On The Sobolev-type Inequality for Lebesgue Spaces with a Variable Exponent

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Abstract

This paper gives Sobolev-type inequality for the generalized Lebesgue space $L^{p(x)}(\Omega)$ and corresponding Sobolev space $W^{1,p(x)}(\Omega)$, consisting of all $f \in L^{p(x)}(\Omega)$ with first-order distributional derivatives in $L^{p(x)}(\Omega)$, where $\Omega$ is a bounded domain $R^N (N \geq 2)$, and $p(x)$ and $q(x)$ measurable function with $1 \leq q^- \leq q(x) \leq q^+ < \infty$, $1 \leq p^+ < N$ and

$$\sup_{x \in \Omega} q(x) \left[ \frac{1}{p(x)} - \frac{1}{N} \right] \leq 1 - \frac{(q^+ - q^-)(N - 1)}{N}.$$ 

In other words, our aim in this study is obtained pre-limit condition for measurable variable exponent functions in Sobolev-type inequality.

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1. Introduction

Let $\Omega \subset R^N$ be an open set and $p : \Omega \to [1, \infty)$ be measurable function. We define the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ consisting of all measurable functions $f : \Omega \to R$ such that

$$\int_{\Omega} |\lambda f(x)|^{p(x)} \, dx < \infty$$

for some $\lambda > 0$. We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$\| f \|_{p(x), \Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}$$

and the $L^{p(x)}(\Omega)$ becomes a Banach spaces [5].

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The corresponding Sobolev space \( W^{1,p(x)}(\Omega) \) is the class of all functions \( f \in L^{p(x)}(\Omega) \) such that all generalized \( D_i f, i = 1, 2, \ldots, n \), belong to \( L^{p(x)}(\Omega) \). Endowed with the norm
\[
\| f \|_{1,p(x)} = \| f \|_{p(x)} + \| \nabla f \|_{p(x)}
\]
it forms a Banach space.

O.Kováč and J. Rákosník \[5\] obtained Sobolev-type inequality
\[
\| u \|_{q(x),\Omega} \leq c \| \nabla u \|_{p(x),\Omega} ; \ u \in C^\infty_0(\Omega).
\]
(1)
for continuous \( p(x) \) function, where a positive constant \( c \) is independent of \( u \) and
\[
1 \leq q(x) \leq \frac{Np(x)}{N - p(x)} - \varepsilon = p^*(x) - \varepsilon , \quad 0 < \varepsilon < \frac{1}{N - 1},
\]
and they showed that, in general, the Sobolev space \( W^{1,p(x)}(\Omega) \) is not embedding in \( L^{p^*(x)}(\Omega) \). Obviously, \( q(x) = \frac{Np(x)}{N - p(x)} \) equality is not valid for above embedding when \( p(x) \) is discontinuous. Naturally, \( q(x) \) must be sufficiently less than \( \frac{Np(x)}{N - p(x)} \).

Later D.E.Edmunds and J.Rákosník \[2,3\] proved \( W^{1,p(x)}(\Omega) \rightarrow L^{p^*(x)}(\Omega) \) for bounded Lipschitz domain, where \( p(x) \in C^{0,1}(\Omega) \) and \( 1 \leq p(x) \leq q < N \).

X.L.Fan, J.Shen and D.Zhao \[4\] investigated Sobolev-type inequality for domain with cone property.

Recently, L.Diening \[1\] obtained above embedding for unbounded domain assuming that \( p(x) \) is constant at infinity and satisfies \((\text{w-Lip.})\)
\[
| p(x) - p(y) | \leq \frac{C}{-\log |x - y|} ; \ x, y \in R^N, \ |x - y| \leq \frac{1}{2}.
\]
That is, under mentioned conditions, Sobolev-type embedding is proved in case of limit by L.Diening \[1\]. S.Samko \[6\] also found the following result for measurable variable exponent functions in generalized Lebesgue space:

**Theorem 1.1** \[6\] Let \( k(x) \in L^Q(B(0,2D)) \) where \( Q \geq 1 \) and \( D = \text{diam} \Omega \).

The convolution operator
\[
K_{\Omega} f = \int_{\Omega} k(x - y)f(y)dy
\]
is bounded from \( L^{p(x)}(\Omega) \) into \( L^{r(x)}(\Omega) \), \( r(x) \geq 1 \), if
\[
\frac{1}{Q} \leq 1 - \frac{1}{p^-} + \frac{1}{p^+}, \quad \frac{1}{r(x)} \geq \frac{1}{Q} + \frac{1}{p^+} - 1.
\]
In this article, our aim is to show Sobolev-type inequality (1) for Lebesgue spaces with a variable exponent where \( \Omega \) is assumed to be a bounded domain in \( \mathbb{R}^N \) and \( p(x) \) is a measurable real-valued function (may be discontinuous) defined on \( \Omega \). That is, we determine “Pre-limit” condition for measurable variable exponent functions in Sobolev-type inequality.

