On Groups with the Same Set of Order Elements

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Abstract

It is proved that up to isomorphism there are only two finite groups with the same set of element orders as the group $PSL_6(3)$.

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1 Introduction

Let G be a finite group. The set of orders of all elements of G is denoted by $\omega(G)$ and is called the spectrum of G . This set is closed under divisibility and therefore is uniquely determined by the set $\mu(G)$ of elements in $\omega(G)$ which are maximal under the divisibility relation. By definition a finite group G is called recognizable by the set $\omega(G)$ (in short recognizable) if for every finite group H with $\omega(H) = \omega(G)$ we have $H \cong G$. If there are k non-isomorphic finite groups H for which $\omega(H) = \omega(G)$ then G is called a k -recognizable group. A recognizable group is therefore 1-recognizable.

Up to present time among the linear groups $PSL_n(q)$ it is proved that only the groups $PSL_4(3)$ and $PSL_5(3)$ are recognizable. (See [3] and [7]). But in [8] it is proved that there are infinitely many non-isomorphic groups G such that $\omega(G) = \omega(PSL_3(3))$. Such groups are called non-recognizable groups. This is because by a result of Shi in [11] every solvable group is a non-recognizable

group. Now by [8] there is a Frobenius group of the form $K : C$ where $K \cong \mathbb{Z}_{13}$ and $C \cong 2.S_4$ and $\mu(K : C) = \{6, 8, 13\} = \mu(PSL_3(3))$.

In this paper we consider the group $PSL_6(3)$ and prove.

Main Theorem. Let G be a finite group. Then $\omega(G) = \omega(PSL_6(3))$ if and only if $G \cong PSL_6(3)$ or $PSL_6(3) : \langle \theta \rangle$, where θ is an automorphism of order 2 of $PSL_6(3)$.

Throughout the paper we use standard notations. In particular n denotes the cyclic group of order n , i.e. $\mathbb{Z}_n$, and $A : B$ denotes the semi-direct product of groups A and B . For a natural number n the set of prime divisors of n is denoted by $\pi(n)$ and for a finite group G we set $\pi(G) = \pi(|G|)$.

2 Preliminary results

For a finite group G the Gruenberg-Kegel graph of G denoted by $GK(G)$ is defined as follows. The vertices of $GK(G)$ are the set of all the prime divisors of $|G|$, and two vertices p and q are adjacent if G contains an element of order pq . We denote by $s = s(G)$ the number of connected components of $GK(G)$, and they are denoted by $\pi_i = \pi_i(G)$, $i \leq 1 \leq s$. If G is of even order then we let $2 \in \pi_1$. By $\omega_i = \omega_i(G)$ and $\mu_i = \mu_i(G)$ we denote the set of numbers in $\omega(G)$ and $\mu(G)$ respectively all of whose prime divisors belong to $\pi_i(G)$. In our study the Gruenberg-Kegel graph plays an important role. If this graph is disconnected, that is $s \geq 2$, then according to the following result quoted in [14] we know the structure of the group in question.

Lemma 1 *Let G be a finite group with disconnected Gruenberg-Kegel graph $GK(G)$. Then one of the following holds:*

(1) $s(G) = 2$, $G = KH$ is a Frobenius group with kernel K and complement H and the connected components of $GK(G)$ are $\pi(K)$ and $\pi(H)$.

(2) $s(G) = 2$, $G = ABC$, where A and AB are normal subgroups of G , B is a normal subgroup of BC , and AB and BC are Frobenius groups, with kernels A and B and complements B and C respectively. The two connected components of $GK(G)$ are $\pi(AC)$ and $\pi(B)$. Moreover the groups B and C are cyclic, $|B|$ is odd, and $|C| \cdot \prod_{r \in \pi(A)} r \in \omega(G)$.

(3) There exists a non-abelian simple group P such that $P \trianglelefteq \overline{G} = \frac{G}{K} \trianglelefteq \text{Aut}(P)$ for some nilpotent normal $\pi_1(G)$ -subgroup K of G , and $\frac{\overline{G}}{P}$ is a $\pi_1(G)$ -group. Moreover $GK(P)$ is disconnected, $s(P) \geq s(G)$, and for every i , $2 \leq i \leq s(G)$, there exists j , $2 \leq j \leq s(P)$, such that $\omega_i(G) = \omega_j(P)$.

