Application of Biorthogonal Wavelets to Preconditioning the 3-Fields Formulation: Numerical Results in Nonconforming Decomposition

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Abstract
The aim of this paper is to implement the wavelet preconditioner for the linear system corresponding to the discrete Poincaré-Steklov operator arising in the three fields formulation. In order to apply the preconditioner, we will introduce multiresolution analysis (MRA) for $L^2(0,2)$ which is suitable in nonconforming domain decomposition. Extensive numerical results show that the condition number of the matrix after preconditioning is stabilized when the number of subdomains and elements per edge are large.

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1 Introduction
In this paper we deal with domain decomposition of elliptic problems. Many domain decomposition approaches aim at solving boundary value problem in
weak formulation stemming from a second order elliptic differential operator
on nonoverlapping subdomains. An important case is when one wants to use
grids with possibly different grid sizes on the different subdomains without
adaptation of the grids at the interface boundaries.

A very general class of domain decomposition with such nonconforming
(non-matching) grids is provided by the three fields formulation (3-FF), as pro-
posed by F. Brezzi and L. D. Marini [2]. By applying a Schur complement
technique, the solution of such a problem can be reduced to the solution of
an equation on the interface unknown, corresponding to the Poincaré-Steklov
operator which is an operator of order one.

Clearly, in order to design efficient solver for 3-FF there is a need for a
good preconditioner for the interface problem. It is well known that if the
"trace" space is discretized by means of suitable wavelets function, there exit
asymptotically optimal diagonal preconditioners [1, 3].

By modifying the biorthogonal MRA for \( L^2(0, 2) \) [1, 6], we will introduce a
MRA for \( L^2(0, 2) \) which is suitable in nonconforming domain decomposition.
In numerical implementation, by using the extension operator on the cross
points of skeleton and special inverse fast wavelet transforms (IFWTs), we can
construct the biorthogonal MRA on the skeleton. For our purpose, we will
consider biorthogonal wavelets based on B-spline and we will make wavelets
bases satisfying special boundary conditions on the unitary interval.

The paper is organized as follows. In Section 2, we briefly review the dis-
crete 3-FF and wavelet preconditioner. The biorthogonal MRA for \( L^2(0, 2) \) in
nonconforming decomposition is introduced in section 3 and finally section 4
is devoted to the numerical results.

2 The Discrete 3-FF and Wavelet Precondi-
tioner.

Let \( \Omega \subset \mathbb{R}^2 \) be a convex polygonal domain, we will consider the following
simple model problem: given \( f \in L^2(\Omega) \), find \( u \) such that

\[
\begin{aligned}
- \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u}{\partial x_i}) + a_0(x)u &= f \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]  

(1)

we assume that for almost all \( x \in \Omega, \) \( 0 \leq a_0(x) \leq R, \) the coefficient matrix
\( (a_{ij}(x))_{i,j=1,2} \) is symmetric, sufficiently smooth and uniformly positive definite.

Let \( \Omega \) be decomposed into a finite number of nonoverlapping polygonal
subdomains \( \Omega_k, \) \( k = 1, \ldots, K, \) where \( \Omega = \bigcup_{k=1}^{K} \Omega_k \) and let \( \Sigma = (\bigcup_{k=1}^{K} \partial \Omega_k) \setminus \partial \Omega \)
be the skeleton of the decomposition. We then consider the following functional
Wavelet Preconditioner for the 3-Fields Formulation

spaces

\[ V = \left\{ (v^1, \ldots, v^K), v^k \in V^k, \quad v^k |_{\partial \Omega \partial \Omega_k} = 0, k = 1, \ldots, K \right\}, \quad \Lambda = \prod_{k=1}^{K} \Lambda^k \]

and

\[ \Phi = \left\{ \varphi \in L^2(\Sigma) : \exists v \in H^1_0(\Omega), \varphi = v |_{\Sigma} \right\} = H^1_0(\Omega)|_{\Sigma} \]

where \( V^k = H^1(\Omega_k) \) and \( \Lambda^k = H^{-\frac{1}{2}}(\partial \Omega_k) \).