**Notation:**

In this study, \(|\Omega|\) is the Lebesgue measure of a set \( \Omega \) in \( \mathbb{R}^N \) and \( g^+ = \sup_{x \in \Omega} g(x) \), \( g^- = \inf_{x \in \Omega} g(x) \), \( \delta = \frac{q^- - q^+}{N^*} \) and \( N^* = \frac{N}{N-1} \).

### 2. A Sobolev-type Inequality

**Theorem 2.1:** Let \( \Omega \subset \mathbb{R}^N \), \( (N \geq 2) \) be a bounded domain. Let \( 1 \leq q^- \leq q(x) \leq q^+ < \infty \) and \( 1 \leq p^+ < N \) be such that

\[
\sup_{x \in \Omega} q(x) \left[ \frac{1}{p(x)} - \frac{1}{N} \right] \leq 1 - \delta
\]

for \( x \in \Omega \). Then, there exist a constant \( C(N, p, q) > 0 \) such that

\[
\| u \|_{q(x), \Omega} \leq C(N, p, q, \Omega) \| \nabla u \|_{p(x), \Omega} ; \ u \in C_0^\infty(\Omega), \quad (2)
\]

where

\[
C(N, p, q, \Omega) = \begin{cases} 
C(N, p, q) \ |\Omega|^\frac{1}{N} - \frac{1}{p^+} + \frac{1}{q^+} & \text{if } |\Omega| \leq 1 \\
C(N, p, q) \ |\Omega|^\frac{1}{N} - \frac{1}{p^+} + \frac{1}{q^+} & \text{if } |\Omega| > 1 
\end{cases}
\]

**Proof:** We suppose that \( \| u \|_{q(x), \Omega} = 1 \) and \( |\Omega| = 1 \). We have

\[
1 = \int_{\Omega} |u(x)|^{q(x)} \, dx = \int_{E_2} |u(x)|^{q(x)} \, dx + \int_{E_0 \cap E_2} |u(x)|^{q(x)} \, dx = I_1 + I_2,
\]

where \( E_t = \{ x \in \Omega : |u(x)| > t \} \). Then it is easy to see that

\[
I_2 = \int_{E_0 \cap E_2} |u(x)|^{q(x)} \, dx \leq 2^{q^+ - 1} \int_{\Omega} |u(x)| \, dx
\]

(3)

and since

\[
|u(x)|^{q(x)} - 1 \geq \frac{1}{2} |u(x)| \cdot q(x)
\]
for \( x \in E_2 \), we can write

\[
I_1 = \int_{E_2} |u(x)|^{q(x)} \, dx \leq 2 \int_{E_2} dx \left( \int_1^{\frac{|u(x)|}{\max_q}} q(x)^{q(x)-1} \, dt \right).
\]

Also by Fubini’s Theorem, we obtain

\[
I_1 \leq 2 \int_1^\infty dt \left( \int_{E_t} t^{q(x)-1} q(x) \, dx \right). \tag{4}
\]

By using Taylor’s expansions of \( e^{x} \), from (4) we get

\[
I_1 \leq 2 \int_1^\infty t^{a-1} dt \left( \int_{E_t} \left( \sum_{n=0}^{\infty} \frac{(q(x) - a)^n q(x) \ln^n t}{n!} \right) \, dx \right) \tag{5}
\]

\[
= 2 \int_1^\infty t^{a-1} dt \left( \sum_{n=0}^{\infty} \frac{\ln^n t}{n!} \int_{E_t} (q(x) - a)^n \, dx \right),
\]

where \( a \in \mathbb{R}^1, \ a < q^- \). On the other hand, by means of Gagliardo’s inequality [7], for \( f \in C_0^\infty(\Omega), w \in L^1_{loc}(\Omega) \)

\[
\left( \int_{\Omega} |f|^{N^*} w(x) \, dx \right)^{\frac{1}{N^*}} \leq C(N) \int_{\Omega} |\nabla f|^{\frac{1}{N^*}} w(x)^{\frac{1}{N^*}} \, dx \tag{6}
\]

