Application of Biorthogonal Wavelets to Preconditioning the 3-Fields Formulation: Numerical Results in Nonconforming Decomposition

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Abstract

The aim of this paper is to implement the wavelet preconditioner for the linear system corresponding to the discrete Poincaré-Steklov operator arising in the three fields formulation. In order to apply the preconditioner, we will introduce multiresolution analysis(MRA) for $L^2(0, 2)$ which is suitable in nonconforming domain decomposition. Extensive numerical results show that the condition number of the matrix after preconditioning is stabilized when the number of subdomains and elements per edge are large.

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1 Introduction

In this paper we deal with domain decomposition of elliptic problems. Many domain decomposition approaches aim at solving boundary value problem in

weak formulation stemming from a second order elliptic differential operator on nonoverlapping subdomains. An important case is when one wants to use grids with possibly different grid sizes on the different subdomains without adaptation of the grids at the interface boundaries.

A very general class of domain decomposition with such nonconforming (non-matching) grids is provided by the three fields formulation (3-FF), as proposed by F. Brezzi and L. D. Marini [2]. By applying a Schur complement technique, the solution of such a problem can be reduced to the solution of an equation on the interface unknown, corresponding to the Poincaré-Steklov operator which is an operator of order one.

Clearly, in order to design efficient solver for 3-FF there is a need for a good preconditioner for the interface problem. It is well known that if the "trace" space is discretized by means of suitable wavelets function, there exist asymptotically optimal diagonal preconditioners [1, 3].

By modifying the biorthogonal MRA for $L^2(0, 2)$ [1, 6], we will introduce a MRA for $L^2(0, 2)$ which is suitable in nonconforming domain decomposition. In numerical implementation, by using the extension operator on the cross points of skeleton and special inverse fast wavelet transforms (IFWTs), we can construct the biorthogonal MRA on the skeleton. For our purpose, we will consider biorthogonal wavelets based on B-spline and we will make wavelets bases satisfying special boundary conditions on the unitary interval.

The paper is organized as follows. In Section 2. we briefly review the discrete 3-FF and wavelet preconditioner. The biorthogonal MRA for $L^2(0, 2)$ in nonconforming decomposition is introduced in section 3 and finally section 4 is devoted to the numerical results.

2 The Discrete 3-FF and Wavelet Preconditioner.

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, we will consider the following simple model problem: given $f \in L^2(\Omega)$, find u such that

$$\begin{cases} -\sum_{i,j=1}^2 \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u}{\partial x_i}) + a_0(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

we assume that for almost all $x \in \Omega$, $0 \leq a_0(x) \leq R$, the coefficient matrix $(a_{ij}(x))_{i,j=1,2}$ is symmetric, sufficiently smooth and uniformly positive definite.

Let Ω be decomposed into a finite number of nonoverlapping polygonal subdomains Ω_k , $k = 1, \dots, K$, where $\overline{\Omega} = \cup_{k=1}^K \overline{\Omega}_k$ and let $\Sigma = (\cup_{k=1}^K \partial\Omega_k) \setminus \partial\Omega$ be the skeleton of the decomposition. We then consider the following functional

spaces

$$V = \left\{ (v^1, \dots, v^K), v^k \in V^k, \quad v^k|_{\partial\Omega\partial\Omega_k} = 0, k = 1, \dots, K \right\}, \quad \Lambda = \prod_{k=1}^K \Lambda^k$$

and

$$\Phi = \left\{ \varphi \in L^2(\Sigma) : \exists v \in H_0^1(\Omega), \varphi = v|_{\Sigma} \right\} = H_0^1(\Omega)|_{\Sigma}$$

where $V^k = H^1(\Omega_k)$ and $\Lambda^k = H^{-\frac{1}{2}}(\partial\Omega_k)$.

To solve the problem (1) by a domain decomposition approach, for each $k = 1, \dots, K$ let $a^k : V^k \times V^k \rightarrow \mathbb{R}$ be the bilinear form induced by the differential operator on the subdomain Ω_k :

$$a^k(v, w) = \int_{\Omega_k} \left(\sum_{i,j=1}^2 (a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j}) + a_0(x)vw \right) dx \quad \forall v, w \in V^k$$