To solve the problem (1) by a domain decomposition approach, for each \( k = 1, \ldots, K \) let \( a^k : V^k \times V^k \to \mathbb{R} \) be the bilinear form induced by the differential operator on the subdomain \( \Omega_k \):

\[ a^k(v, w) = \int_{\Omega_k} \left( \sum_{i,j=1}^{2} (a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}) + a_0(x) vw \right) dx \quad \forall v, w \in V^k \]

The 3-FF of the problem (1) is the following: find \((u, \lambda, \tau) \in (V \times \Lambda \times \Phi)\) such that

\[
\begin{cases}
\sum_{k=1}^{K} a^k(u^k, v^k) - \sum_{k=1}^{K} < \lambda^k, v^k >_k = (f, v)_L^2(\Omega) & \forall v \in V \\
\sum_{k=1}^{K} < \tau - u^k, \mu^k >_k = 0 & \forall \mu \in \Lambda \\
\sum_{k=1}^{K} < \lambda^k, \sigma >_k = 0 & \forall \sigma \in \Phi
\end{cases}
\] (2)

where \(< , >_k\) is duality pairing between \( H^{-\frac{1}{2}}(\partial \Omega_k) \) and \( H^{\frac{1}{2}}(\partial \Omega_k) \). For every \( f \in L^2(\Omega) \), this problem has a unique solution \((u, \lambda, \tau)\) where \( u \) is indeed the solution of problem (1), \( \lambda^k = \frac{\partial u^k}{\partial n^k} (\eta^k: \text{outward conormal derivative to } \Omega_k) \) on \( \partial \Omega_k \) and \( \phi = u \) on the skeleton \( \Sigma \).

The problem (2) can now be approximated by a Galerkin scheme. To this end, one chooses finite dimensional subspaces \( V_h = \prod_{k=1}^{K} V^k_h \subset V \), \( \Lambda_h = \prod_{k=1}^{K} \Lambda^k_h \subset \Lambda \) and \( \Phi_h \subset \Phi \). Then the discretized system reads as follows: find \((u_h, \tau_h, \lambda_h) \in (V_h \times \Phi_h \times \Lambda_h)\) such that

\[
\begin{cases}
\sum_{k=1}^{K} a^k(u^k_h, v^k_h) - \sum_{k=1}^{K} < \lambda^k_h, v^k_h >_k = (f, v_h)_L^2(\Omega) & \forall v_h \in V_h, \\
\sum_{k=1}^{K} < \tau_h - u^k_h, \mu^k_h >_k = 0 & \forall \mu_h \in \Lambda_h, \\
\sum_{k=1}^{K} < \lambda^k_h, \sigma_h >_k = 0 & \forall \sigma_h \in \Phi_h
\end{cases}
\] (3)
It is clear that suitable inf-sup condition [1] should be assumed on the choice of $V_h$, $\Lambda_h$ and $\Phi_h$. The linear system stemming from such an approximation takes the form

\[
\begin{cases}
Au_h - B^T \lambda_h &= f_h, \\
-Bu_h + C^T \tau_h &= 0, \\
C \lambda_h &= 0,
\end{cases}
\] (4)

where $u_h$, $\lambda_h$ and $\tau_h$ are the coefficients vectors of $u_h$, $\lambda_h$ and $\tau_h$ in the considered bases for $V_h$, $\Lambda_h$ and $\Phi_h$. By a Schur complement argument, the solution of (4) can be reduced to a system in the unknown $\tau_h$ of the form

\[
CA^{-1} C^T \tau_h = CA^{-1} \begin{pmatrix} f_h \\ 0 \end{pmatrix}, \quad A := \begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 0 \\ C \end{pmatrix}.\] (5)

Here the matrix $S := CA^{-1} C^T$ (the discrete Poincaré-Steklov operator on $\Sigma$) does not need to be assembled. It is an operator of order one, therefore the matrix $S$ needs to be preconditioned[2].

By using Biorthogonal wavelet bases for $\Phi_h$, we will consider asymptotically optimal diagonal preconditioner for discrete operator $S$. Suppose that we have a couple of wavelet bases for $L^2(\Sigma)$

\[
\psi := \{\psi_{j,m}, (j, m) \in \nabla := \bigcup_{j \geq j_0} \nabla_j\}, \\
\tilde{\psi} := \{\tilde{\psi}_{j,m}, (j, m) \in \tilde{\nabla} := \bigcup_{j \geq j_0} \tilde{\nabla}_j\}
\]

such that $\nabla_j \sim 2^j$ and also

(P1) $\psi$ and $\tilde{\psi}$ satisfy on the biorthogonal property, i.e.,

\[
(\psi_{j,m}, \tilde{\psi}_{j',m'})_{L^2(\Sigma)} = \delta_{j,j'} \delta_{m,m'} \quad m \in \nabla_j, m' \in \nabla_{j'}
\]

(P2) any function $\sigma \in L^2(\Sigma)$ can be expanded in term of $\psi$ (or $\tilde{\psi}$):

\[
\sigma = \sum_{j \geq j_0} \sum_{m \in \nabla_j} (\sigma, \psi_{j,m})_{L^2(\Sigma)} \psi_{j,m}
\]

(P3) The following norm equivalence holds

\[
\| \sigma \|_\Phi \sim \sum_{j \geq j_0} 2^j \sum_{m \in \nabla_j} |(\sigma, \tilde{\psi}_{j,m})_{L^2(\Sigma)}|^2, \quad \sigma \in \Phi
\]

(P4) $\psi$ and $\tilde{\psi}$ have local support, i.e.,

\[
diam(\text{supp}\psi_{j,m}) \sim diam(\text{supp}\tilde{\psi}_{j,m}) \sim 2^{-j}.
\]
where the notation $\alpha \sim \beta$ express that $\alpha$ can be bounded from above and below by a constant multiple of $\beta$ uniformly in any parameters on which $\alpha$ and $\beta$ my depend.

