On The Sobolev-type Inequality for Lebesgue Spaces with a Variable Exponent

B.CEKIC^{a, 1}, R.MASHIYEV^a and G.T.ALISOY^b

^aDicle University, Dept. of Mathematics, 21280, Diyarbakır, Turkey

^bInonu University, Dept. of Mathematical Sciences, 44280, Malatya, Turkey

Abstract

This paper gives Sobolev-type inequality for the generalized Lebesgue space $L^{p(x)}(\Omega)$ and corresponding Sobolev space $W^{1,p(x)}(\Omega)$, consisting of all $f \in L^{p(x)}(\Omega)$ with first-order distributional derivatives in $L^{p(x)}(\Omega)$, where Ω is a bounded domain $R^N (N \geq 2)$, and $p(x)$ and $q(x)$ measurable function with $1 \leq q^- \leq q(x) \leq q^+ < \infty$, $1 \leq p^+ < N$ and

$$\sup_{x \in \Omega} q(x) \left[\frac{1}{p(x)} - \frac{1}{N} \right] \leq 1 - \frac{(q^+ - q^-)(N - 1)}{N}.$$

In other words, our aim in this study is obtained pre-limit condition for measurable variable exponent functions in Sobolev-type inequality.

Mathematics Subject Classification: 46E35, 26D10

Keywords: Variable exponent, Sobolev-type inequality

1. Introduction

Let $\Omega \subset R^N$ be an open set and $p : \Omega \rightarrow [1, \infty)$ be measurable function. We define the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ consisting of all measurable functions $f : \Omega \rightarrow R$ such that

$$\int_{\Omega} |\lambda f(x)|^{p(x)} dx < \infty$$

for some $\lambda > 0$. We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$\|f\|_{p(x), \Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

and the $L^{p(x)}(\Omega)$ becomes a Banach spaces [5].

---

¹Corresponding Author : bilalc@dicle.edu.tr

This research is supported by the DUAPK 04-FF-40.

The corresponding Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $f \in L^{p(x)}(\Omega)$ such that all generalized $D_i f$, $i = 1, 2, \dots, n$, belong to $L^{p(x)}(\Omega)$. Endowed with the norm

$$\|f\|_{1,p(x)} = \|f\|_{p(x)} + \|\nabla f\|_{p(x)}$$

it forms a Banach space.

O.Kováč and J. Rákosník [5] obtained Sobolev-type inequality

$$\|u\|_{q(x),\Omega} \leq c \|\nabla u\|_{p(x),\Omega} \quad ; \quad u \in C_0^\infty(\Omega). \quad (1)$$

for continuous $p(x)$ function, where a positive constant c is independent of u and

$$1 \leq q(x) \leq \frac{Np(x)}{N-p(x)} - \varepsilon = p^*(x) - \varepsilon, \quad 0 < \varepsilon < \frac{1}{N-1},$$

and they showed that, in general, the Sobolev space $W^{1,p(x)}(\Omega)$ is not embedding in $L^{p^*(x)}(\Omega)$. Obviously, $q(x) = \frac{Np(x)}{N-p(x)}$ equality is not valid for above embedding when $p(x)$ is discontinuous. Naturally, $q(x)$ must be sufficiently less than $\frac{Np(x)}{N-p(x)}$.

Later D.E.Edmunds and J.Rákosník [2,3] proved $W^{1,p(x)}(\Omega) \rightarrow L^{p^*(x)}(\Omega)$ for bounded Lipschitz domain, where $p(x) \in C^{0,1}(\overline{\Omega})$ and $1 \leq p(x) \leq q < N$.

X.L.Fan, J.Shen and D.Zhao [4] investigated Sobolev-type inequality for domain with cone property.

Recently, L.Diening [1] obtained above embedding for unbounded domain assuming that $p(x)$ is constant at infinity and satisfies (w -Lip.)