The 3-FF of the problem (1) is the following: find $(u, \lambda, \tau) \in (V \times \Lambda \times \Phi)$ such that

$$\left\{ \begin{array}{ll} \sum_{k=1}^K a^k(u^k, v^k) - \sum_{k=1}^K \langle \lambda^k, v^k \rangle_k = (f, v)_{L^2(\Omega)} & \forall v \in V \\ \sum_{k=1}^K \langle \tau - u^k, \mu^k \rangle_k = 0 & \forall \mu \in \Lambda \\ \sum_{k=1}^K \langle \lambda^k, \sigma \rangle_k = 0 & \forall \sigma \in \Phi \end{array} \right. \quad (2)$$

where $\langle \cdot, \cdot \rangle_k$ is duality pairing between $H^{-\frac{1}{2}}(\partial\Omega_k)$ and $H^{\frac{1}{2}}(\partial\Omega_k)$. For every $f \in L^2(\Omega)$, this problem has a unique solution (u, λ, τ) where u is indeed the solution of problem (1), $\lambda^k = \frac{\partial u^k}{\partial \eta^k} (\eta^k$:outward conormal derivative to Ω_k) on $\partial\Omega_k$ and $\phi = u$ on the skeleton Σ .

The problem (2) can now be approximated by a Galerkin scheme. To this end, one chooses finite dimensional subspaces $V_h = \prod_{k=1}^K V_h^k \subset V$, $\Lambda_h = \prod_{k=1}^K \Lambda_h^k \subset \Lambda$ and $\Phi_h \subset \Phi$. Then the discretized system reads as follows: find $(u_h, \tau_h, \lambda_h) \in (V_h \times \Phi_h \times \Lambda_h)$ such that

$$\left\{ \begin{array}{ll} \sum_{k=1}^K a^k(u_h^k, v_h^k) - \sum_{k=1}^K \langle \lambda_h^k, v_h^k \rangle_k = (f, v_h)_{L^2(\Omega)} & \forall v_h \in V_h, \\ \sum_{k=1}^K \langle \tau_h - u_h^k, \mu_h^k \rangle_k = 0 & \forall \mu_h \in \Lambda_h, \\ \sum_{k=1}^K \langle \lambda_h^k, \sigma_h \rangle_k = 0 & \forall \sigma_h \in \Phi_h, \end{array} \right. \quad (3)$$

It is clear that suitable *inf* – *sup* condition [1] should be assumed on the choice of V_h , Λ_h and Φ_h . The linear system stemming from such an approximation takes the form

$$\begin{cases} A\underline{u}_h - B^T \underline{\lambda}_h &= \underline{f}_h, \\ -B\underline{u}_h + C^T \underline{\tau}_h &= 0, \\ C\underline{\lambda}_h &= 0, \end{cases} \quad (4)$$

where $\underline{u}_h$, $\underline{\lambda}_h$ and $\underline{\tau}_h$ are the coefficients vectors of u_h , λ_h and τ_h in the considered bases for V_h , Λ_h and Φ_h . By a Schur complement argument, the solution of (4) can be reduced to a system in the unknown $\underline{\tau}_h$ of the form

$$\mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T \underline{\tau}_h = \mathbb{C}\mathbb{A}^{-1} \begin{pmatrix} \underline{f}_h \\ 0 \end{pmatrix}, \mathbb{A} := \begin{pmatrix} A & -B^T \\ -B & 0 \end{pmatrix}, \quad \mathbb{C} := (0 \ C). \quad (5)$$

Here the matrix $S := \mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T$ (the discrete Poincaré-Steklov operator on Σ) does not need to be assembled. It is an operator of order one, therefore the matrix S needs to be preconditioned[2].

By using Biorthogonal wavelet bases for Φ_h , we will consider asymptotically optimal diagonal preconditioner for discrete operator S . Suppose that we have a couple of wavelet bases for $L^2(\Sigma)$

$$\begin{aligned} \psi &:= \{\psi_{j,m}, (j, m) \in \nabla := \cup_{j \geq j_0} \nabla_j\} \\ \tilde{\psi} &:= \{\tilde{\psi}_{j,m}, (j, m) \in \nabla := \cup_{j \geq j_0} \nabla_j\} \end{aligned}$$

such that $\nabla_j \sim 2^j$ and also

(P1) ψ and $\tilde{\psi}$ satisfy on the biorthogonal property, i.e.,

$$(\psi_{j,m}, \tilde{\psi}_{j',m'})_{L^2(\Sigma)} = \delta_{j,j'} \delta_{m,m'} \quad m \in \nabla_j, m' \in \nabla_{j'}$$

(P2) any function $\sigma \in L^2(\Sigma)$ can be expanded in term of ψ (or $\tilde{\psi}$):

$$\sigma = \sum_{j \geq j_0} \sum_{m \in \nabla_j} (\sigma, \tilde{\psi}_{j,m})_{L^2(\Sigma)} \psi_{j,m}$$