Let $D_J$ be the diagonal matrix defined by

$$(D_J)_{(j,m),(j',m')} = 2^{j/2} \delta_{j,j'} \delta_{m,m'}, \quad j_0 \leq j, j' \leq J, \ m, m' \in \nabla_j.$$ 

**Theorem 1.** Let $\Phi_h = \text{span}_{\psi_J} \{\psi_J\}$ with $
abla_J := \bigcup_{j \geq j_0} \nabla_j$, and $S_{\psi_J}$ be the matrix corresponding to the multiscale basis $\psi_J$ then

$$\text{cond}_2(D_J^{-1} S_{\psi_J} D_J^{-1}) = O(1), \quad J \to \infty.$$ 

**Proof:** [1, 3].

Therefore by choosing the wavelet base $\psi_J$ for $\Phi_h$, the discrete Poincaré-Steklov operator can be preconditioned by the asymptotically optimal preconditioner $D_J^{-1}$.

## 3 Biorthogonal MRA for $L^2(0, 2)$

The aim of this section is to construct a MRA for $L^2(0, 2)$. By using this MRA, we can construct a MRA for $L^2(\Sigma)$ where $\Sigma$ will be the union of one dimensional segments.

The idea is to start from a biorthogonal MRA for $L^2(0, 1)$ and introducing a MRA for $L^2(0, 2)$ by considering the interval $(0, 2)$ as a closure of $(0, 1) \cup (1, 2)$. We want to point out that the primal scaling and wavelet functions must be continue on $[0, 2]$. However, for our purpose, the continuity of the dual functions is not strictly necessary. We note that the number of scaling and wavelet functions on the intervals $(0, 1)$ and $(1, 2)$ must be different. We start with considering two families of scaling functions

$$\varphi^1_{j_1} = \{\varphi^1_{j_1,k}, k \in \Delta_{j_1}\} \subset L^2(0, 1), \quad \tilde{\varphi}^1_{j_1} = \{\tilde{\varphi}^1_{j_1,k}, k \in \Delta_{j_1}\} \subset L^2(0, 1), \quad \text{(6)}$$

where $\Delta_{j_1} = \{0, 1, \ldots, N_{j_1}\}, \#\Delta_{j_1} \sim 2^{j_1}$ and $j_1 \geq j_0$.

These families are generator bases of a biorthogonal MRA for $L^2(0, 1)$ [4, 5]

$$T^1_{j_1} = \text{span}\varphi^1_{j_1}, \quad \tilde{T}^1_{j_1} = \text{span}\tilde{\varphi}^1_{j_1}, \quad \text{(7)}$$

such that the following conditions are fulfilled:
(G1) The two families \( \varphi_{j_1}^1, \tilde{\varphi}_{j_1}^1 \) are refineable and we have
\[
\text{clos}_{H^1}(\bigcup_{j_1 \geq j_0} T^1_{j_1}) = H^1(0,1), \quad \text{clos}_{L^2}(\bigcup_{j_1 \geq j_0} \tilde{T}^1_{j_1}) = L^2(0,1).
\]
furthermore they consist of functions with local support,
\[
\text{diam}(\text{supp} \varphi_{j_1,k}^1) \sim \text{diam}(\text{supp} \tilde{\varphi}_{j_1,k}^1) \sim 2^{-j_1}, \quad \text{for all} \ k \in \Delta_{j_1},
\]
which are \( L^2 \)-Stable.

(G2) They are biorthogonal and satisfy following boundary conditions
\[
\varphi_{j_1,0}^1(0) = 2^{j_1/2}, \quad \varphi_{j_1,m}^1(0) = 0, \quad 0 < m \leq N_{j_1},
\]
\[
\varphi_{j_1,N_{j_1}}^1(1) = 2^{j_1/2}, \quad \varphi_{j_1,m}^1(1) = 0, \quad 0 \leq m < N_{j_1}.
\]

(G3) All of polynomials with order less than \( d, \tilde{d} \) are contained in \( T^1_{j_1}, \tilde{T}^1_{j_1} \).