The finite simple groups G with disconnected $GK(G)$ are classified in [14] and [6]. A list of these groups can be found in [8]. Most of the projective

special linear groups have connected Gruenberg-Kegel graph. But the ones with $s(G) \geq 2$ are tabulated bellow

Table I
Simple groups $G = PSL_n(q)$ with $s(G) \geq 2$ (p is a prime number)

G	Condition	π_1	π_2	π_3	π_4
$PSL_p(q)$	$(p, q) \neq (3, 2),$ $(3, 4)$	$\pi(q \prod_{i=1}^{p-1} (q^i - 1))$	$\pi(\frac{q^p - 1}{(q-1)(p, q-1)})$	—	—
$PSL_{p+1}(q)$	$q - 1 \mid p + 1$	$\pi(q(q^{p+1} - 1) \prod_{i=1}^{p-1} (q^i - 1))$	$\pi(\frac{q^p - 1}{q-1})$	—	—
$PSL_2(q)$	$q \equiv 1 \pmod{4}$	$\pi(q - 1)$	$\{p\}$	$\pi(\frac{q+1}{2})$	—
$PSL_2(q)$	$q \equiv -1 \pmod{4}$	$\pi(q + 1)$	$\{p\}$	$\pi(\frac{q-1}{2})$	—
$PSL_2(q)$	$q > 2, q \text{ even}$	$\{2\}$	$\pi(q - 1)$	$\pi(q + 1)$	—
$PSL_3(4)$	—	$\{2\}$	$\{3\}$	$\{5\}$	$\{7\}$

Therefore if $q = 3$ and p is an odd prime number then we have $s(PSL_n(3)) = 2$ if $n = p$ or $p + 1$. The connected components of the Gruenberg-Kegel graph of $PSL_n(3)$ are $\pi_1 = \pi(3 \prod_{i=1}^{p-1} (3^i - 1))$ or $\pi_1 = \pi(3(3^{p+1} - 1) \prod_{i=1}^{p-1} (3^i - 1))$ in the respective cases $n = p$ or $p + 1$. In any case the second component is $\pi_2 = \pi(\frac{3^p - 1}{2})$.

The group $PSL_6(3)$ has order $2^{11}.3^{15}.5.7.11^2.13^2$ and its components are $\pi_1 = \{2, 3, 5, 7, 13\}$ and $\pi_2 = \{11\}$. To find the set of orders of elements of $PSL_6(3)$ we use [2].

Lemma 2 *Let $G = PSL_6(3)$. Then*

$$\begin{aligned} \mu(G) &= \{36, 78, 80, 104, 120, 121, 182\} \\ \omega(G) &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 24, 26, 30, \\ &\quad 36, 39, 40, 52, 60, 80, 91, 104, 120, 121, 182\}. \end{aligned}$$

According to Lemma 2 the graph $GK(PSL_6(3))$ may be drawn as follows:

Lemma 3 *If G is a simple group with $\{2, 11\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 11, 13\}$ then G is isomorphic to one of the following groups*

- (1) $\mathbb{A}_n$, $11 \leq n \leq 16$,
- (2) $M_{11}, M_{12}, M^cL, Fi_{22}, HS, Suz, M_{22}$,
- (3) $PSL_2(11), PSL_5(3), PSL_6(3), PSU_5(2^2), PSU_6(2^2)$.

Proof. Clearly G is a non-abelian simple group. According to the classification of the finite simple groups we consider the following cases:

Case 1 G is isomorphic to an alternating group. Since $11 \mid |G|$ but $7 \nmid |G|$ we must have $G \cong \mathbb{A}_n$, $11 \leq n \leq 16$.

Case 2 G is isomorphic to a sporadic simple group. In this case by inspecting the list of sporadic simple groups in [1] the possible groups listed above are obtained.

Case 3 G is isomorphic to a simple group of Lie type. We use the list of orders of these groups given in [1]. If G is defined over a field with characteristic p , then since $p \mid |G|$ we must have $p = 2, 3, 5, 7, 11$ or 13 . Using the fact that the orders of G is divisible by numbers of the form $p^k \pm 1$ and $11 \in \pi(G)$ we continue as follows.

If $p = 2$, then the order of 2 modulo 11 is 10 and we have $11 \mid 2^5 + 1$. Since $2^7 - 1 = 127$ is a prime not in $\pi(G)$ we must have $G \cong PSU_5(2^2)$ or $PSU_6(2^2)$.

If $p = 3$, then $11 \mid 3^5 - 1$, hence $G \cong PSL_5(3)$ or $PSL_6(3)$.