we obtain

\[
\left( \int_{\Omega} q(x) (q(x) - a)^{n} |f|^{N^*} \, dx \right)^{\frac{1}{N^*}} \leq K^{N^*}(n, a) \left( \int_{\Omega} |\nabla f| \, (q(x) - a) \, q(x)^{\frac{n}{N^*}} \, dx \right) \tag{7}
\]

where

\[
K(n, a) = C(N) \left( \frac{q^+ - a}{q^- - a} \right)^n, \ n = 0, 1, 2, \ldots
\]
If we take \( Z(x) = \min \left( \frac{|u(x)| - \frac{t}{2}}{t}, 1 \right) \), \( t > 0 \) in place of the test function \( f \) in (7) then, we find
\[
\int_{E_t} q(x) (q(x) - a)^n dx \\
\leq \frac{K(n, a)}{t^{N^*}} \left( \int_{G_t} |\nabla u| (q(x) - a)^{\frac{n}{N^*}} q(x)^{\frac{1}{N^*}} dx \right)^{N^*}
\]
(8)

where
\[
G_t = \left\{ x \in \Omega : \frac{t}{2} < |u(x)| \leq t \right\}.
\]
Hence, from (5) and (8) we have
\[
I_1 \leq 2 \int_1^\infty t^{a-1} dt \times \\
\left( \sum_{n=0}^\infty \frac{K(n, a) \ln^n t}{n! t^{N^*}} \left( \int_{G_t} |\nabla u| (q(x) - a)^{\frac{n}{N^*}} q(x)^{\frac{1}{N^*}} dx \right)^{N^*} \right).
\]
By using Minkowski’s inequalities for sums, we obtain
\[
I_1 \leq 2 \int_1^\infty t^{a-1-N^*} dt \times \\
\left( \int_{G_t} |\nabla u| q(x)^{\frac{1}{N^*}} \left[ \sum_{n=0}^\infty \frac{K(n, a) \ln^n t}{n!} (q(x) - a)^n \right]^{\frac{1}{N^*}} dx \right)^{N^*}
\]
\[
= 2 C^{N^*}(N) \int_1^\infty t^{a-1-N^*} dt \left( \int_{G_t} |\nabla u| q(x)^{\frac{1}{N^*}} t^{\frac{(q(x) - a) \beta(a)}{N^*}} dx \right)^{N^*},
\]
(9)
where \( \beta(a) = K^{\frac{1}{n}}(n, a) \). Again by using Minkowski’s inequality for inte-
From \((3)\) and \((10)\), we get

\[
1 = I_1 + I_2 
\leq C(N, q^+, q^-, a) \left( \int_{E_1} |\nabla u| \sqrt[\frac{q^+(x) - a}{N^*}]^{\beta(a)} + \frac{\alpha - 1}{N^*} dx \right)^{N^*} 
+ 2^{q^+ - 1} \int_{E_1} |u| \sqrt[\frac{q^+(x) - a}{N^*}]^{\beta(a)} + \frac{\alpha - 1}{N^*} dx \tag{11}
\]

and for \(a \to -\infty\),

\[
\frac{q^+(x) - a}{N^*} + \frac{a}{N^*} - 1 \to \frac{q^-(x)}{N^*} - 1 + \frac{q^+ - q^-}{N^*} 
\]

where \(C(N, q^+, q^-)\) is finite. If we apply Fato's Lemma to \((11)\), then we write

\[
1 \leq C \left( \int_{E_1} |\nabla u| |u|^{\frac{q^+ - 1}{N^*}} dx \right)^{N^*} 
+ 2^{q^+ - 1} \left( \int_{\Omega} |u|^{N^*} dx \right)^{1/N^*} \tag{12}
\]

where \(C > 0\) only depends on \(N, q^+\) and \(q^-\).

If we apply Young inequality \(db \leq \varepsilon d^p + c(\varepsilon) b^p\); \(d > 0, b > 0, \varepsilon > 0, p > 1\) to first term of inequality \((12)\) and inequality \((6)\) to second term for \(w(x) = 1,\)
we obtain
\[ 1 \leq C \left( C(\varepsilon) \int_\Omega |\nabla u|^{p(x)} \, dx + \varepsilon \int_\Omega |u|^\left(\frac{q(x)}{N^*} - 1 + \delta\right)^{p'(x)} \right)^{N^*} + 2^{q^+ - 1} \int_\Omega |\nabla u| \, dx, \]  
(13)
where \( p(x) : \frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \ 1 \leq p(x) < \infty. \)