(G4) Projectors \( P^1_{j_1} : L^2(0,1) \to T^1_{j_1} \) can be defined by
\[
P^1_{j_1}v = \sum_{k \in \Delta_{j_1}} (v, \tilde{\varphi}_{j_1,k}^1)_{L^2(0,1)} \varphi_{j_1,k}^1,
\]
which are uniformly bounded on \( L^2(0,1) \).

(G5) A Bernstein-type inequality for \( T^1_{j_1} \) holds
\[
\|f_{j_1}\|_{H^1(0,1)} \lesssim 2^{j_1} \|f_{j_1}\|_{L^2(0,1)} \quad \text{for all} \ f_{j_1} \in T^1_{j_1}.
\]

By using the stable completion technique [4], biorthogonal wavelets can be constructed
\[
\psi_{j_1}^1 = \{ \psi_{j_1,k}^1, k \in \nabla_{j_1} \}, \quad \tilde{\psi}_{j_1}^1 = \{ \tilde{\psi}_{j_1,k}^1, k \in \nabla_{j_1} \},
\]
where \( \nabla_{j_1} = \{1, \ldots, M_{j_1}\} \), and \#\nabla_{j_1} \sim 2^{j_1} \). Now we will consider other generator bases for \( T^1_{j_1}, \tilde{T}^1_{j_2} \)
\[
\varphi_{j_2}^1 = \{ \varphi_{j_2,k}^1, k \in \Delta_{j_2} \}, \quad \tilde{\varphi}_{j_2}^1 = \{ \tilde{\varphi}_{j_2,k}^1, k \in \Delta_{j_2} \},
\]
satisfying (G1)-(G5) and following properties [5]
(i) \( \varphi_{j_2,0}^1(0) = \tau 2^{j_2/2}, \tilde{\varphi}_{j_2,k}^1(0) = 0, 0 < k \leq N_{j_2} \)
where $\tau$ is a nonzero constant.

(ii) The corresponding wavelet base $\{\varphi_{j_2,k}, k \in \mathcal{N}_{j_2}\}$ are zero at $x = 0$.

Now we want to define a couple $\{T_{j_1,j_2}\}, \{\tilde{T}_{j_1,j_2}\}$ of biorthogonal MRA for $L^2(0,2)$. By the aid of extension operator $E_{j_1,j_2}$

\begin{equation}
(E_{j_1,j_2}f)(x) := \begin{cases} 
  f(x) = \sum_{m \in \Delta_{j_1}} f_m \varphi_{j_1,m}^1(x), & x \in [0,1), \\
  2^{(j_1-j_2)/2} f_{N_{j_1}} \varphi_{j_2,0}^2(x), & x \in [1,2], 
\end{cases}
\end{equation}

where $f \in T_{j_1}^1$. The generator base for $T_{j_1,j_2}$ is the following

\begin{equation}
\varphi_{j_1,j_2} := \{\varphi_{j_1,j_2,m}, m = 0, 1, \ldots, N_{j_1} + N_{j_2}\},
\end{equation}

where for $m = 0, \ldots, N_{j_1} - 1$, 

\[
\varphi_{j_1,j_2,m} = \begin{cases} 
  \varphi_{j_1,m}^1, & \text{in } [0,1), \\
  0, & \text{in } [1,2].
\end{cases}
\]

and for $m = \ell + N_{j_1}, \ell = 1, \ldots, N_{j_2}$,

\[
\varphi_{j_1,j_2,m} = \begin{cases} 
  0, & \text{in } [0,1), \\
  \varphi_{j_2,\ell}^2, & \text{in } [1,2],
\end{cases}
\]

where $\varphi_{j_2,\ell}^2(x) = \varphi_{j_2,\ell}^1(x-1)$ for all $x \in [1,2]$. The function $\varphi_{j_1,j_2,n_1}$ at the cross point $x = 1$ is defined by

\[
\varphi_{j_1,j_2,n_1} := \begin{cases} 
  \varphi_{j_1,n_1}^1, & \text{in } [0,1), \\
  2^{(j_1-j_2)/2} \varphi_{j_2,0}^2, & \text{in } [1,2].
\end{cases}
\]

The generator basis of dual spaces $\tilde{T}_{j_1,j_2}$ can be defined by

\[
\tilde{\varphi}_{j_1,j_2,k} = \begin{cases} 
  \tilde{\varphi}_{j_1,k}^1, & \text{in } [0,1) \\
  0, & \text{in } [1,2],
\end{cases} \quad \tilde{\varphi}_{j_1,j_2,\ell+n_1} = \begin{cases} 
  d_{j_1,j_2,\ell} \tilde{\varphi}_{j_1,n_1}^1, & \text{in } [0,1) \\
  \tilde{\varphi}_{j_2,\ell}^2, & \text{in } [1,2],
\end{cases}
\]