If $p = 5$, then $11 \mid 5^5 - 1$, but $5^3 - 1 = 4 \times 31$, and since $31 \notin \pi(G)$ we don't obtain a possibility.

If $p = 7$, then $11 \mid 7^5 + 1$. Since $7^3 + 1 = 8 \times 43$ and $43 \notin \pi(G)$ there is no possibility in this case as well.

If $p = 11$, then G is defined over the Galois field $GF(11)$ and since $11^2 + 1 = 2 \times 61$ and $11^3 - 1 = 70 \times 19$ and $61, 19 \notin \pi(G)$ the only possibility is $G \cong PSL_2(11)$.

If $p = 13$, then $11 \mid 13^5 + 1$ and since $13^4 - 1 = 1680 \times 17$ and $17 \notin \pi(G)$ we don't obtain a possibility. Therefore all the cases are examined and the Lemma is proved now. ■

Lemma 4 *If G is a finite group and $\omega(G) = \omega(PSL_6(3))$, then G is a non-solvable group.*

Proof. Suppose G is a solvable group. Let H be a $\{5, 7, 11\}$ -Hall subgroup of G . By Lemma 2 the group G does not contain elements of order 5.7, 5.11 and 7.11. Hence by Theorem 1 in [5] we must have $|\pi(G)| \leq 2$, a contradiction. The Lemma is proved now. ■

3 Proof of the Main Theorem

Theorem 5 *If G is a finite group such that $\omega(G) = \omega(PSL_6(3))$, then $G \cong PSL_6(3)$ or $PSL_6(3) : \langle \theta \rangle$, where θ is the graph automorphism of $PSL_6(3)$.*

Proof. By Lemma 2 the graph $GK(G)$ has two connected components $\pi_1 = \{2, 3, 5, 7, 13\}$ and $\pi_2 = \{11\}$. Therefore by Lemma 1 we have one of the following three cases that we are going to deal with in separate Lemmas:

Lemma 6 *Case 1 is impossible.*

Proof. If case 1 holds then $G = KH$ is a Frobenius group with kernel K and complement H and the connected components of $GK(G)$ are $\omega(K)$ and $\omega(H)$. By a famous theorem of Thompson K is nilpotent. If H is a solvable group, then G will be solvable which is not the case by Lemma 4. Therefore H is a non-solvable Frobenius complement. Now by [10] the group H contains a normal subgroup H_0 such that $[H : H_0] \leq 2$ and $H_0 \cong Z \times SL_2(5)$ where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \emptyset$. Since the group G does not contain elements of order 5.7 and 5.11, therefore $\{7, 11\} \subseteq \pi(K)$. Now being a nilpotent group, K must contain an element of order 77 which contradicts Lemma 2. Hence case 1 does not hold and the Lemma is proved. ■

Lemma 7 *Case 2 is impossible.*

Proof. If case 2 holds, then $G = ABC$, where A and AB are normal subgroups of G , B is a normal subgroup of BC , B and C are cyclic groups, and AB and BC are Frobenius groups with kernels A and B and complements B and C respectively. The connected components of $GK(G)$ are $\omega(B)$ and $\omega(AC)$. Since A and B are Frobenius kernels hence they are nilpotent. Therefore the group AB is solvable. Since $AB \trianglelefteq G$ and C is also solvable we deduce that G is also solvable which violates Lemma 4. Therefore case 2 is impossible as claimed. ■

Thus case 3 holds, that is there exists a non-abelian simple group P such that $P \trianglelefteq \overline{G} = \frac{G}{K} \trianglelefteq Aut(P)$ for some nilpotent normal $\pi_1(G)$ -subgroup K of G , and $\frac{\overline{G}}{P}$ is a $\pi_1(G)$ -group. Moreover $GK(P)$ is disconnected, $s(P) \geq s(G)$, and for every i , $2 \leq i \leq s(G)$, there exists j , $2 \leq j \leq s(P)$, such that $\omega_i(G) = \omega_j(P)$. Since $s(G) = 2$ we obtain $\omega_2(P) = \omega_2(G) = \{11\}$. Obviously $\pi(P) \subseteq \pi(G)$ and since P is a non-abelian simple group we have $\{2, 11\} \subseteq \pi(P) \subseteq \{2, 3, 7, 11, 13\}$. By Lemma 3 one of the following possibilities occur for P that we are going to deal with them separately: $\mathbb{A}_{11}$, $\mathbb{A}_{12}$, $\mathbb{A}_{13}$, $\mathbb{A}_{14}$, $\mathbb{A}_{15}$, $\mathbb{A}_{16}$, M_{11} , M_{12} , M^{cl} , Fi_{22} , HS , Suz , M_{22} , $PSL_2(11)$, $PSL_5(3)$, $PSL_6(3)$, $PSU_5(2^2)$, $PSU_6(2^2)$.

