Intuitionistic $Q$-Fuzzy Subalgebras of BCK/BCI-Algebras

Eun Hwan Roh
Department of Mathematics Education
Chinju National University of Education
Jinju 660-756, Korea
ehroh@cue.ac.kr

Kyung Ho Kim and Jong Geol Lee
Department of Mathematics, Chungju National University
Chungju 380-702, Korea
ghkim@chungju.ac.kr

Abstract. The intuitionistic $Q$-fuzzification of the concept of subalgebras in $BCK/BCI$-algebra is considered, and some related properties are investigated.

Keywords: BCK/BCI-algebra, $Q$-fuzzy set, Intuitionistic $Q$-fuzzy set, intuitionistic $Q$-fuzzy subalgebra, preimage

Mathematics Subject Classification: 06F35, 03G25, 03E72

1. Introduction

The notion of BCK-algebras was proposed by Imai and Iséki in 1966. In the same year, Iséki [4] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. After the introduction of the concept of fuzzy sets by Zadeh [6], several researches were conducted on the generalization of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. In this paper, using the Atanassov’s idea, we establish the intuitionistic $Q$-fuzzification of the concept of subalgebras in BCK/BCI-algebras, and investigate some of their properties.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Recall that a $BCI$-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(I) $((x*y)*(x*z))*(z*y) = 0$,
(II) $(x*(x*y))*y = 0$,
(III) $x*x = 0$, and
(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$
for every \( x, y, z \in X \). A BCI-algebra \( X \) satisfying the condition

(V) \( 0 \ast x = 0 \) for all \( x \in X \)

is called a BCK-algebra. In any BCK/BCI-algebra \( X \) one can define a partial order “\( \leq \)” by putting \( x \leq y \) if and only if \( x \ast y = 0 \).

A BCK/BCI-algebra \( X \) has the following properties:

\[
\begin{align*}
(2.1) & \ x \ast 0 = x, \\
(2.2) & \ (x \ast y) \ast z = (x \ast z) \ast y, \\
(2.3) & \ x \leq y \text{ implies that } x \ast z \leq y \ast z \text{ and } z \ast y \leq z \ast x, \\
(2.4) & \ (x \ast z) \ast (y \ast z) \leq x \ast y
\end{align*}
\]

for all \( x, y, z \in X \).

A non-empty subset \( S \) of a BCK/BCI-algebra \( X \) is called a subalgebra of \( X \) if \( x \ast y \in S \) whenever \( x, y \in S \). A mapping \( f : X \to Y \) of BCK/BCI-algebras is called a homomorphism if \( f(x \ast y) = f(x) \ast f(y) \) for all \( x, y \in X \). Let \( X \) be a BCK/BCI-algebra. A fuzzy set \( f \) in \( X \), i.e., a mapping \( f : X \to [0, 1] \), is called a fuzzy subalgebra of \( X \) if \( f(x \ast y) \geq f(x) \land f(y) \) for all \( x, y \in X \). Note that if \( f \) is a fuzzy subalgebra of a BCK/BCI-algebra \( X \), then \( f(0) \geq f(x) \) for all \( x \in X \).

3. Intuitionistic \( Q \)-fuzzy subalgebras

In what follows, let \( Q \) and \( X \) denote a set and a BCK/BCI-algebra, respectively, unless otherwise specified. A mapping \( H : X \times Q \to [0, 1] \) is called a \( Q \)-fuzzy set in \( X \).

A \( Q \)-fuzzy set \( H : X \times Q \to [0, 1] \) is called a fuzzy subalgebra of \( X \) over \( Q \) (briefly, \( Q \)-fuzzy subalgebra of \( X \)) if \( H(x \ast y, q) \geq H(x, q) \lor H(y, q) \) for all \( x, y \in X \) and \( q \in Q \).

**Definition 3.1.** Let \( Q \) and \( X \) denote a set and a BCK/BCI-algebra, respectively. An intuitionistic \( Q \)-fuzzy set (IQFS for short) \( A \) is an object having the form

\[
A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\}
\]

where the functions \( \mu_A : X \times Q \to [0, 1] \) and \( \gamma_A : X \times Q \to [0, 1] \) denote the degree of membership (namely \( \mu_A(x, q) \)) and the degree of nonmembership (namely \( \gamma_A(x, q) \)) of each element \((x, q) \in X \times Q\) to the set \( A \), respectively, and \( 0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1 \) for all \( x \in X \) and \( q \in Q \).