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \quad ; \quad x, y \in R^N, \quad |x-y| \leq \frac{1}{2}.$$

That is, under mentioned conditions, Sobolev-type embedding is proved in case of limit by L.Diening [1]. S.Samko [6] also found the following result for measurable variable exponent functions in generalized Lebesgue space:

Theorem 1.1 [6] Let $k(x) \in L^Q(B(0, 2D))$ where $Q \geq 1$ and $D = \text{diam } \Omega$. The convolution operator

$$K_\Omega f = \int_\Omega k(x-y)f(y)dy$$

is bounded from $L^{p(x)}(\Omega)$ into $L^{r(x)}(\Omega)$, $r(x) \geq 1$, if

$$\frac{1}{Q} \leq 1 - \frac{1}{p^-} + \frac{1}{p^+}, \quad \frac{1}{r(x)} \geq \frac{1}{Q} + \frac{1}{p^+} - 1.$$

In this article, our aim is to show Sobolev-type inequality (1) for Lebesgue spaces with a variable exponent where Ω is assumed to be a bounded domain in R^N and $p(x)$ is a measurable real-valued function (may be discontinuous) defined on Ω . That is, we determine “Pre-limit” condition for measurable variable exponent functions in Sobolev-type inequality.

Notation:

In this study, $|\Omega|$ is the Lebesgue measure of a set Ω in R^N and $g^+ = \sup_{x \in \Omega} g(x)$, $g^- = \inf_{x \in \Omega} g(x)$, $\delta = \frac{q^+ - q^-}{N^*}$ and $N^* = \frac{N}{N-1}$.

2. A Sobolev-type Inequality

Theorem 2.1: Let $\Omega \subset R^N$, ($N \geq 2$) be a bounded domain. Let $1 \leq q^- \leq q(x) \leq q^+ < \infty$ and $1 \leq p^+ < N$ be such that

$$\sup_{x \in \Omega} q(x) \left[\frac{1}{p(x)} - \frac{1}{N} \right] \leq 1 - \delta$$

for $x \in \Omega$. Then, there exist a constant $C(N, p, q) > 0$ such that

$$\|u\|_{q(x), \Omega} \leq C(N, p, q, \Omega) \|\nabla u\|_{p(x), \Omega} \quad ; \quad u \in C_0^\infty(\Omega), \quad (2)$$

where

$$C(N, p, q, \Omega) = \begin{cases} C(N, p, q) |\Omega|^{\frac{1}{N} - \frac{1}{p^-} + \frac{1}{q^+}} & \text{if } |\Omega| \leq 1 \\ C(N, p, q) |\Omega|^{\frac{1}{N} - \frac{1}{p^+} + \frac{1}{q^-}} & \text{if } |\Omega| > 1 \end{cases}.$$

Proof: We suppose that $\|u\|_{q(x), \Omega} = 1$ and $|\Omega| = 1$. We have

$$\begin{aligned} 1 = \int_{\Omega} |u(x)|^{q(x)} dx &= \int_{E_2} |u(x)|^{q(x)} dx + \int_{E_0/E_2} |u(x)|^{q(x)} dx \\ &= I_1 + I_2, \end{aligned}$$

where $E_t = \{x \in \Omega : |u(x)| > t\}$. Then it is easy to see that

$$I_2 = \int_{E_0/E_2} |u(x)|^{q(x)} dx \leq 2^{q^+ - 1} \int_{\Omega} |u(x)| dx \quad (3)$$

and since

$$|u(x)|^{q(x)} - 1 \geq \frac{1}{2} |u(x)|^{q(x)}$$

for $x \in E_2$, we can write

$$I_1 = \int_{E_2} |u(x)|^{q(x)} dx \leq 2 \int_{E_2} dx \left(\int_1^{|u(x)|} q(x) t^{q(x)-1} dt \right).$$

Also by Fubini's Theorem, we obtain

$$I_1 \leq 2 \int_1^\infty dt \left(\int_{E_t} t^{q(x)-1} q(x) dx \right). \quad (4)$$