(P3) The following *norm equivalence* holds

$$\|\sigma\|_{\Phi} \sim \sum_{j \geq j_0} 2^j \sum_{m \in \nabla_j} |(\sigma, \tilde{\psi}_{j,m})_{L^2(\Sigma)}|^2, \quad \sigma \in \Phi$$

(P4) ψ and $\tilde{\psi}$ have local support, i.e.,

$$\text{diam}(\text{supp} \psi_{j,m}) \sim \text{diam}(\text{supp} \tilde{\psi}_{j,m}) \sim 2^{-j}.$$

where the notation $\alpha \sim \beta$ express that α can be bounded from above and below by a constant multiple of β uniformly in any parameters on which α and β may depend.

Let D_J be the diagonal matrix defined by

$$(D_J)_{(j,m),(j',m')} = 2^{j/2} \delta_{j,j'} \delta_{m,m'}, \quad j_0 \leq j, j' \leq J, \quad m \in \nabla_j, \quad m' \in \nabla_{j'}.$$

Theorem 1. let $\Phi_h = \text{span}_{\Phi}(\psi_J)$ with $\psi_J = \{\psi_{j,m}, (j, m) \in \nabla_J := \cup_{j \geq j_0}^J \nabla_j\}$ and S_{ψ_J} be the matrix corresponding to the multiscale basis ψ_J then

$$\text{cond}_2(D_J^{-1} S_{\psi_J} D_J^{-1}) = \mathbf{O}(1), \quad J \rightarrow \infty.$$

Proof: [1, 3].

Therefore by choosing the wavelet base ψ_J for Φ_h , the discrete Poincaré-Steklov operator can be preconditioned by the asymptotically optimal preconditioner D_J^{-1} .

3 Biorthogonal MRA for $L^2(0, 2)$

The aim of this section is to construct a MRA for $L^2(0, 2)$. By using this MRA, we can construct a MRA for $L^2(\Sigma)$ where Σ will be the union of one dimensional segments.

The idea is to start from a biorthogonal MRA for $L^2(0, 1)$ and introducing a MRA for $L^2(0, 2)$ by considering the interval $(0, 2)$ as a closure of $(0, 1) \cup (1, 2)$. We want to point out that the primal scaling and wavelet functions must be continue on $[0, 2]$. However, for our purpose, the continuity of the dual functions is not strictly necessary. We note that the number of scaling and wavelet functions on the intervals $(0, 1)$ and $(1, 2)$ must be different. We start with considering two families of scaling functions

$$\begin{aligned} \varphi_{j_1}^1 &= \{\varphi_{j_1,k}^1, k \in \Delta_{j_1}\} \subset L^2(0, 1), \\ \tilde{\varphi}_{j_1}^1 &= \{\tilde{\varphi}_{j_1,k}^1, k \in \Delta_{j_1}\} \subset L^2(0, 1), \end{aligned} \tag{6}$$

where $\Delta_{j_1} = \{0, 1, \dots, N_{j_1}\}$, $\#\Delta_{j_1} \sim 2^{j_1}$ and $j_1 \geq j_0$.

These families are generator bases of a biorthogonal MRA for $L^2(0, 1)$ [4, 5]

$$T_{j_1}^1 = \text{span} \varphi_{j_1}^1, \quad \tilde{T}_{j_1}^1 = \text{span} \tilde{\varphi}_{j_1}^1, \tag{7}$$

such that the following conditions are fulfilled:

(G1) The two families $\varphi_{j_1}^1, \tilde{\varphi}_{j_1}^1$ are refineable and we have

$$\text{clos}_{H^1}(\cup_{j_1 \geq j_0} T_{j_1}^1) = H^1(0, 1), \quad \text{clos}_{L^2}(\cup_{j_1 \geq j_0} \tilde{T}_{j_1}^1) = L^2(0, 1).$$

furthermore they consist of functions with local support,

$$\text{diam}(\text{supp} \varphi_{j_1, k}^1) \sim \text{diam}(\text{supp} \tilde{\varphi}_{j_1, k}^1) \sim 2^{-j_1}, \quad \text{for all } k \in \Delta_{j_1}.$$

which are L^2 -Stable.