But before continuing our case by case analysis we will mention a useful result taken from [9].

Lemma 8 *Let H be a finite group, $K \trianglelefteq H$ and $\frac{H}{K}$ be a Frobenius group with kernel F and cyclic complement C . If $(|F|, |K|) = 1$ and F is not contained in $\frac{KC_H(K)}{K}$, then $p|C| \in \omega(H)$ for some prime divisor p of $|K|$.*

Lemma 9 *The group $PSL_n(q)$ contains a Frobenius group with kernel an abelian group of order q^{n-1} and complement a cyclic group of order $\frac{q^{n-1}-1}{(n, q-1)}$*

Proof. Let $H = \langle A \rangle$, $A \in GL_{n-1}(q)$, $O(A) = q^{n-1} - 1$, and set

$$L = \left\{ \left[\begin{array}{c|c} (\det X)^{-1} & d_1 \cdots d_{n-1} \\ \hline 0 & X \end{array} \right] \mid X \in H, a_i \in GF(q), 1 \leq i \leq n-1 \right\} \leq SL_n(q).$$

Then L is a Frobenius group with kernel $S = \left\{ \left[\begin{array}{c|c} 1 & a_1 \cdots a_{n-1} \\ \hline 0 & I \end{array} \right] \mid a_i \in GF(q), 1 \leq i \leq n-1 \right\}$ of order q^{n-1} and complement $T = \left\{ \left[\begin{array}{c|c} (\det X)^{-1} & 0 \\ \hline 0 & X \end{array} \right] \mid X \in H \right\}$ of order $q^{n-1} - 1$. Obviously S is abelian and T is cyclic. Let Z denote the centre of $SL_n(q)$. The image of $L = ST$ in $PSL_n(q)$ is the group $\bar{L} = \frac{ZST}{Z} = \frac{(ZS)(ZT)}{Z} \cong \left(\frac{ZS}{Z} \right) \left(\frac{ZT}{Z} \right) = \bar{S} \bar{T}$, where $\bar{S} \cong S$ and $\bar{T} \cong \frac{T}{Z}$ and $\bar{L}$ is a Frobenius group with kernel $\bar{S}$ and complement $\bar{T}$. ■

Next we start the examination of each possibilities for the simple group P with the conditions stated above. From $\bar{G} = \frac{G}{K}$ we know that $\mu(\bar{G}) \subseteq \mu(G)$ and $\omega(\bar{G}) \subseteq \omega(G)$. For the above simple groups P we use [1] for the inspection of the set $\mu(P)$.

Assume $P \cong A_n$, $11 \leq n \leq 16$. Since these groups contain elements of order 35 and by Lemma 2 $35 \notin \omega(G)$, hence these possibilities for P lead to a contradiction.

Suppose $P \cong M_{11}, M_{12}, M^c l, HS, M_{22}, PSL_2(11), PSU_5(2^2)$, or $PSL_5(2^2)$. By [1] we have $Aut(P) = P : S$ where the order of S is at most 6. Therefore $\bar{G} \cong \frac{G}{K} = P : A$ where $|A| \mid 6$. We may assume K is an elementary abelian p -group for some $p \in \pi_1(G)$. For the above groups by [1] we have $13 \nmid |P|$, hence $13 \mid |K|$. Again by [1] all the above groups contain a group isomorphic to the linear groups $PSL_2(11)$. But $PSL_2(11)$ has a Frobenius group $\frac{H}{K}$ with kernel $F \cong \mathbb{Z}_{11}$ and complement $C \cong \mathbb{Z}_5$. Since $\frac{KC_H(K)}{K} \cong \frac{C_H(K)}{K \cap C_H(K)}$ and by Lemma 2 the group G contains no element of order 11.13 we deduce that F is not contained in $\frac{KC_H(K)}{K}$, hence all the conditions of Lemma 8 are fulfilled. Therefore $13.5 = 65 \in \omega(H)$ contradicting Lemma 2. In this way the above possibilities are ruled out.

Next suppose $P \cong Fi_{22}$ or Suz . In both cases by [1] we observe that P contains an element of order 21 which violates Lemma 2.