By using Taylor's expansions of e^x , from (4) we get

$$\begin{aligned} I_1 &\leq 2 \int_1^\infty t^{a-1} dt \left(\int_{E_t} t^{q(x)-a} q(x) dx \right) \\ &= 2 \int_1^\infty t^{a-1} dt \left(\int_{E_t} \left(\sum_{n=0}^\infty \frac{(q(x)-a)^n q(x) \ln^n t}{n!} \right) dx \right) \\ &= 2 \int_1^\infty t^{a-1} dt \left(\sum_{n=0}^\infty \frac{\ln^n t}{n!} \int_{E_t} q(x) (q(x)-a)^n dx \right), \end{aligned} \quad (5)$$

where $a \in R^1$, $a < q^-$. On the other hand, by means of Gagliardo's inequality [7], for $f \in C_0^\infty(\Omega)$, $w \in L_{loc}^1(\Omega)$

$$\left(\int_\Omega |f|^{N^*} w(x) dx \right)^{\frac{1}{N^*}} \leq C(N) \int_\Omega |\nabla f| w(x)^{\frac{1}{N^*}} dx \quad (6)$$

we obtain

$$\begin{aligned} &\left(\int_\Omega q(x) (q(x)-a)^n |f|^{N^*} dx \right)^{\frac{1}{N^*}} \\ &\leq K^{\frac{1}{N^*}}(n, a) \left(\int_\Omega |\nabla f| (q(x)-a)^{\frac{n}{N^*}} q(x)^{\frac{1}{N^*}} dx \right) \end{aligned} \quad (7)$$

where

$$K(n, a) = C(N) \left(\frac{q^+ - a}{q^- - a} \right)^n, \quad n = 0, 1, 2, \dots$$

If we take $Z(x) = \min\left(\frac{|u(x)| - \frac{t}{2}}{\frac{t}{2}}, 1\right)$, $t > 0$ in place of the test function f in (7) then, we find

$$\begin{aligned} & \int_{E_t} q(x) (q(x) - a)^n dx \\ & \leq \frac{K(n, a)}{t^{N^*}} \left(\int_{G_t} |\nabla u| (q(x) - a)^{\frac{n}{N^*}} q(x)^{\frac{1}{N^*}} dx \right)^{N^*} \end{aligned} \quad (8)$$

where

$$G_t = \left\{ x \in \Omega : \frac{t}{2} < |u(x)| \leq t \right\}.$$

Hence, from (5) and (8) we have

$$\begin{aligned} I_1 & \leq 2 \int_1^\infty t^{a-1} dt \times \\ & \left(\sum_{n=0}^\infty \frac{K(n, a) \ln^n t}{n! t^{N^*}} \left(\int_{G_t} |\nabla u| (q(x) - a)^{\frac{n}{N^*}} q(x)^{\frac{1}{N^*}} dx \right)^{N^*} \right). \end{aligned}$$

By using Minkowski's inequalities for sums, we obtain

$$\begin{aligned} I_1 & \leq 2 \int_1^\infty t^{a-1-N^*} dt \times \\ & \left(\int_{G_t} |\nabla u| q(x)^{\frac{1}{N^*}} \left[\sum_{n=0}^\infty \frac{K(n, a) \ln^n t}{n!} (q(x) - a)^n \right]^{\frac{1}{N^*}} dx \right)^{N^*} \\ & = 2 C^{N^*}(N) \int_1^\infty t^{a-1-N^*} dt \left(\int_{G_t} |\nabla u| q(x)^{\frac{1}{N^*}} t^{\frac{(q(x)-a)\beta(a)}{N^*}} dx \right)^{N^*}, \end{aligned} \quad (9)$$

where $\beta(a) = K^{\frac{1}{n}}(n, a)$. Again by using Minkowski's inequality for inte-

grals we have

$$\begin{aligned}
 I_1 &\leq 2C^{N^*}(N) \times \\
 &\quad \left(\int_{E_1} |\nabla u| q(x)^{\frac{1}{N^*}} dx \left(\int_{|u(x)|}^{2|u(x)|} t^{(q(x)-a)\beta(a)+a-1-N^*} dt \right)^{\frac{1}{N^*}} \right)^{N^*} \\
 &\leq \frac{2q^+ C^{N^*}(N) \left(2^{\frac{(q^+-a)\beta(a)}{N^*} + \frac{a}{N^*} - 1} - 1 \right)}{(q^- - a)\beta(a) + a - N^*} \times \\
 &\quad \left(\int_{E_1} |\nabla u| |u|^{\frac{(q(x)-a)\beta(a)}{N^*} + \frac{a}{N^*} - 1} dx \right)^{N^*}. \tag{10}
 \end{aligned}$$