for $k = 0, \ldots, N_{j_1}, \ell = 1, \ldots, N_{j_2}$ and $d_{j_1,j_2,\ell} = -\langle \varphi_{j_2,0}^2, \tilde{\varphi}_{j_2,\ell} \rangle_{L^2(1,2)}$. We observe that $\varphi_{j_1,j_2}$ and $\tilde{\varphi}_{j_1,j_2}$ are biorthogonal in the interval $(0,2)$. We note that The spaces $T_{j_1,j_2}$ and $\tilde{T}_{j_1,j_2}$ are nested, i.e.,

\[
T_{j_1,j_2} \subset T_{j_1,j_2}', \quad \tilde{T}_{j_1,j_2} \subset \tilde{T}_{j_1,j_2}', \quad j_1' > j_1, j_2' > j_2.
\]
The corresponding operator $P_{j_1,j_2} : L^2(0, 2) \to T_{j_1,j_2}$ can be defined by

$$P_{j_1,j_2}f = \sum_{k \in \Delta_{j_1,j_2}} (f, \tilde{\varphi}_{j_1,j_2,k})L^2(0,2)\varphi_{j_1,j_2,k}$$

(13)

where $\Delta_{j_1,j_2} = \{0, \ldots, N_{j_1} + N_{j_2}\}$.

**Remark 1** The operator $P_{j_1,j_2}$ has the following properties

(i) $P_{j_1,j_2}v = v$ for all $v \in T_{j_1,j_2}$

(ii) $P_{j_1,j_2}P_{j_1,j_2} = P_{j_1,j_2}P_{j_1,j_2}$ for all $\ell_1 > j_1, \ell_2 > j_2$

(iii) $P_{j_1,j_2}$ is bounded on $L^2(0, 2)$

The following wavelet functions on the interval $(0, 2)$ can be constructed

$$\psi_{j_1,j_2} = \{\psi_{j_1,j_2,m}, m = 1, \ldots, M_{j_1} + M_{j_2}\};$$

$$\tilde{\psi}_{j_1,j_2} = \{\tilde{\psi}_{j_1,j_2,m}, m = 1, \ldots, M_{j_1} + M_{j_2}\}$$

where for $m = 1, \ldots, M_{j_1}$,

$$\psi_{j_1,j_2,m} = \begin{cases} 
\psi_{j_1,m}^1 & \text{in } [0, 1) \\
c_{j_1,j_2,m}(\varphi_{j_2,0}^2 - P_{j_2,0}^2\varphi_{j_2,0}^2) & \text{in } [1, 2],
\end{cases}$$

$$\tilde{\psi}_{j_1,j_2,m} = \begin{cases} 
\tilde{\psi}_{j_1,m}^1 & \text{in } [0, 1) \\
0 & \text{in } [1, 2],
\end{cases}$$

such that $c_{j_1,j_2,m} = 2^{-j_1/2}\psi_{j_1,m}^1(1)$ and $P_{j_1,0}^2f = \sum_{k=1}^{N_{j_2}} (f, \tilde{\varphi}_{j_2,k})L^2(1,2)\tilde{\varphi}_{j_2,k}^2$.

while for $\tilde{c}_{j_1,j_2,m} = -2^{(j_1-j_2)/2}(\varphi_{j_2,0}^2, \tilde{\psi}_{j_2,m})L^2(1,2)$,

$$\begin{cases} 
0, & \text{in } [0, 1) \\
\tilde{\varphi}_{j_2,m}^2 & \text{in } [1, 2],
\end{cases}$$

$$\begin{cases} 
\tilde{c}_{j_1,j_2,m}\varphi_{j_1,N_{j_1}}^1, & \text{in } [0, 1) \\
\tilde{\varphi}_{j_2,m}^2 & \text{in } [1, 2].
\end{cases}$$

**Remark 2** It is simple to check that

(i) $P_{j_1,j_2}\psi_{j_1,j_2,m} = 0, \ m \in \nabla_{j_1,j_2}$.

(ii) the following new biorthogonal wavelets are $L^2$-stable on the interval $(0, 2)$

$$\psi = \{\psi_{j_1,j_2,m}, (j_1, j_2, m) \in \nabla = \bigcup_{j_1 \geq j_0, j_2 \geq j_0} \nabla_{j_1,j_2}\},$$
such that in this case $N$ is the trace of $V$ in kind $00$.

L

Now we construct the MRA for $L^2(0, 1)$ and $L^2(0, 2)$, we can obtain a biorthogonal MRA for $L^2(\Sigma)$ satisfying in (P1)-(P4)[1,6].