(G2) They are biorthogonal and satisfy following boundary conditions

$$\begin{aligned} \varphi_{j_1, 0}^1(0) &= 2^{j_1/2}, \quad \varphi_{j_1, m}^1(0) = 0, \quad 0 < m \leq N_{j_1}, \\ \varphi_{j_1, N_{j_1}}^1(1) &= 2^{j_1/2}, \quad \varphi_{j_1, m}^1(1) = 0, \quad 0 \leq m < N_{j_1}. \end{aligned}$$

(G3) All of polynomials with order less than $d, \tilde{d}$ are contained in $T_{j_1}^1, \tilde{T}_{j_1}^1$.

(G4) Projectors $P_{j_1}^1 : L^2(0, 1) \rightarrow T_{j_1}^1$ can be defined by

$$P_{j_1}^1 v = \sum_{k \in \Delta_{j_1}} (v, \tilde{\varphi}_{j_1, k}^1)_{L^2(0, 1)} \varphi_{j_1, k}^1, \quad (8)$$

which are uniformly bounded on $L^2(0, 1)$.

(G5) A Bernstein-type inequality for $T_{j_1}^1$ holds

$$\|f_{j_1}\|_{H^1(0, 1)} \lesssim 2^{j_1} \|f_{j_1}\|_{L^2(0, 1)} \quad \text{for all } f_{j_1} \in T_{j_1}^1. \quad (9)$$

By using the stable completion technique [4], biorthogonal wavelets can be constructed

$$\psi_{j_1}^1 = \{\psi_{j_1, k}^1, k \in \nabla_{j_1}\}, \quad \tilde{\psi}_{j_1}^1 = \{\tilde{\psi}_{j_1, k}^1, k \in \nabla_{j_1}\},$$

where $\nabla_{j_1} = \{1, \dots, M_{j_1}\}$, and $\#\nabla_{j_1} \sim 2^{j_1}$. Now we will consider other generator bases for $T_{j_2}^1, \tilde{T}_{j_2}^1$

$$\check{\varphi}_{j_2}^1 = \{\check{\varphi}_{j_2, k}^1, k \in \Delta_{j_2}\}, \quad \check{\tilde{\varphi}}_{j_2}^1 = \{\check{\tilde{\varphi}}_{j_2, k}^1, k \in \Delta_{j_2}\}, \quad (10)$$

satisfying (G1)-(G5) and following properties [5]

(i) $\check{\varphi}_{j_2, 0}^1(0) = \tau 2^{j_2/2}, \check{\tilde{\varphi}}_{j_2, k}^1(0) = 0, 0 < k \leq N_{j_2}$

where τ is a nonzero constant.

(ii) The corresponding wavelet base $\{\check{\psi}_{j_2,k}^1, k \in \nabla_{j_2}\}$ are zero at $x = 0$

Now we want to define a couple $\{T_{j_1,j_2}\}, \{\tilde{T}_{j_1,j_2}\}$ of biorthogonal MRA for $L^2(0, 2)$. By the aid of extension operator E_{j_1,j_2}

$$(E_{j_1,j_2}f)(x) := \begin{cases} f(x) = \sum_{m \in \Delta_{j_1}} f_m \varphi_{j_1,m}^1(x), & x \in [0, 1), \\ 2^{(j_1-j_2)/2} f_{N_{j_1}} \varphi_{j_2,0}^2(x), & x \in [1, 2], \end{cases} \quad (11)$$

where $f \in T_{j_1}^1$. The generator base for T_{j_1,j_2} is the following

$$\varphi_{j_1,j_2} := \{\varphi_{j_1,j_2,m}, m = 0, 1, \dots, N_{j_1} + N_{j_2}\}, \quad (12)$$

where for $m = 0, \dots, N_{j_1} - 1$,

$$\varphi_{j_1,j_2,m} = \begin{cases} \varphi_{j_1,m}^1, & \text{in } [0, 1), \\ 0, & \text{in } [1, 2]. \end{cases}$$

and for $m = \ell + N_{j_1}$, $\ell = 1, \dots, N_{j_2}$,

$$\varphi_{j_1,j_2,m} = \begin{cases} 0, & \text{in } [0, 1), \\ \check{\varphi}_{j_2,\ell}^2, & \text{in } [1, 2], \end{cases}$$

where $\check{\varphi}_{j_2,\ell}^2(x) = \check{\varphi}_{j_2,\ell}^1(x - 1)$ for all $x \in [1, 2]$. The function $\varphi_{j_1,j_2,N_{j_1}}$ at the cross point $x = 1$ is defined by

$$\varphi_{j_1,j_2,N_{j_1}} := \begin{cases} \varphi_{j_1,N_{j_1}}^1, & \text{in } [0, 1), \\ 2^{(j_1-j_2)/2} \varphi_{j_2,0}^2, & \text{in } [1, 2]. \end{cases}$$