By assumption of theorem, we have
\[ \left[ \frac{q(x)}{N^*} - 1 + \delta \right] p'(x) \leq q(x) \]
and by using Young's inequality again to last integral in the inequality (13), then we write
\[ 1 \leq C \left[ C(\varepsilon) \int_\Omega |\nabla u|^{p(x)} \, dx + \varepsilon \int_\Omega |\nabla u|^{p(x)} \, dx + \varepsilon 2^{q^+ - 1}. \right. \]

Hence, for sufficiently small \( \varepsilon > 0 \) we obtain
\[ 1 \leq C \left[ \int_\Omega |\nabla u|^{p(x)} \, dx + \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\frac{1}{N^*}} \right]. \]  \( 14 \)

If \( C \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\frac{1}{N^*}} < \frac{1}{2}, \) then we have from (14)
\[ 1 \leq C \int_\Omega |\nabla u|^{p(x)} \, dx. \]  \( 15 \)

To contrary, if \( C \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\frac{1}{N^*}} \geq \frac{1}{2}, \) it follows that \( 2C \int_\Omega |\nabla u|^{p(x)} \, dx \geq 1. \)

Therefore we have \( 2C \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right)^{\frac{1}{N^*}} \leq \left( 2C \right)^{N^*} \left( \int_\Omega |\nabla u|^{p(x)} \, dx \right) \) and from (14) we can obtain inequality (15) again.

Now we suppose that \( \| u \|_{q(x), \Omega} \) is any number. Then, inequality (15) for function
\[ z = \frac{u}{\| u \|_{q(x), \Omega}} \]
follows that \( 1 \leq C \int_{\Omega} \left( \frac{\| \nabla u \|}{\| u \|_{q(x), \Omega}} \right)^{p(x)} \, dx \). Hence,

\[
\| u \|_{q(x), \Omega} \leq C \left( N, p, q \right) \| \nabla u \|_{p(x), \Omega}.
\]  

(16)

We suppose that \( |\Omega| \) is any positive number. Then we consider the mapping \( x := Ty = x_0 + ry \), where \( r > 0 \), \( x_0 \in \Omega \) and \( x_0 \) is fixed. It is easy to see that range of \( \Omega \) is \( \Omega^* \) as \( x \to y \). We assume that \( \tilde{u}(y) = u(x_0 + ry) \), \( \tilde{p}(y) = p(x_0 + ry) \) and \( \tilde{q}(y) = q(x_0 + ry) \). We choose \( r \) such that \( |\Omega^*| = 1 \).

i.e. \( r = C_N |\Omega|^{1 \over N} \).

By the definition of norm, we obtain

\[
\int_{\Omega^*} \left( \frac{|u(y)|}{\| u \|_{\tilde{q}(y), \Omega^*} |\Omega|^{1/q^*}} \right)^{\tilde{q}(y)} \, dy = 1. \text{ If } |\Omega| \leq 1,
\]

by using inverse mapping we obtain

\[
\int_{\Omega} \left( \frac{|u(x)|}{\| u \|_{\tilde{q}(y), \Omega^*} |\Omega|^{1/q^*}} \right)^{q(x)} \, dx \leq 1.
\]

Therefore,

\[
\| u \|_{q(x), \Omega} \leq C_N |\Omega|^{1 \over q^*} \| \tilde{u} \|_{\tilde{q}(y), \Omega^*}.
\]  

(17)

Again, with respect to definition of the norm of gradient of function, we find

\[
\int_{\Omega} \left( \frac{\| \nabla u \|}{\| \nabla u \|_{p(x), \Omega}} \right)^{p(x)} \, dx = 1.
\]

Then, we have

\[
|\Omega| \int_{\Omega^*} \left( \frac{\| \nabla \tilde{u} \|}{r \| \nabla u \|_{p(x), \Omega}} \right)^{\tilde{p}(y)} \, dy \leq 1.
\]

and

\[
\int_{\Omega^*} \left( \frac{\| \nabla \tilde{u} \|}{|\Omega|^{1 \over N - p^*} \| \nabla u \|_{p(x), \Omega}} \right)^{\tilde{p}(y)} \, dy \leq 1
\]  

(18)

where \( |\Omega| \leq 1 \).

As a result of (18), we obtain

\[
\| \nabla \tilde{u} \|_{\tilde{p}(y), \Omega^*} \leq |\Omega|^{1 \over N - p^*} \| \nabla u \|_{p(x), \Omega}.
\]  

(19)
From (16), (17) and (19) for $u$, we have

$$\|u\|_{q(x),\Omega} \leq C(N, p, q) |\Omega|^{\frac{1}{N-p} + \frac{1}{q}} \| \nabla u \|_{p(x),\Omega},$$

where $|\Omega| \leq 1$.