From (3) and (10), we get

$$\begin{aligned}
 1 &= I_1 + I_2 \\
 &\leq C(N, q^+, q^-, a) \left(\int_{E_1} |\nabla u| |u|^{\frac{(q(x)-a)\beta(a)}{N^*} + \frac{a}{N^*} - 1} dx \right)^{N^*} \\
 &\quad + 2^{q^+-1} \int_{\Omega} |u| dx. \tag{11}
 \end{aligned}$$

and for $a \rightarrow -\infty$,

$$\begin{aligned}
 \frac{(q(x)-a)\beta(a)}{N^*} + \frac{a}{N^*} - 1 &\rightarrow \frac{q(x)}{N^*} - 1 + \frac{q^+ - q^-}{N^*} \\
 C(N, q^+, q^-, a) &\rightarrow C(N, q^+, q^-)
 \end{aligned}$$

where $C(N, q^+, q^-)$ is finite. If we apply Fato's Lemma to (11), then we write

$$1 \leq C \left(\int_{E_1} |\nabla u| |u|^{\frac{q}{N^*} - 1 + \delta} dx \right)^{N^*} + 2^{q^+-1} \left(\int_{\Omega} |u|^{N^*} dx \right)^{1/N^*}, \tag{12}$$

where $C > 0$ only depends on N, q^+ and q^- .

If we apply Young inequality $db \leq \varepsilon d^p + c(\varepsilon) b^{p'}$; $d > 0, b > 0, \varepsilon > 0, p > 1$ to first term of inequality (12) and inequality (6) to second term for $w(x) = 1$,

we obtain

$$1 \leq C \left(C(\varepsilon) \int_{\Omega} |\nabla u|^{p(x)} dx + \varepsilon \int_{\Omega} |u|^{\left(\frac{q}{N^*} - 1 + \delta\right)p'(x)} dx \right)^{N^*} + 2^{q^+ - 1} \int_{\Omega} |\nabla u| dx, \quad (13)$$

where $p(x) : \frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $1 \leq p(x) < \infty$.

By assumption of theorem, we have

$$\left[\frac{q(x)}{N^*} - 1 + \delta \right] p'(x) \leq q(x)$$

and by using Young's inequality again to last integral in the inequality (13), then we write

$$1 \leq C \left[C(\varepsilon) \int_{\Omega} |\nabla u|^{p(x)} dx + \varepsilon \right]^{N^*} + C(\varepsilon) \int_{\Omega} |\nabla u|^{p(x)} dx + \varepsilon 2^{q^+ - 1}.$$

Hence, for sufficiently small $\varepsilon > 0$ we obtain

$$1 \leq C \left[\int_{\Omega} |\nabla u|^{p(x)} dx + \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{N^*}} \right]. \quad (14)$$

If $C \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{N^*}} < \frac{1}{2}$, then we have from (14)

$$1 \leq C \int_{\Omega} |\nabla u|^{p(x)} dx. \quad (15)$$

To contrary, if $C \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{N^*}} \geq \frac{1}{2}$, it follows that $2C \int_{\Omega} |\nabla u|^{p(x)} dx \geq 1$.

Therefore we have $2C \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)^{\frac{1}{N^*}} \leq (2C)^{N^*} \left(\int_{\Omega} |\nabla u|^{p(x)} dx \right)$ and from (14) we can obtain inequality (15) again.

Now we suppose that $\|u\|_{q(x), \Omega}$ is any number. Then, inequality (15) for function

$$z = \frac{u}{\|u\|_{q(x), \Omega}}$$

follows that $1 \leq C \int_{\Omega} \left(\frac{|\nabla u|}{\|u\|_{q(x), \Omega}} \right)^{p(x)} dx$. Hence,

$$\|u\|_{q(x), \Omega} \leq C(N, p, q) \|\nabla u\|_{p(x), \Omega}. \quad (16)$$

We suppose that $|\Omega|$ is any positive number. Then we consider the mapping $x := Ty = x_0 + ry$, where $r > 0$, $x_0 \in \Omega$ and x_0 is fixed. It is easy to see that range of Ω is Ω^* as $x \rightarrow y$. We assume that $\tilde{u}(y) = u(x_0 + ry)$, $\tilde{p}(y) = p(x_0 + ry)$ and $\tilde{q}(y) = q(x_0 + ry)$. We choose r such that $|\Omega^*| = 1$. i.e. $r = C_N |\Omega|^{\frac{1}{N}}$.