4 Numerical results

We will test the diagonal preconditioner on the following simple model problem: given $f \in L^2(\Omega)$, find $u$ satisfying

$$
\begin{align*}
-\Delta u + a_0 u &= f & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{align*}
$$

with $\Omega = (0, 1)^2$. We consider a uniform decomposition of $\Omega = (0, 1)^2$ in $K = N \times N$ equal subdomain of size $H \times H, H = 1/N$.

In each subdomain $\Omega_k$, by considering a nonconforming domain decomposition, we take a uniform mesh composed of $n_k \times n_k$, equal square element of size $\delta_k \times \delta_k, \delta_k = H/n_k$ and $n_k = 2^k$. We then define $V^k_h$ to be the corresponding space of $Q1$ finite element on the mesh. The multiplier $\Lambda^k_h$ is then defined as the trace of $V^k_h$ on $\partial \Omega_k$. The skeleton $\Sigma$ is split as $\Sigma = \bigcup_{i=1}^{2(N-1)} e_i$ with $e_i = (\partial \Omega_k \cap \partial \Omega_\ell)$ for some $k = k(i)$ and $\ell = \ell(i)$. By considering a grid $\mathcal{E}$ on $\Sigma$ obtained by uniformly splitting each $e_i$ in to $2^J_i (J_i = \min\{s_k, s_\ell\})$ elements, we define $\Phi_h$ to be the space of $P1$ finite elements on such a grid

$$
\Phi_h = \{ \varphi_h \in C^0(\Sigma); \varphi_h|_{\partial \Omega} = \varphi_h|_{\partial \Omega}, \forall \kappa \in \mathcal{E}, \varphi_h|_{\partial \Omega} = 0 \}
$$

Now we construct the MRA for $L^2(\Sigma)$ with $N = 2$. Therefore the skeleton $\Sigma$ has the Figure 1.

we consider biorthogonal spline with order $d = \tilde{d} = 2$

$$
\begin{align*}
\theta(x) &= \frac{1}{2} \theta(2x + 1) + \theta(2x) + \frac{1}{2} \theta(x - 1) \\
\tilde{\theta}(x) &= -\frac{1}{4} \tilde{\theta}(2x + 2) + \frac{1}{2} \tilde{\theta}(2x + 1) + \frac{3}{2} \tilde{\theta}(2x) + \frac{1}{2} \tilde{\theta}(x - 1) - \frac{1}{4} \tilde{\theta}(2x - 2)
\end{align*}
$$

such that in this case $N_i = M_i = 2^j$. By using [4,5], we can construct two types of the following function families

(i) Biorthogonal generator and wavelet bases on the edge $e_i = [a_i, b_i], i = 2, 3, 4$ in kind "00"

$$
\begin{align*}
\phi^{e_i}_{j_k} &= \{ \phi^{e_i}_{j_k, k} \mid k = 1, \ldots, 2^J_i - 1 \}, & \tilde{\phi}^{e_i}_{j_k} &= \{ \tilde{\phi}^{e_i}_{j_k, k} \mid k = 1, \ldots, 2^J_i - 1 \} \\
\psi^{e_i}_{j_k} &= \{ \psi^{e_i}_{j_k, k} \mid k = 1, \ldots, 2^J_i \}, & \tilde{\psi}^{e_i}_{j_k} &= \{ \tilde{\psi}^{e_i}_{j_k, k} \mid k = 1, \ldots, 2^J_i \}
\end{align*}
$$

(15)
Figure 1. Example of skeleton $\Sigma$ when $\Omega = (0, 1)^2$ is divided to $2 \times 2$ equal squares subdomains satisfying boundary conditions

$$
\tilde{\phi}_{J_1}^{e_1}(a_i) = \phi_{J_1}^{e_i}(a_i) = \phi_{J_1}^{e_i}(b_i) = 0, \quad 1 \leq k < 2^{J_1} \\
\psi_{J_1}^{e_1}(a_i) = \psi_{J_1}^{e_i}(b_i) = 0, \quad 1 \leq k \leq 2^{J_1}
$$

(ii) Biorthogonal generator and wavelet bases on the edge $e_1 = [a_1, b_1]$ in kind “01”

$$
\phi_1^{e_1} = \{\phi_{J_1,k}^{e_1}, k = 1, \ldots, 2^{J_1}\}, \quad \tilde{\phi}_1^{e_1} = \{\tilde{\phi}_{J_1,k}^{e_1}, k = 1, \ldots, 2^{J_1}\} \\
\psi_1^{e_1} = \{\psi_{J_1,k}^{e_1}, k = 1, \ldots, 2^{J_1}\}, \quad \tilde{\psi}_1^{e_1} = \{\tilde{\psi}_{J_1,k}^{e_1}, k = 1, \ldots, 2^{J_1}\}
$$

satisfying boundary condition

$$
\tilde{\phi}_{J_1}^{e_1}(a_1) = \phi_{J_1}^{e_1}(a_1) = 0, \quad 1 \leq k \leq 2^{J_1} \\
\phi_{J_1,2^{J_1}}^{e_1}(b_1) = 2^{J_1/2}, \phi_{J_1,k}^{e_1}(b_1) = 0 \quad 1 \leq k < 2^{J_1} \\
\psi_{J_1,k}^{e_1}(a_1) = 0, \quad 1 \leq k \leq 2^{J_1}
$$