The generator basis of dual spaces $\tilde{T}_{j_1,j_2}$ can be defined by

$$\tilde{\varphi}_{j_1,j_2,k} = \begin{cases} \tilde{\varphi}_{j_1,k}^1 & \text{in } [0, 1) \\ 0, & \text{in } [1, 2], \end{cases} \quad \tilde{\varphi}_{j_1,j_2,\ell+N_{j_1}} = \begin{cases} d_{j_1,j_2,\ell} \tilde{\varphi}_{j_1,N_{j_1}}^1, & \text{in } [0, 1) \\ \check{\varphi}_{j_2,\ell}^2, & \text{in } [1, 2], \end{cases}$$

for $k = 0, \dots, N_{j_1}$, $\ell = 1, \dots, N_{j_2}$ and $d_{j_1,j_2,\ell} = -(\varphi_{j_1,0}^2, \check{\varphi}_{j_2,\ell}^2)_{L^2(1,2)}$. We observe that φ_{j_1,j_2} and $\tilde{\varphi}_{j_1,j_2}$ are biorthogonal in the interval $(0, 2)$. We note that The spaces T_{j_1,j_2} and $\tilde{T}_{j_1,j_2}$ are nested, i.e.,

$$T_{j_1,j_2} \subset T_{j'_1,j'_2}, \quad \tilde{T}_{j_1,j_2} \subset \tilde{T}_{j'_1,j'_2}, \quad j'_1 > j_1, j'_2 > j_2.$$

The corresponding operator $P_{j_1, j_2} : L^2(0, 2) \rightarrow T_{j_1, j_2}$ can be defined by

$$P_{j_1, j_2} f = \sum_{k \in \Delta_{j_1, j_2}} (f, \tilde{\varphi}_{j_1, j_2, k})_{L^2(0, 2)} \varphi_{j_1, j_2, k} \quad (13)$$

where $\Delta_{j_1, j_2} = \{0, \dots, N_{j_1} + N_{j_2}\}$.

Remark 1 The operator P_{j_1, j_2} has the following properties

- (i) $P_{j_1, j_2} v = v$ for all $v \in T_{j_1, j_2}$
- (ii) $P_{\ell_1, \ell_2} P_{j_1, j_2} = P_{j_1, j_2} P_{\ell_1, \ell_2} = P_{j_1, j_2}$ for all $\ell_1 > j_1, \ell_2 > j_2$
- (iii) P_{j_1, j_2} is bounded on $L^2(0, 2)$

The following wavelet functions on the interval $(0, 2)$ can be constructed

$$\begin{aligned} \psi_{j_1, j_2} &= \{\psi_{j_1, j_2, m}, m = 1, \dots, M_{j_1} + M_{j_2}\}, \\ \tilde{\psi}_{j_1, j_2} &= \{\tilde{\psi}_{j_1, j_2, m}, m = 1, \dots, M_{j_1} + M_{j_2}\} \end{aligned}$$

where for $m = 1, \dots, M_{j_1}$,

$$\begin{aligned} \psi_{j_1, j_2, m} &= \begin{cases} \psi_{j_1, m}^1, & \text{in } [0, 1) \\ c_{j_1, j_2, m}(\varphi_{j_2, 0}^2 - P_{j_2, 0}^2 \varphi_{j_2, 0}^2) & \text{in } [1, 2], \end{cases} \\ \tilde{\psi}_{j_1, j_2, m} &= \begin{cases} \tilde{\psi}_{j_1, m}^1, & \text{in } [0, 1) \\ 0 & \text{in } [1, 2], \end{cases} \end{aligned}$$

such that $c_{j_1, j_2, m} = 2^{-j_2/2} \psi_{j_1, m}^1(1)$ and $P_{j_2, 0}^2 f = \sum_{k=1}^{N_{j_2}} (f, \tilde{\varphi}_{j_2, k}^2)_{L^2(1, 2)} \tilde{\varphi}_{j_2, k}^2$.

while for $\tilde{c}_{j_1, j_2, m} = -2^{(j_1 - j_2)/2} (\varphi_{j_2, 0}^2, \tilde{\psi}_{j_2, m}^2)_{L^2(1, 2)}$,

$$\begin{aligned} \psi_{j_1, j_2, m + M_{j_1}} &= \begin{cases} 0, & \text{in } [0, 1) \\ \tilde{\psi}_{j_2, m}^2, & \text{in } [1, 2], \end{cases} \\ \tilde{\psi}_{j_1, j_2, m + M_{j_1}} &= \begin{cases} \tilde{c}_{j_1, j_2, m} \tilde{\varphi}_{j_1, N_{j_1}}^1, & \text{in } [0, 1) \\ \tilde{\psi}_{j_2, m}^2 & \text{in } [1, 2]. \end{cases} \end{aligned}$$