For the sake of simplicity, we shall use the symbol \( A = (\mu_A, \gamma_A) \) for the IQFS \( A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\} \).

**Definition 3.2.** An IQFS \( A = (\mu_A, \gamma_A) \) in \( X \) is called an intuitionistic \( Q \)-fuzzy subalgebra of \( X \) if

\[ (IQF1) \ \mu_A(x \ast y, q) \geq \mu_A(x, q) \land \mu_A(y, q) \text{ and } \gamma_A(x \ast y, q) \leq \gamma_A(x, q) \lor \gamma_A(y, q) \]

for all \( x, y \in X \) and \( q \in Q \).
Example 3.3. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Define an IQFS $= (\mu_A, \gamma_A)$ in $X$ as follows: for every $q \in Q$,

$\mu_A(0, q) = \mu_A(b, q) = 0.6, \mu_A(a, q) = \mu_A(c, q) = 0.2$

and

$\gamma_A(0, q) = \gamma_A(b, q) = 0.3, \gamma_A(a, q) = \gamma_A(c, q) = 0.7$

It is easy to verify that $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subalgebra of $X$.

Example 3.4. Consider a BCI-algebra $X = \{0, x\}$ with Cayley table as follows (Iséki [2]):

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td>x</td>
<td>x</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $Q = \{1, 2\}$ and let $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy set in $X$ defined by

$\mu_A(0, 1) = \mu_A(0, 2) = 1, \mu_A(x, 1) = 0.8, \mu_A(x, 2) = 0.5$

and

$\gamma_A(0, 1) = \gamma_A(0, 2) = 0, \gamma_A(x, 1) = 0.1, \gamma_A(x, 2) = 0.2$

It is easy to verify that $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subalgebra of $X$.

Proposition 3.5. If $A = (\mu_A, \gamma_A)$ in $X$ is an intuitionistic fuzzy $Q$-subalgebra of $X$, then $\mu_A(0, q) \geq \mu_A(x, q)$ and $\gamma_A(0, q) \leq \gamma_A(x, q)$ for all $x \in X$ and $q \in Q$.

Proof. Let $x \in X$ and $q \in Q$. Then $\mu_A(0, q) = \mu_A(x \ast x, q) \geq \mu_A(x, q)$ and $\gamma_A(0, q) = \gamma_A(x \ast x, q) \leq \gamma_A(x, q)$.

Proposition 3.6. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy subalgebra of $X$. Define an intuitionistic $Q$-fuzzy set $B = (\mu_B, \gamma_B)$ in $X$ by

$\mu_B(x, q) = \frac{\mu_A(x, q)}{\mu_A(0, q)}, \gamma_B(x, q) = \frac{\gamma_A(x, q)}{\gamma_A(0, q)}$

for all $x \in X$ and $q \in Q$. Then $B = (\mu_B, \gamma_B)$ is an intuitionistic $Q$-fuzzy subalgebra of $X$. 

Thus
\[ \mu_B(x \ast y, q) = \frac{\mu_A(x \ast y, q)}{\mu_A(0, q)} \geq \frac{1}{\mu_A(0, q)} \{ \mu_A(x, q) \land \mu_A(y, q) \} \]
\[ = \{ \mu_A(x, q) \land \mu_A(y, q) \} = \mu_B(x, q) \land \mu_B(y, q) \]
and
\[ \gamma_B(x \ast y, q) = \frac{\gamma_A(x \ast y, q)}{\gamma_A(0, q)} \leq \frac{1}{\gamma_A(0, q)} \{ \gamma_A(x, q) \lor \gamma_A(y, q) \} \]
\[ = \{ \gamma_A(x, q) \lor \gamma_A(y, q) \} = \gamma_B(x, q) \lor \gamma_B(y, q). \]
Hence \( B = (\mu_B, \gamma_B) \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( X \). \hfill \Box

