence the result.

(3) $\Rightarrow$ (1), (2) The universal covering of M is a one of the following unimodular Lie groups $SU(2)$, $\tilde{SL}(2, \mathbb{R})$, $\tilde{E}(2)$. Denote by G one of the above Lie groups. Then, G admits a left-invariant non-Sasakian contact metric structure (η, g) with ξ eigenvector of the Ricci operator and satisfying $\nabla_\xi \tau = 0$, moreover there exists an orthonormal basis $\{\xi, e_1, e_2 = Je_1\}$ of eigenvectors of h , satisfying

$$[e_1, e_2] = 2\xi, \quad [e_2, \xi] = \lambda_1 e_1, \quad [\xi, e_1] = \lambda_2 e_2,$$

with $\lambda_1 + \lambda_2 = 2$ (i.e., $\nabla_\xi \tau = 0$) and $\lambda_1 \neq \lambda_2$ (i.e., $\tau \neq 0$). We refer to [10] for the explicit construction of this left-invariant non-Sasakian contact metric structure on G . Using the notations introduced at the beginning of this section, by straightforward computations we find

$$\begin{aligned} (\eta_1 \wedge d\eta_1)(\xi, e_1, e_2) &= (\eta_2 \wedge d\eta_2)(\xi, e_1, e_2) = -1 = (\eta \wedge d\eta)(\xi, e_1, e_2), \\ (\eta_1 \wedge d\eta)(\xi, e_1, e_2) &= 0 = (\eta \wedge d\eta_1)(\xi, e_1, e_2). \\ (\eta_2 \wedge d\eta)(\xi, e_1, e_2) &= 0 = (\eta \wedge d\eta_2)(\xi, e_1, e_2). \end{aligned}$$

Therefore, (η, η_1) and (η, η_2) are Cartan structures, in particular taut contact circles. $\diamond$

Let M be a compact 3-manifold. If M admits a non-Sasakian H-contact structure satisfying $\nabla_\xi \tau = 0$, Theorem 3.2 implies that M admits a taut contact circle. Conversely, suppose that M admits a taut contact circle. Then, Theorems 3.1 yields that M is diffeomorphic to a left quotient of the Lie group G under a discrete subgroup Γ , where G is one of $SU(2)$, $\tilde{SL}(2, \mathbb{R})$ or $\tilde{E}(2)$. Now, each left-invariant vector field on G descends to $M = \Gamma \backslash G$, or equivalently, if X is left-invariant, then $\pi_* X_{ba} = \pi_* X_a$, for all $a \in G$ and $b \in \Gamma$, where π is the natural projection. In a similar way, a left invariant contact metric structure on G and, in general, all its left-invariant tensor fields, descend to the quotient space. Hence, if we consider on G a left-invariant non-Sasakian H-contact structure satisfying $\nabla_\xi \tau = 0$ (see [10]), then $\Gamma \backslash G$ has a contact metric structure with the same properties on G . Therefore, we obtain at once the following

Corollary 3.3 *A compact 3-manifold M admits a taut contact circle if and only if it admits a non-Sasakian H -contact metric structure satisfying $\nabla_\xi \tau = 0$.*

Theorem 3.4 *Let (M, η, g) be a non-Sasakian contact metric 3-manifold. Then, the following properties are equivalent:*

- (1) (η, η_i) , $i = 1, 2$, are taut contact circles;
- (2) (η, η_i) are Cartan structures;
- (3) ξ is a harmonic vector field and (η, η_1) (or, equivalently, (η, η_2)) is a taut contact circle.

Proof. Let (M, η, g) be a non-Sasakian contact metric 3-manifold. In this case, using Lemma 2.2, by the same computations made in the proof of Theorem 3.2, we find (3.2) and (3.3). From (3.2) it follows that $(1) \Leftrightarrow (2)$. Moreover, we find

$$(3.9) \quad (\eta \wedge d\eta_1)(\xi, e_1, e_2) = \frac{e_2(\lambda) - e_1(\lambda) + A_1 - A_2}{4\lambda},$$

$$(3.10) \quad (\eta \wedge d\eta_2)(\xi, e_1, e_2) = -\frac{e_1(\lambda) + e_2(\lambda) + A_1 + A_2}{4\lambda}.$$

Therefore, (3.2), (3.3) and (3.9), (3.10) imply that (η, η_i) , $i = 1, 2$, are taut contact circles if and only if

$$\begin{cases} a = 0, \\ e_1(\lambda) + e_2(\lambda) + A_1 + A_2 = 0, \\ e_2(\lambda) - e_1(\lambda) + A_1 - A_2 = 0, \end{cases}$$

that is,

$$(3.11) \quad \begin{cases} a = 0, \\ e_2(\lambda) + A_1 = 0, \\ e_1(\lambda) + A_2 = 0. \end{cases}$$

Next, Lemma 2.2, taking into account (3.11), implies

$$[\xi, e_1] = (\lambda + 1)e_2, \quad [\xi, e_2] = (\lambda - 1)e_1, \quad [e_1, e_2] = 2\xi.$$

From these formulas, by using the Jacobi identity

$$[[\xi, e_1], e_2] + [[e_1, e_2], \xi] + [[e_2, \xi], e_1] = 0,$$

it follows

$$e_1(\lambda)e_1 + e_2(\lambda)e_2 = 0,$$

and hence, $e_1(\lambda) = e_2(\lambda) = 0$. Moreover, $2\xi(\lambda) = [e_1, e_2](\lambda) = 0$. Therefore, λ is constant and, by (3.11), $A_1 = A_2 = 0$, namely, ξ is a harmonic vector field with (η, η_i) , $i = 1, 2$, taut contact circles. Thus, $(1) \Rightarrow (3)$. The implication $(3) \Rightarrow (1)$ is given in the proof of Theorem 3.2. $\diamond$

Remark 3.5 Our argument uses three orthogonal contact forms. J. Gonzalo [5] showed that a 3-dimensional compact orientable manifold admits three independent contact forms.

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