Remark 2 It is simple to check that

- (i) $P_{j_1, j_2} \psi_{j_1, j_2, m} = 0, m \in \nabla_{j_1, j_2}$.
- (ii) the following new biorthogonal wavelets are L^2 -stable on the interval $(0, 2)$

$$\psi = \{\psi_{j_1, j_2, m}, (j_1, j_2, m) \in \nabla = \cup_{j_1 \geq j_0, j_2 \geq j_0} \nabla_{j_1, j_2}\},$$

$$\tilde{\psi} = \{\tilde{\psi}_{j_1, j_2, m}, (j_1, j_2, m) \in \nabla = \cup_{j_1 \geq j_0, j_2 \geq j_0} \nabla_{j_1, j_2}\},$$

where $\nabla_{j_1, j_2} = \{1, \dots, M_{j_1} + M_{j_2}\}$.

By using the biorthogonal MRA for $L^2(0, 1)$ and $L^2(0, 2)$, we can obtain a biorthogonal MRA for $L^2(\Sigma)$ satisfying in **(P1)**-(**P4**)[1,6].

4 Numerical results

We will test the diagonal preconditioner on the following simple model problem: given $f \in L^2(\Omega)$, find u satisfying

$$\begin{cases} -\Delta u + a_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (14)$$

with $\Omega = (0, 1)^2$. We consider a uniform decomposition of $\Omega = (0, 1)^2$ in $K = N \times N$ equal subdomain of size $H \times H$, $H = 1/N$.

In each subdomain Ω_k , by considering a nonconforming domain decomposition, we take a uniform mesh composed of $n_k \times n_k$, equal square element of size $\delta_k \times \delta_k$, $\delta_k = H/n_k$ and $n_k = 2^{s_k}$. We then define V_h^k to be the corresponding space of $\mathbb{Q}1$ finite element on the mesh. The multiplier Λ_h^k is then defined as the trace of V_h^k on $\partial\Omega_k$. The skeleton Σ is split as $\Sigma = \cup_{i=1}^{2^{(N-1)N}} e_i$ with $e_i = (\partial\Omega_k \cap \partial\Omega_\ell)$ for some $k = k(i)$ and $\ell = \ell(i)$. By considering a grid $\mathcal{D}$ on Σ obtained by uniformly splitting each e_i into 2^{J_i} ($J_i = \min\{s_k, s_\ell\}$) elements, we define Φ_h to be the space of $\mathbb{P}1$ finite elements on such a grid

$$\Phi_h = \{\varphi_h \in C^0(\Sigma); \varphi_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{D}, \varphi_h|_{\partial\Omega} = 0\}$$

Now we construct the MRA for $L^2(\Sigma)$ with $N = 2$. Therefore the skeleton Σ has the Figure 1.

we consider biorthogonal spline with order $d = \tilde{d} = 2$

$$\begin{aligned} \theta(x) &= \frac{1}{2}\theta(2x+1) + \theta(2x) + \frac{1}{2}\theta(x-1) \\ \tilde{\theta}(x) &= -\frac{1}{4}\tilde{\theta}(2x+2) + \frac{1}{2}\tilde{\theta}(2x+1) + \frac{3}{2}\tilde{\theta}(2x) + \frac{1}{2}\tilde{\theta}(x-1) - \frac{1}{4}\tilde{\theta}(2x-2) \end{aligned}$$

such that in this case $N_j = M_j = 2^j$. By using [4,5], we can construct two types of the following function families

(i) Biorthogonal generator and wavelet bases on the edge $e_i = [a_i, b_i]$, $i = 2, 3, 4$ in kind "00"

$$\begin{aligned} \phi_{J_i}^{e_i} &= \{\phi_{J_i, k}^{e_i}, k = 1, \dots, 2^{J_i} - 1\}, \quad \tilde{\phi}_{J_i}^{e_i} = \{\tilde{\phi}_{J_i, k}^{e_i}, k = 1, \dots, 2^{J_i} - 1\} \\ \psi_{J_i}^{e_i} &= \{\psi_{J_i, k}^{e_i}, k = 1, \dots, 2^{J_i}\}, \quad \tilde{\psi}_{J_i}^{e_i} = \{\tilde{\psi}_{J_i, k}^{e_i}, k = 1, \dots, 2^{J_i}\} \end{aligned} \quad (15)$$