As similarly,

$$\|u\|_{q(x),\Omega} \leq C(N, p, q) |\Omega|^{\frac{1}{N-p} + \frac{1}{q}} \| \nabla u \|_{p(x),\Omega}$$

where $|\Omega| > 1$. The theorem is proved. ■

**Remark**

Let $D$ and $E$ be any regular bounded domains in $\mathbb{R}^2$ and $E \subset D$. Consider functions

$$p(x) = \begin{cases} 
4/3 & \text{if } x \in E \\
1 & \text{if } x \in D/E 
\end{cases}$$

and

$$q(x) = \begin{cases} 
2 & \text{if } x \in E \\
1 & \text{if } x \in D/E 
\end{cases}.$$

It is clear that conditions of Theorem 1.1 do not hold in this case. But Theorem 2.1 holds. ■

**Theorem 2.2.** Let $\Omega \in \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $p(x) \in w-Lip.$ and $1 < p^- \leq p^+ < N$. Then, for all measurable $q : \Omega \to [1, \infty)$ with $q(x) \leq p^*(x) - \delta$ ($p^* = \frac{Np(x)}{N-p(x)}$) and some $\delta > 0$, there holds $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. i.e. the embedding is compact.

**Proof:** There exist $m$ such that $q \leq m \leq p^*$, where $0 < p(p - m) < m(p - p^*)$. Note that $m$ is a bounded exponent. Let $u_n, u \in W^{1,p(\cdot)}(\Omega)$ with $u_n \rightharpoonup u$ (weak lim). We have to show that $u_n \to u$ in $L^{q(\cdot)}(\Omega)$ (strong lim). Theorem 2.1 implies chains of embeddings

$$W^{1,p(x)} \hookrightarrow W^{1,m(x)} \hookrightarrow W^{1,m^-} \hookrightarrow L^{(m^-)^*} \hookrightarrow L^{q^*} \hookrightarrow L^{q(x)}, \quad (m^-)^* \geq q^*$$

and from the generalized Hölder’s inequality [1], we obtain

$$\|u_n - u\|_m \to 0$$

or

$$u_n \to u$$
in \( L^q(\Omega) \).

Now, we consider of the classes

\[
\| u \|_{W^{m,p(x)}(\Omega)} = \| u \|_{L^p(\Omega)} + \left\| \frac{\partial^m u}{\partial x_j^m} \right\|_{L^p(\Omega)}, \quad (j = 1, \ldots, n).
\]

**Theorem 2.3.** Let \( \Omega_{k|h} \subset \subset \Omega \) and \( kh < \text{dist}(\Omega_{k|h}, \partial \Omega) \), if \( u \in W^{k,p(x)}(\Omega) \) where \( 1 \leq p^- \leq p^+ < \infty \) and \( p(x) \in w-Lip. \), then \( \Delta^{k}_{x_j,h} u(x) \in L^p(\Omega_{k|h}) \) and we have

\[
\left\| \Delta^{k}_{x_j,h} u \right\|_{p(x), (\Omega_{k|h})} \leq C_k \left\| \frac{\partial^k u}{\partial x_j^k} \right\|_{p(x), (\Omega)}.
\]

**Proof.** For \( k = 1 \), \( (x_1, \ldots, x_n) \in \Omega_{[h]} \) and \( \Delta^{1}_{x_1,h} u(x) = \int_0^h u^{'}_{x_1}(x_1 + t, x_2, \ldots, x_n)dt \), we can write

\[
\left\| \Delta^{1}_{x_1,h} u \right\|_{p(x), (\Omega_{[h]})} \leq \int_0^h \left\| u^{'}_{x_1}(x_1 + t, x_2, \ldots, x_n) \right\|_{p(x), (\Omega_{[h]})} dt
\]

\[
\leq C \left\| u^{'}_{x_1} \right\|_{p(x), (\Omega)}.
\]

Therefore, for any \( k \) we obtain

\[
\left\| \Delta^{k}_{x_j,h} u \right\|_{p(x), (\Omega_{k|h})} = \left\| \Delta^{k}_{x_j,h} (\Delta^{k-1}_{x_j,h} u) \right\|_{p(x), (\Omega_{k|h})}
\]

\[
\leq C_1 \left\| \Delta^{k-1}_{x_j,h} u^{'}_{x_j} \right\|_{p(x), (\Omega_{(k-1)|h})}
\]

\[
\vdots
\]

\[
\leq C_k \left\| u^{(k)}_{x_j} \right\|_{p(x), (\Omega)}.
\]

Theorem 2.3. is proved.■

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