By the definition of norm, we obtain $\int_{\Omega^*} \left(\frac{|\tilde{u}(y)|}{\|\tilde{u}\|_{\tilde{q}(y), \Omega^*}} \right)^{\tilde{q}(y)} dy = 1$. If $|\Omega| \leq 1$, by using inverse mapping we obtain

$$\int_{\Omega} \left(\frac{|u(x)|}{\|\tilde{u}\|_{\tilde{q}(y), \Omega^*} |\Omega|^{1/q^+}} \right)^{q(x)} dx \leq 1.$$

Therefore,

$$\|u\|_{q(x), \Omega} \leq C_N |\Omega|^{\frac{1}{q^+}} \|\tilde{u}\|_{\tilde{q}(y), \Omega^*}. \quad (17)$$

Again, with respect to definition of the norm of gradient of function, we find

$$\int_{\Omega} \left(\frac{|\nabla u|}{\|\nabla u\|_{p(x), \Omega}} \right)^{p(x)} dx = 1.$$

Then, we have

$$|\Omega| \int_{\Omega^*} \left(\frac{|\nabla \tilde{u}|}{r \|\nabla u\|_{p(x), \Omega}} \right)^{\tilde{p}(y)} dy \leq 1.$$

and

$$\int_{\Omega^*} \left(\frac{|\nabla \tilde{u}|}{|\Omega|^{\frac{1}{N} - \frac{1}{p^+}} \|\nabla u\|_{p(x), \Omega}} \right)^{\tilde{p}(y)} dy \leq 1 \quad (18)$$

where $|\Omega| \leq 1$.

As a result of (18), we obtain

$$\|\nabla \tilde{u}\|_{\tilde{p}(y), \Omega^*} \leq |\Omega|^{\frac{1}{N} - \frac{1}{p^+}} \|\nabla u\|_{p(x), \Omega}. \quad (19)$$

From (16) , (17) and (19) for u , we have

$$\|u\|_{q(x),\Omega} \leq C(N, p, q) |\Omega|^{\frac{1}{N} - \frac{1}{p^-} + \frac{1}{q^+}} \|\nabla u\|_{p(x),\Omega},$$

where $|\Omega| \leq 1$.

As similarly,

$$\|u\|_{q(x),\Omega} \leq C(N, p, q) |\Omega|^{\frac{1}{N} - \frac{1}{p^+} + \frac{1}{q^-}} \|\nabla u\|_{p(x),\Omega}$$

where $|\Omega| > 1$. The theorem is proved. ■

Remark

Let D and E be any regular bounded domains in R^2 and $E \subset D$. Consider functions

$$p(x) = \begin{cases} 4/3 & \text{if } x \in E \\ 1 & \text{if } x \in D/E \end{cases}$$

and

$$q(x) = \begin{cases} 2 & \text{if } x \in E \\ 1 & \text{if } x \in D/E \end{cases}.$$

It is clear that conditions of Theorem 1.1 do not hold in this case. But Theorem 2.1 holds. ■

Theorem 2.2. Let $\Omega \in R^N$ be an open bounded set with Lipschitz boundary, $p(x) \in w-Lip$. and $1 < p^- \leq p^+ < N$. Then, for all measurable $q : \Omega \rightarrow [1, \infty)$ with $q(x) \leq p^*(x) - \delta$ ($p^* = \frac{Np(x)}{N-p(x)}$) and some $\delta > 0$, there holds $W^{1,p(\cdot)}(\Omega) \rightarrow L^{q(\cdot)}(\Omega)$. i.e. the embedding is compact.