Figure 1. Example of skeleton Σ when $\Omega = (0, 1)^2$ is divided to 2×2 equal squares subdomains

satisfying boundary conditions

$$\begin{aligned} \tilde{\phi}_{J_i,k}^{e_i}(a_i) &= \phi_{J_i,k}^{e_i}(a_i) = \phi_{J_i,k}^{e_i}(b_i) = 0, & 1 \leq k < 2^{J_i} \\ \psi_{J_i,k}^{e_i}(a_i) &= \psi_{J_i,k}^{e_i}(b_i) = 0, & 1 \leq k \leq 2^{J_i} \end{aligned}$$

(ii) Biorthogonal generator and wavelet bases on the edge $e_1 = [a_1, b_1]$ in kind "01"

$$\begin{aligned} \phi_{J_1}^{e_1} &= \{\phi_{J_1,k}^{e_1}, k = 1, \dots, 2^{J_1}\}, & \tilde{\phi}_{J_1}^{e_1} &= \{\tilde{\phi}_{J_1,k}^{e_1}, k = 1, \dots, 2^{J_1}\} \\ \psi_{J_1}^{e_1} &= \{\psi_{J_1,k}^{e_1}, k = 1, \dots, 2^{J_1}\}, & \tilde{\psi}_{J_1}^{e_1} &= \{\tilde{\psi}_{J_1,k}^{e_1}, k = 1, \dots, 2^{J_1}\} \end{aligned} \quad (16)$$

satisfying boundary condition

$$\begin{aligned} \tilde{\phi}_{J_1,k}^{e_1}(a_1) &= \phi_{J_1,k}^{e_1}(a_1) = 0, & 1 \leq k \leq 2^{J_1} \\ \phi_{J_1,2^{J_1}}^{e_1}(b_1) &= 2^{J_1/2}, \phi_{J_1,k}^{e_1}(b_1) = 0 & 1 \leq k < 2^{J_1} \\ \psi_{J_1,k}^{e_1}(a_1) &= 0, & 1 \leq k \leq 2^{J_1} \end{aligned}$$

By using the function families (15), (16) and extension operator on the cross point C , we can construct multiscale basis

$$\psi_J^\Sigma = \phi_{J_0}^\Sigma \bigcup_{i=0}^{b-1} \psi_{J_0+(i,i,i,i)}^\Sigma$$

on the skeleton Σ , where $J = (J_1, J_2, J_3, J_4)$, $J_0 = (J_1 - b, J_2 - b, J_3 - b, J_4 - b)$ and $b = \min_{1 \leq i \leq 4} \{J_i - j_0\}$. Then we can write

$$\psi_J^\Sigma = P^T \Phi_h, \quad S_{\psi_J^\Sigma} = P^T S P$$

where the matrix P has the following form

$$P = \begin{pmatrix} \text{IFWT}_{01}^{e_1} & \text{Ext}_{e_1}^{e_2} & \text{Ext}_{e_1}^{e_3} & \text{Ext}_{e_1}^{e_4} \\ 0 & \text{IFWT}_{00}^{e_2} & 0 & 0 \\ 0 & 0 & \text{IFWT}_{00}^{e_3} & 0 \\ 0 & 0 & 0 & \text{IFWT}_{00}^{e_4} \end{pmatrix}$$

The matrices $\text{IFWT}_{01}^{e_1}, \text{IFWT}_{00}^{e_i}, i = 1, 2, 3, 4$ can be obtained by IFWT on the unitary interval (in kind "01" and "00"). By using the extension operator (11), the matrices $\text{Ext}_{e_1}^{e_i}, i = 2, 3, 4$ can be computed. The diagonal preconditioner D_J^{-1} has the following form

$$D_J^{-1} = \text{diag}(D_{e_1}^{-1}, D_{e_2}^{-1}, D_{e_3}^{-1}, D_{e_4}^{-1})$$

such that

$$D_{e_1}^{-1} = \left(\underbrace{2^{-J_0(1)/2}, \dots, 2^{-J_0(1)/2}}_{2^{J_0(1)+1}}, \underbrace{2^{-(J_0(1)+1)/2}, \dots, 2^{-(J_0(1)+1)/2}}_{2^{J_0(1)+1}}, \dots, \underbrace{2^{-(J_1-1)/2}, \dots, 2^{-(J_1-1)/2}}_{2^{J_1-1}} \right)$$

$$D_{e_i}^{-1} = \left(\underbrace{2^{-J_0(i)/2}, \dots, 2^{-J_0(i)/2}}_{2^{J_0(i)+1-1}}, \underbrace{2^{-(J_0(i)+1)/2}, \dots, 2^{-(J_0(i)+1)/2}}_{2^{J_0(i)+1}}, \dots, \underbrace{2^{-(J_i-1)/2}, \dots, 2^{-(J_i-1)/2}}_{2^{J_i-1}} \right) \quad i = 2, 3, 4.$$