Let \( A = (\mu_A, \gamma_A) \) be an IQFS in a set \( X \) and let \( \alpha, \beta \in [0, 1] \) be such that \( \alpha + \beta \leq 1 \). Then we define the set
\[ X_A^{(\alpha, \beta)} = \{ x \in X \mid \mu_A(x, q) \geq \alpha, \gamma_A(x, q) \leq \beta, q \in Q \}. \]

**Theorem 3.7.** Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic \( Q \)-fuzzy subalgebra of \( X \). Then \( X_A^{(\alpha, \beta)} \) is a subalgebra of \( X \) for every \( (\alpha, \beta) \in \text{Im}(\mu_A) \times \text{Im}(\gamma_A) \) with \( \alpha + \beta \leq 1 \).

**Proof.** Let \( x, y \in X_A^{(\alpha, \beta)} \) and \( q \in Q \). Then \( \mu_A(x, q) \geq \alpha, \gamma_A(x, q) \leq \beta, \mu_A(y, q) \geq \alpha, \gamma_A(y, q) \leq \beta \) which imply that
\[ \mu_A(x \ast y, q) \geq \mu_A(x, q) \land \mu_A(y, q) \geq \alpha \]
\[ \gamma_A(x \ast y, q) \leq \gamma_A(x, q) \lor \gamma_A(y, q) \leq \beta. \]
Thus \( xy \in X_A^{(\alpha, \beta)} \). Therefore \( X_A^{(\alpha, \beta)} \) is a subalgebra of \( X \). \hfill \Box

For the convenience of notation, we denote
\[ (\ldots ((x \ast y_1) \ast y_2) \ast \ldots) \ast y_n = x \ast \prod_{i=1}^{n} y_i, \text{ and} \]
\[ y_n \ast (y_{n-1} \ast (\cdots \ast (y_1 \ast x) \cdots)) = x \ast \prod_{i=1}^{n} y_i. \]

**Proposition 3.8.** Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic \( Q \)-fuzzy subalgebra of a BCK-algebra \( X \) and let \( x_1, x_2, \ldots, x_n \) be arbitrary elements of \( X \). If there exists \( k \in \{1, 2, \ldots, n\} \) such that \( x_k = x_1 \), then \( \mu_A(x_1 \ast \prod_{i=2}^{n} x_i, q) \geq \mu_A(x, q) \) and \( \gamma_A(x_1 \ast \prod_{i=2}^{n} x_i, q) \leq \gamma_A(x, q) \) for all \( x \in X \) and \( q \in Q \).

**Proof.** Let \( k \) be a fixed number in \( \{1, 2, \ldots, n\} \) such that \( x_k = x_1 \). Using (III), (V) and (2.2), one can deduce
\[ \mu_A(x_1 \ast \prod_{i=2}^{n} x_i, q) = \mu_A(0, q) \text{ and } \gamma_A(x_1 \ast \prod_{i=2}^{n} x_i, q) = \gamma_A(0, q). \]
It follows from Proposition 3.5 that
\[ \mu_{\Lambda}(x_1 \ast \prod_{i=2}^{n} x_i, q) \geq \mu_{\Lambda}(x, q) \] and \[ \gamma_{\Lambda}(x_1 \ast \prod_{i=2}^{n} x_i, q) \leq \gamma_{\Lambda}(x, q) \]
for all \( x \in X \) and \( q \in Q \).

**Proposition 3.9.** Let \( A = (\mu_{\Lambda}, \gamma_{\Lambda}) \) be an intuitionistic Q-fuzzy subalgebra of a BCK-algebra \( X \). Then

(i) \( \mu_{\Lambda}(x_1 \ast \prod_{i=1}^{2k} x_i, q) \geq \mu_{\Lambda}(x, q) \) and \( \gamma_{\Lambda}(x_1 \ast \prod_{i=1}^{2k} x_i, q) \leq \gamma_{\Lambda}(x, q) \) for all \( x \in X, q \in Q \) and for \( k = 1, 2, \cdots \).