Now let $P \cong PSL_5(3)$. In this case $Aut(P) = P : 2$ and from the order of P we deduce that $7 \mid |K|$. By [4] the group $PSL_5(3)$ contains a group isomorphic to M_{11} , hence it contains a copy of the group $PSL_2(11)$. Now again we consider the Frobenius subgroup $\mathbb{Z}_{11} : \mathbb{Z}_5$ and drive a contradiction.

Finally we are left with $P \cong PSL_6(3)$. As we mentioned above by [13] we may assume that K is an elementary abelian p -group for some $p \in \pi_1(G) = \{2, 3, 5, 7, 13\}$ and K is a module for P on which P acts faithfully and $C_G(K) = K$. Our first aim is to prove $K = 1$ is the identity group. Let us consider the Frobenius subgroup of $PSL_6(3)$ of the shape $3^5 : \frac{3^5-1}{(6,3-1)} = 3^5 : 121$ which is constructed in Lemma 9. If $p \neq 3$, then by Lemma 8 the group G must contain an element of order $121p$ which is not possible by Lemma 2. Therefore we may assume $p = 3$. Let $|K| = 3^k$. Since $7 \in \omega(P)$ and the order of 7 modulo 2 is 6, hence $6 \mid k$. We put $K = 3^k$, $k \geq 1$, an elementary abelian 3-group. In this case $\frac{G}{K}$ is isomorphic to a subgroup of $GL_k(3)$. But $\frac{G}{K}$ also contains a group isomorphic to $P \cong PSL_6(3)$, hence $3^k.PSL_6(3) \leq G \leq 3^k.GL_k(3)$. Now if a subgroup H of $PSL_6(3)$ splits over 3^k , then $3^k : H$ would be a subgroup of the affine group of the vector space of dimension $k+1$ over $\mathbb{Z}_3$ and the elements of the group $3^k : H$ would be of the form $\begin{bmatrix} 1 & v \\ 0 & A \end{bmatrix}$ in the group $GL_{k+1}(3)$.

Our notation is chosen so that $\bar{A} = ZA$, where $Z = \{\pm I\}$ is the centre of $SL_6(3)$ and $A \in SL_6(3)$. If $X = \begin{bmatrix} 1 & v \\ 0 & \bar{A} \end{bmatrix}$, then it is easy to show that for all $k \in \mathbb{N}$ we have

$$X^k = \begin{bmatrix} 1 & v + v\bar{A} + \cdots + v(\bar{A})^{k-1} \\ 0 & (\bar{A})^k \end{bmatrix}$$

where $v\bar{A} = vA$ and so on. Therefore if $O(\bar{A}) = n$ and v is chosen so that $u = v + v\bar{A} + \cdots + v(\bar{A})^{n-1} \neq 0$, then the order of $X = \begin{bmatrix} 1 & v \\ 0 & \bar{A} \end{bmatrix}$ is $3n$.

Now the following matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

is in $SL_6(3)$ and $O(\bar{A}) = 16$. Moreover $\langle \bar{A} \rangle$ is a subgroup of order 16 in $PSL_6(3)$ which splits over 3^k . By direct calculation we see that $I + \bar{A} + (\bar{A})^2 +$

$\cdots + (\overline{A})^{15} \neq 0$, hence the order of $X = \begin{bmatrix} 1 & v \\ 0 & \overline{A} \end{bmatrix}$, where $v = (1, 0, 0, 0, 0, 0)$, as an element of $3^6.PSL_6(3)$ is 48, contradicting Lemma 2. Hence $K = 1$ as we claimed. Therefore we have $P \leq G \leq Aut(P)$ where $P \cong PSL_6(3)$. By [12] every automorphism of the group $PSL_n(q)$ is a product of inner, field, graph and diagonal automorphism of the group. Let $q = p^f$, p prime, and let σ_p denote the Frobenius automorphism of the Galois field $GF(q)$. Then σ_p induces an automorphism of order f of the group $PSL_n(q)$ in the natural way. This automorphism is denoted by σ_p again and is called a field automorphism of $PSL_n(q)$. If $A \in GL_n(q)$, then $\delta = \delta_A$ defined by $\delta_A(B) = A^{-1}BA$, $\forall B \in SL_n(q)$, is an automorphism of $SL_n(q)$ and induces an automorphism of $PSL_n(q)$ in the usual way which is denoted by δ_A again. The automorphisms of $PSL_n(q)$ of the above type are called diagonal automorphisms and it is known that they form a group whose order modulo $PSL_n(q)$ is $d = (n, q-1)$. Finally if $n \geq 3$, then the mapping defined by $\theta(A) = (A^{-1})^t$, $A \in SL_n(q)$, is an outer automorphism of $SL_n(q)$ of order 2 which induces an outer automorphism of $PSL_n(q)$ which is denoted by θ again and is called the graph automorphism of $PSL_n(q)$.