Remark 3. Now we will test the diagonal preconditioner D_J^{-1} on the problem (5). For all of tests we set $f = 1$. We solved the linear system by Conjugate Gradient (CG) and preconditioned Conjugate Gradient (PCG) methods, using $\varphi_h = 0$ as an initial guess. We will report the number of iterations needed to reduce the residual a factor 10^{-5} . The condition number of the assembled matrices S and $D_J^{-1} S_{\psi_J^\Sigma} D_J^{-1}$ are computed in 2-norm. In first example, we set $a_0 = 0$ and Tables 1-3 are related to this example. In second example, we set $a_0 = 1$ and Tables 4-6 are related to this example.

Table 1. Domain Ω is split into $K = 4$. We set $n_1 = n_3 = 2^J, n_2 = n_4 = 2^{J+1}, j_0 = 2$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1} S_{\psi_J^\Sigma} D_J^{-1})$	#Iter CG	#Iter PCG
3	17.60	7.5234	16	17
4	36.18	11.5125	22	20
5	72.47	13.2964	29	21
6	148.10	14.0015	37	22
7	297.40	14.2985	48	23
8	604.33	14.3125	64	23

Table 2. Domain Ω is split into $K = 9$. we set $n_k = 2^{J+1}, k \neq 4, n_4 = 2^J, j_0 = 2$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1}S_{\psi_J^\Sigma}D_J^{-1})$	#Iter CG	#Iter PCG
3	47.74	7.6537	23	21
4	97.36	11.2582	30	24
5	196.67	12.9069	40	25
6	395.33	13.5404	56	26
7	799.23	13.8502	78	27
8	1602.23	13.9614	119	28

Table 3. Domain Ω is split into $K = 16$. We set $n_k = 2^J, k = 1, \dots, 8, n_k = 2^{J+1}, k = 9, \dots, 16$ and $j_0 = 2$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1}S_{\psi_J^\Sigma}D_J^{-1})$	#Iter CG	#Iter PCG
3	68.85	9.4552	30	25
4	140.75	13.7455	44	26
5	284.61	15.5751	63	27
6	572.25	16.3578	89	28
7	1156.79	16.7649	124	29
8	2318.03	16.9891	178	29

Table 4. Domain Ω is split into $K = 25$. We set $n_k = 2^J, k \neq 3, 13, 18, n_3 = n_{13} = n_{18} = 2^{J+1}$ and $j_0 = 3$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1}S_{\psi_J^\Sigma}D_J^{-1})$	#Iter CG	#Iter PCG
4	160.56	9.4188	44	29
5	325.75	13.6806	64	31
6	656.19	15.4092	89	32
7	1311.22	16.3921	129	33
8	2628.34	16.8532	191	33

Table 5. Domain Ω is split into $K = 36$. We set $n_k = 2^J, k \neq 15, 16, 21, 22, n_{15} = n_{16} = n_{21} = n_{22} = 2^{J+1}$ and $j_0 = 2$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1}S_{\psi_J^\Sigma}D_J^{-1})$	#Iter CG	#Iter PCG
3	118.92	9.4596	38	25
4	244.34	13.7545	55	29
5	495.13	15.5865	80	30
6	1002.26	16.3754	117	31
7	2014.92	16.8012	170	32
8	4030.85	16.9815	221	33

Table 6. Domain Ω is split into $K = 9$. We set $n_k = 2^{J+1}$, $k \neq 2, 5, 8$, $n_2 = n_5 = n_8 = 2^J$ and $j_0 = 3$.

J	$\text{Cond}(S)$	$\text{Cond}(D_J^{-1} S_{\psi_J} D_J^{-1})$	#Iter CG	#Iter PCG
4	65.12	8.6724	29	23
5	132.18	12.6405	39	25
6	265.17	14.5469	53	26
7	535.48	15.4832	79	27
8	1062.86	15.8592	112	28

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