By using the function families (15), (16) and extension operator on the cross point $C$, we can construct multiscale basis

$$
\psi_J^\Sigma = \phi_J^\Sigma \bigcup_{i=0}^{b-1} \psi_{J_0 + (i,i,i)}^\Sigma
$$

on the skeleton $\Sigma$, where $J = (J_1, J_2, J_3, J_4), J_0 = (J_1 - b, J_2 - b, J_3 - b, J_4 - b)$ and $b = \min_{1 \leq i \leq 4} \{J_i - j_0\}$. Then we can write

$$
\psi_J^\Sigma = P^T \Phi_h, \quad S_{\psi_J^\Sigma} = P^T SP
$$

where the matrix $P$ has the following form
The matrices $\text{IFWT}_{01}^{e_i}, \text{IFWT}_{00}^{e_i}, i = 1, 2, 3, 4$ can be obtained by IFWT on the unitary interval (in kind “01” and “00”). By using the extension operator (11), the matrices $\text{Ext}_{e_i}^{e_1}, i = 2, 3, 4$ can be computed. The diagonal preconditioner $D_J^{-1}$ has the following form

$$ D_J^{-1} = \text{diag}(D_{e_1}^{-1}, D_{e_2}^{-1}, D_{e_3}^{-1}, D_{e_4}^{-1}) $$

such that

$$ D_{e_1}^{-1} = \left( \begin{array}{cccc}
2^{-J_0(1)/2}, & \ldots, & 2^{-J_0(1)/2}, & 2^{-J_0(1)+1/2}, \ldots, & 2^{-J_0(1)+1/2} \\
2^{J_0(1)+1}, & \ldots, & 2^{-J_1(1)/2}, & \ldots, & 2^{-J_1(1)/2} \\
& \ldots, & 2^{-J_1(1)/2}, & \ldots, & 2^{-J_1(1)/2} \\
2^{J_1(1)-1}, & \ldots, & 2^{-J_i(1)/2}, & \ldots, & 2^{-J_i(1)/2}
\end{array} \right) $$

$$ D_{e_i}^{-1} = \left( \begin{array}{cccc}
2^{-J_0(i)/2}, & \ldots, & 2^{-J_0(i)/2}, & 2^{-J_0(i)+1/2}, \ldots, & 2^{-J_0(i)+1/2} \\
2^{J_0(i)+1-1}, & \ldots, & 2^{-J_1(i)/2}, & \ldots, & 2^{-J_1(i)/2} \\
& \ldots, & 2^{-J_1(i)/2}, & \ldots, & 2^{-J_1(i)/2} \\
2^{J_1(i)-1}, & \ldots, & 2^{-J_i(i)/2}, & \ldots, & 2^{-J_i(i)/2}
\end{array} \right) \quad i = 2, 3, 4. $$