(ii) \( \mu_{\Lambda}(x_1 \ast \prod_{i=1}^{n} x_i, q) \geq \mu_{\Lambda}(x, q) \) and \( \gamma_{\Lambda}(x_1 \ast \prod_{i=1}^{n} x_i, q) \leq \gamma_{\Lambda}(x, q) \) for all \( x \in X, q \in Q \) and for \( n = 1, 2, \cdots \).

**Proof.** It follows immediately from (III), (2.1) and Proposition 3.5.

**Theorem 3.10.** If \( \{A_i : i \in A\} \) is an arbitrary family of intuitionistic Q-fuzzy subalgebras of \( X \), then \( \bigcap A_i \) is an intuitionistic Q-fuzzy subalgebra of \( X \) where \( \bigcap A_i = \{(x, \wedge_{\Lambda_i}(x, q), \vee_{\Lambda_i}(x, q)) : x \in X, q \in Q\} \).

**Proof.** Let \( x, y \in X, q \in Q \). Then
\[ \wedge_{\Lambda_i}(x \ast y, q) \geq \wedge(\mu_{\Lambda_i}(x, q) \wedge \mu_{\Lambda_i}(y, q)) = (\wedge_{\Lambda_i}(x, q)) \wedge (\wedge_{\Lambda_i}(y, q)) \] and
\[ \vee_{\Lambda_i}(x \ast y, q) \leq \vee(\gamma_{\Lambda_i}(x, q) \vee \gamma_{\Lambda_i}(y, q)) = (\vee_{\Lambda_i}(x, q)) \vee (\vee_{\Lambda_i}(y, q)). \]

Hence \( \bigcap A_i = (\wedge_{\Lambda_i}, \vee_{\Lambda_i}) \) is an intuitionistic Q-fuzzy subalgebra of \( X \).

**Theorem 3.11.** If an IQFS \( A = (\mu_{\Lambda}, \gamma_{\Lambda}) \) in \( X \) is an intuitionistic Q-fuzzy subalgebra of \( X \), then so is \( \square A \), where \( \square A = \{(x, q, \mu_{\Lambda}(x, q), 1 - \mu_{\Lambda}(x, q)) : x \in X, q \in Q\} \).

**Proof.** It is sufficient to show that \( \mu_{\Lambda} \) satisfies the second condition in (IQF1). Let \( x, y \in X \) and \( q \in Q \). Then
\[ \mu_{\Lambda}(x \ast y, q) = 1 - \mu_{\Lambda}(x, q) \leq 1 - (\mu_{\Lambda}(x, q) \wedge \mu_{\Lambda}(y, q)) \]
\[ = (1 - \mu_{\Lambda}(x, q)) \vee (1 - \mu_{\Lambda}(y, q)) \]
\[ = \mu_{\Lambda}(x, q) \vee \mu_{\Lambda}(y, q). \]

Hence \( \square A \) is an intuitionistic Q-fuzzy subalgebra of \( X \).

**Theorem 3.12.** If an IQFS \( A = (\mu_{\Lambda}, \gamma_{\Lambda}) \) in \( X \) is an intuitionistic Q-fuzzy subalgebra of \( X \), then the sets
\[ X_\mu := \{x \in X : \mu_{\Lambda}(x, q) = \mu_{\Lambda}(0, q)\} \]
and
\[ X_\gamma := \{ x \in X : \gamma_A(x, q) = \gamma_A(0, q) \} \]
are subalgebras of \( X \) for all \( q \in Q \).

Proof. Let \( x, y \in X_\mu \) and \( q \in Q \). Then \( \mu_A(x, q) = \mu_A(0, q) = \mu_A(y, q) \), and so
\[
\mu_A(x * y, q) \geq \mu_A(x, q) \land \mu_A(y, q) = \mu_A(0, q).
\]
By using Proposition 3.5, we know that \( \mu_A(x * y, q) = \mu_A(0, q) \) or equivalently \( x * y \in X_\mu \). Now let \( x, y \in X_\gamma \). Then
\[
\gamma_A(x * y, q) \leq \gamma_A(x, q) \lor \gamma_A(y, q) = \gamma_A(0, q),
\]
and by applying Proposition 3.5, we conclude that \( \gamma_A(x * y, q) = \gamma_A(0, q) \) and hence \( x * y \in X_\gamma \). □

Definition 3.13. Let \( A = (\mu_A, \gamma_A) \) be an IQFS in \( X \) and let \( \alpha \in [0, 1] \). Then the set \( \mu_{A, \alpha}^\geq := \{ x \in X : \mu_A(x, q) \geq \alpha, q \in Q \} \) is called a \( \mu \)-level \( \alpha \)-cut and \( \gamma_{A, \alpha}^\leq := \{ x \in X : \gamma_A(x, q) \leq \alpha, q \in Q \} \) is called a \( \gamma \)-level \( \alpha \)-cut of \( A \).