Proof: There exist m such that $q \leq m \leq p^*$, where $0 < p(p-m) < m(p-p^*)$. Note that m is a bounded exponent. Let $u_n, u \in W^{1,p(\cdot)}(\Omega)$ with $u_n \rightarrow u$ (weak lim). We have to show that $u_n \rightarrow u$ in $L^{q(\cdot)}(\Omega)$ (strong lim). Theorem 2.1 implies chains of embeddings

$$W^{1,p(x)} \rightarrow W^{1,m(x)} \rightarrow W^{1,m^-} \rightarrow L^{(m^-)^*} \rightarrow L^{q^*} \rightarrow L^{q(x)}, \quad (m^-)^* \geq q^*$$

and from the generalized Hölder's inequality [1], we obtain

$$\|u_n - u\|_m \rightarrow 0$$

or

$$u_n \rightarrow u$$

in $L^{q(\cdot)}(\Omega)$. ■

Now, we consider of the classes

$$\|u\|_{W_{x_j}^{m,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \left\| \frac{\partial^m u}{\partial x_j^m} \right\|_{L^{p(x)}(\Omega)}, \quad (j = 1, \dots, n).$$

Theorem 2.3. Let $\Omega_{k|h|} \subset \subset \Omega$ and $kh < \text{dist}(\Omega_{k|h|}, \partial\Omega)$, if $u \in W_{x_j}^{k,p(x)}(\Omega)$ where $1 \leq p^- \leq p^+ < \infty$ and $p(x) \in w\text{-Lip.}$, then $\Delta_{x_j,h}^k u(x) \in L^{p(x)}(\Omega_{k|h|})$ and we have

$$\left\| \Delta_{x_j,h}^k u \right\|_{p(x), (\Omega_{k|h|})} \leq C_k \left\| \frac{\partial^k u}{\partial x_j^k} \right\|_{p(x), (\Omega)}.$$

Proof. For $k = 1$, $(x_1, \dots, x_n) \in \Omega_{|h|}$ and $\Delta_{x_1,h} u(x) = \int_0^h u'_{x_1}(x_1 + t, x_2, \dots, x_n) dt$, we can write

$$\begin{aligned} \left\| \Delta_{x_1,h} u \right\|_{p(x), (\Omega_{|h|})} &\leq \left| \int_0^h \|u'_{x_1}(x_1 + t, x_2, \dots, x_n)\|_{p(x), (\Omega_{|h|})} dt \right| \\ &\leq \left| \int_0^h \|u'_{x_1}\|_{p(x), (\Omega)} dt \right| \\ &= C \|u'_{x_1}\|_{p(x), (\Omega)}. \end{aligned}$$

Therefore, for any k we obtain

$$\begin{aligned} \left\| \Delta_{x_j,h}^k u \right\|_{p(x), (\Omega_{k|h|})} &= \left\| \Delta_{x_j,h} (\Delta_{x_j,h}^{k-1} u) \right\|_{p(x), (\Omega_{k|h|})} \\ &\leq C_1 \left\| \Delta_{x_j,h}^{k-1} u'_{x_j} \right\|_{p(x), (\Omega_{(k-1)|h|})} \\ &\cdot \\ &\cdot \\ &\cdot \\ &\leq C_k \left\| u_{x_j}^{(k)} \right\|_{p(x), (\Omega)}. \end{aligned}$$

Theorem 2.3. is proved. ■

ACKNOWLEDGEMENTS. The authors would like to thank S.G.Samko for a useful discussion.

References

- [1] L.Diening, Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. *Math. Nachr.*, October 8, 2003, 112-115.
- [2] D.E. Edmunds, J. Rákosník, Sobolev embedding with variable exponent. *Studia Math.*, **143** no.3 2000, 267-293.
- [3] D.E. Edmunds, J. Rákosník, Sobolev embedding with variable exponent II. *Math.Nachr.*, 246-247, 2002, 53-67.
- [4] X.L. Fan, J.S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J.Math. Anal. Appl.*, **262**, 2001, 749-760.
- [5] O. Kovác, J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. *Czechoslovak Math.J.*, **116** 1991, 592-618.
- [6] S.G. Samko, Convolution and potential type operators in $L^{p(x)}(R^n)$. *Integr. Transform. and Special Funct.* v.7, **3-4** 1998, 261-284.
- [7] E. Sawyer, R.L. Wheeden, Weighted inequalities for Fractional Integrals on Euclidian and Homogeneous Spaces. *American Journal of Math.*, **114**, 1992, 813-874.

Received: October 14, 2005