**Remark 3.** Now we will test the diagonal preconditioner $D_J^{-1}$ on the problem (5). For all of tests we set $f = 1$. We solved the linear system by Conjugate Gradient(CG) and preconditioned Conjugate Gradient(PCG) methods, using $\varphi_h = 0$ as an initial guess. We will report the number of iterations needed to reduce the residual a factor $10^{-5}$. The condition number of the assembled matrices $S$ and $D_J^{-1}S \psi J D_J^{-1}$ are computed in 2-norm. In first example, we set $a_0 = 0$ and Tables 1-3 are related to this example. In second example, we set $a_0 = 1$ and Tables 4-6 are related to this example.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond($S$)</th>
<th>Cond($D_J^{-1}S \psi J D_J^{-1}$)</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>17.60</td>
<td>7.5234</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>36.18</td>
<td>11.5125</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>72.47</td>
<td>13.2964</td>
<td>29</td>
<td>21</td>
</tr>
<tr>
<td>6</td>
<td>148.10</td>
<td>14.0015</td>
<td>37</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>297.40</td>
<td>14.2985</td>
<td>48</td>
<td>23</td>
</tr>
<tr>
<td>8</td>
<td>604.33</td>
<td>14.3125</td>
<td>64</td>
<td>23</td>
</tr>
</tbody>
</table>
Table 2. Domain Ω is split into $K = 9$. We set $n_k = 2^{J+1}$, $k \neq 4, n_4 = 2^J$, $j_0 = 2$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond(S)</th>
<th>Cond$(D_{J+1}^{-1}S_{\psi_{2J}}D_{J}^{-1})$</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>47.74</td>
<td>7.6537</td>
<td>23</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>97.36</td>
<td>11.2582</td>
<td>30</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>196.67</td>
<td>12.9069</td>
<td>40</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>395.33</td>
<td>13.5404</td>
<td>56</td>
<td>26</td>
</tr>
<tr>
<td>7</td>
<td>799.23</td>
<td>13.8502</td>
<td>78</td>
<td>27</td>
</tr>
<tr>
<td>8</td>
<td>1602.23</td>
<td>13.9614</td>
<td>119</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 3. Domain Ω is split into $K = 16$. We set $n_k = 2^{J}$, $k = 1, \ldots, 8$, $n_k = 2^{J+1}$, $k = 9, \ldots, 16$ and $j_0 = 2$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond(S)</th>
<th>Cond$(D_{J+1}^{-1}S_{\psi_{2J}}D_{J}^{-1})$</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>68.85</td>
<td>9.4552</td>
<td>30</td>
<td>25</td>
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<tr>
<td>4</td>
<td>140.75</td>
<td>13.7455</td>
<td>44</td>
<td>26</td>
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<tr>
<td>5</td>
<td>284.61</td>
<td>15.5751</td>
<td>63</td>
<td>27</td>
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<tr>
<td>6</td>
<td>572.25</td>
<td>16.3578</td>
<td>89</td>
<td>28</td>
</tr>
<tr>
<td>7</td>
<td>1156.79</td>
<td>16.7649</td>
<td>124</td>
<td>29</td>
</tr>
<tr>
<td>8</td>
<td>2318.03</td>
<td>16.9891</td>
<td>178</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 4. Domain Ω is split into $K = 25$. We set $n_k = 2^{J}$, $k \neq 3, 13, 18$, $n_3 = n_{13} = n_{18} = 2^{J+1}$ and $j_0 = 3$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond(S)</th>
<th>Cond$(D_{J+1}^{-1}S_{\psi_{2J}}D_{J}^{-1})$</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>160.56</td>
<td>9.4188</td>
<td>44</td>
<td>29</td>
</tr>
<tr>
<td>5</td>
<td>325.75</td>
<td>13.6806</td>
<td>64</td>
<td>31</td>
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<tr>
<td>6</td>
<td>656.19</td>
<td>15.4092</td>
<td>89</td>
<td>32</td>
</tr>
<tr>
<td>7</td>
<td>1311.22</td>
<td>16.3921</td>
<td>129</td>
<td>33</td>
</tr>
<tr>
<td>8</td>
<td>2628.34</td>
<td>16.8532</td>
<td>191</td>
<td>33</td>
</tr>
</tbody>
</table>

Table 5. Domain Ω is split into $K = 36$. We set $n_k = 2^{J}$, $k \neq 15, 16, 21, 22$, $n_{15} = n_{16} = n_{21} = n_{22} = 2^{J+1}$ and $j_0 = 2$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond(S)</th>
<th>Cond$(D_{J+1}^{-1}S_{\psi_{2J}}D_{J}^{-1})$</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>118.92</td>
<td>9.4596</td>
<td>38</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>244.34</td>
<td>13.7545</td>
<td>55</td>
<td>29</td>
</tr>
<tr>
<td>5</td>
<td>495.13</td>
<td>15.5865</td>
<td>80</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>1002.26</td>
<td>16.3754</td>
<td>117</td>
<td>31</td>
</tr>
<tr>
<td>7</td>
<td>2014.92</td>
<td>16.8012</td>
<td>170</td>
<td>32</td>
</tr>
<tr>
<td>8</td>
<td>4030.85</td>
<td>16.9815</td>
<td>221</td>
<td>33</td>
</tr>
</tbody>
</table>
Table 6. Domain $\Omega$ is split into $K = 9$. We set $n_k = 2^{J+1}, k \neq 2, 5, 8$, $n_2 = n_5 = n_8 = 2^J$ and $j_0 = 3$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Cond($S$)</th>
<th>Cond($D^{-1}<em>J S</em>{\psi J} D^{-1}_J$)</th>
<th>#Iter CG</th>
<th>#Iter PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>65.12</td>
<td>8.6724</td>
<td>29</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>132.18</td>
<td>12.6405</td>
<td>39</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>265.17</td>
<td>14.5469</td>
<td>53</td>
<td>26</td>
</tr>
<tr>
<td>7</td>
<td>535.48</td>
<td>15.4832</td>
<td>79</td>
<td>27</td>
</tr>
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<td>8</td>
<td>1062.86</td>
<td>15.8592</td>
<td>112</td>
<td>28</td>
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References


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