Theorem 3.14. If an IQFS \( A = (\mu_A, \gamma_A) \) in \( X \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( X \), then the \( \mu \)-level \( \alpha \)-cut and \( \gamma \)-level \( \alpha \)-cut of \( A \) are subalgebras of \( X \) for every \( \alpha \in [0, 1] \) such that \( \alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \), which are called a \( \mu \)-level subalgebra and a \( \gamma \)-level subalgebra respectively.

Proof. Let \( x, y \in \mu_{A, \alpha}^\geq \) and \( q \in Q \). Then \( \mu_A(x, q) \geq \alpha \) and \( \mu_A(y, q) \geq \alpha \). It follows that \( \mu_A(x * y, q) \geq \mu_A(x, q) \land \mu_A(y, q) \geq \alpha \) so that \( x * y \in \mu_{A, \alpha}^\geq \). Hence \( \mu_{A, \alpha}^\geq \) is a subalgebra of \( X \). Now let \( x, y \in \gamma_{A, \alpha}^\leq \). Then \( \gamma_A(x * y, q) \leq \gamma_A(x, q) \lor \gamma_A(y, q) \leq \alpha \) and so \( x * y \in \gamma_{A, \alpha}^\leq \). Therefore \( \gamma_{A, \alpha}^\leq \) is a subalgebra of \( X \). □

Theorem 3.15. Let \( A = (\mu_A, \gamma_A) \) be an IQFS in \( X \) such that the sets \( \mu_{A, \alpha}^\geq \) and \( \gamma_{A, \alpha}^\leq \) are subalgebras of \( X \). Then \( A = (\mu_A, \gamma_A) \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( X \).

Proof. We need to show that \( A = (\mu_A, \gamma_A) \) satisfies the condition (IQF1). If the first condition of (IQF1) is not true, then there exist \( x_0, y_0 \in X \) such that \( \mu_A(x_0 * y_0, q) < \mu_A(x_0, q) \land \mu_A(y_0, q) \) for all \( q \in Q \). Let
\[
\alpha_0 := \frac{1}{2} (\mu_A(x_0 * y_0, q) + (\mu_A(x_0, q) \land \mu_A(y_0, q)))
\]
Then \( \mu_A(x_0 * y_0, q) < \alpha_0 < \mu_A(x_0, q) \land \mu_A(y_0, q) \), and so \( x_0 * y_0 \notin \mu_{A, \alpha_0}^\geq \), but \( x_0, y_0 \in \mu_{A, \alpha_0}^\geq \). This leads to a contradiction. Now assume that the second condition of (IQF1) does not hold. Then \( \gamma_A(x_0 * y_0, q) > \gamma_A(x_0, q) \lor \gamma_A(y_0, q) \) for some \( x_0, y_0 \in X \) and \( q \in Q \). Taking
\[
\beta_0 := \frac{1}{2} (\gamma_A(x_0 * y_0, q) + (\gamma_A(x_0, q) \lor \gamma_A(y_0, q)))
\]
then \( \gamma_A(x_0, q) \lor \gamma_A(y_0, q) < \beta_0 < \gamma_A(x_0 * y_0, q) \). It follows that \( x_0, y_0 \notin \gamma_{A, \alpha_0}^\leq \) and \( x_0 * y_0 \notin \gamma_{A, \alpha_0}^\leq \). This leads to a contradiction. This completes the proof. □
Theorem 3.16. Any subalgebra of \( X \) can be realized as both a \( \mu \)-level subalgebra and a \( \gamma \)-level subalgebra of some intuitionistic \( Q \)-fuzzy subalgebra of \( X \).