Now in the case of $P \cong PSL_6(3)$ we have $Aut(P) = P : S$ where $S = \langle \delta, \theta \rangle$ is generated by a diagonal automorphism δ and the graph automorphism θ . We have $\delta^2 = \theta^2 = 1$, $\delta\theta = \theta\delta$, hence S is isomorphic to the Klein fourgroup. In this case we define $\delta = \delta_A$, where $A = diag(-1, 1, 1, 1, 1, 1)$ the diagonal matrix with -1 and 1 along the main diagonal. Therefore for G we have the following possibilities: $P, P : \langle \theta \rangle, P : \langle \delta \rangle, P : \langle \theta\delta \rangle, P : \langle \theta, \delta \rangle$.

Assume $G = P : \langle \delta \rangle$. First we consider the group $SL_6(3) : \langle \delta \rangle$. Then using $\delta = \delta_A$ with $A = diag(-1, 1, 1, 1, 1, 1)$ we obtain $C_{SL_6(3)}(\delta) = \left\{ \begin{bmatrix} a & 0 \\ 0 & A \end{bmatrix} \mid a \det(A) = 1, A \in GL_5(3) \right\}$. Now the group $SL_5(3)$ has an element of order 121 and from the above we obtain an element of order 242 in $SL_6(3) : \langle \delta \rangle$ and consequently in $PSL_6(3) : \langle \delta \rangle$, contradicting Lemma 2 again.

Next we will assume $G = P : \langle \theta\delta \rangle$. If $A \in SL_6(3)$, then using the action of θ we obtain $(A\theta\delta)^2 = A\theta\delta A\theta\delta = A\delta\theta A\theta\delta = A\delta(A^{-1})^t\delta$. Now we choose $A = \begin{bmatrix} \tau & 0 \\ 0 & \eta \end{bmatrix}$ a block diagonal matrix such that $\tau \in SL_2(3)$ and $\eta \in SL_4(3)$.

Using $\delta = diag(-1, 1, 1, 1, 1, 1)$ we obtain $A\delta(A^{-1})^t\delta = \begin{bmatrix} \tau\alpha(\tau^{-1})^t\alpha & 0 \\ 0 & \eta(\eta^{-1})^t \end{bmatrix}$ where $\alpha = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Using [1] we see that the group $SL_4(3) : \langle \theta \rangle$, where θ is again the graph automorphism of $SL_4(3)$, has an element $\eta\theta$ of order 36. Therefore $(\eta\theta)^2 = \eta\theta\eta\theta = \eta(\eta^{-1})^t$ has order 18. Now we chose $\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\in SL_2(3)$ and calculate $\tau\alpha(\tau^{-1})^t\alpha = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ which is a matrix of order 4. Therefore $(A\theta\delta)^2$ has order 36 and consequently $A\theta\delta$ is an element of order 72 in $SL_6(3) : \langle \theta\delta \rangle$. The image of $A\theta\delta$ in $PSL_6(3) : \langle \theta\delta \rangle$ also has order 72 which is against Lemma 2.

Since $P : \langle \theta, \delta \rangle$ contains the group $P : \langle \delta \rangle$, hence it is not a possibility.

Finally we are left with $G = P \cong PSL_6(3)$ or $G = P : \langle \theta \rangle$. But if $G = P : \langle \theta \rangle$, then $G = P \cup P\theta$, and if we set $Z = Z(SL_6(3)) = \{\pm I\}$, then for $ZA\theta \in P\theta$, we have $(ZA\theta)^2 = ZA(A^{-1})^t$, where $A \in SL_6(3)$. Therefore order of $ZA\theta$ is twice the order of $ZA(A^{-1})$. But using a computer program we found the order of all matrices of the forms $A(A^{-1})^{-1}$, where $A \in SL_6(3)$, and found that $O(ZA\theta) \in \omega(G)$. Therefore $\omega(P) = \omega(P : \langle \theta \rangle)$ and the theorem is proved. ■

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