Proof. Let \( S \) be a subalgebra of \( X \) and let \( \mu_A \) and \( \gamma_A \) be \( Q \)-fuzzy sets in \( X \) defined by

\[
\mu_A(x, q) := \begin{cases} 
\alpha, & \text{if } x \in S, \\
0, & \text{otherwise}, 
\end{cases} \quad \text{and} \quad \gamma_A(x, q) := \begin{cases} 
\beta, & \text{if } x \in S, \\
1, & \text{otherwise}, 
\end{cases}
\]

for all \( x \in X \) and \( q \in Q \) where \( \alpha \) and \( \beta \) are fixed numbers in \((0, 1)\) such that \( \alpha + \beta < 1 \). Let \( x, y \in X \) for all \( q \in Q \). If \( x, y \in S \), then \( x \cdot y \in S \). Hence \( \mu_A(x \cdot y, q) = \mu_A(x, q) \land \mu_A(y, q) \) and \( \gamma_A(x \cdot y, q) = \gamma_A(x, q) \lor \gamma_A(y, q) \). If at least one of \( x \) and \( y \) does not belong to \( S \), then at least one of \( \mu_A(x, q) \) and \( \mu_A(y, q) \) is equal to 0, and at least one of \( \gamma_A(x, q) \) and \( \gamma_A(y, q) \) is equal to 1. It follows that \( \mu_A(x \cdot y, q) \geq 0 = \mu_A(x, q) \land \mu_A(y, q) \) and \( \gamma_A(x \cdot y, q) \leq 1 = \gamma_A(x, q) \lor \gamma_A(y, q) \). Hence \( A = (\mu_A, \gamma_A) \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( X \). Obviously, \( \mu_{A,\alpha}^\geq = S = \gamma_{A,\beta}^\leq \). This completes the proof. \( \square \)

Definition 3.17. Let \( f \) be a map from a set \( X \) to a set \( Y \). If \( A = (\mu_A, \gamma_A) \) and \( B = (\mu_B, \gamma_B) \) are IQFSs in \( X \) and \( Y \) respectively, then the preimage of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is an IQFS in \( X \) defined by

\[
f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)),
\]

and the image of \( A \) under \( f \), denoted by \( f(A) \), is an IQFS of \( Y \) defined by

\[
f(A) = (f_{\sup}(\mu_A), f_{\inf}(\gamma_A)),
\]

where

\[
f_{\sup}(\mu_A)(y, q) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu_A(x, q), & \text{if } f^{-1}(y, q) \neq \emptyset, \\
0, & \text{otherwise}, 
\end{cases}
\]

and

\[
f_{\inf}(\gamma_A)(y, q) = \begin{cases} 
\inf_{x \in f^{-1}(y)} \gamma_A(x, q), & \text{if } f^{-1}(y, q) \neq \emptyset, \\
1, & \text{otherwise}, 
\end{cases}
\]

for each \( y \in Y \) and \( q \in Q \).

Theorem 3.18. Let \( f : X \to Y \) be a homomorphism of \( BCK/BCI \)-algebras. If \( B = (\mu_B, \gamma_B) \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( Y \), then the preimage \( f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \) of \( B \) under \( f \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( X \).

Proof. Assume that \( B = (\mu_B, \gamma_B) \) is an intuitionistic \( Q \)-fuzzy subalgebra of \( Y \) and let \( x_1, x_2 \in X \) and \( q \in Q \). Then

\[
f^{-1}(\mu_B)(x_1 \cdot x_2, q) = \mu_B(f(x_1 \cdot x_2), q) = \mu_B(f(x_1) \cdot f(x_2), q) \\
\geq \mu_B(f(x_1), q) \land \mu_B(f(x_2), q) \\
= f^{-1}(\mu_B)(x_1, q) \land f^{-1}(\mu_B)(x_2, q)
\]
Proof. Let $f$ be an intuitionistic fuzzy subalgebra of $X$. We have

$$f^{-1}(\gamma_B)(x_1 \ast x_2, q) = \gamma_B(f(x_1 \ast x_2), q) = \gamma_B(f(x_1) \ast f(x_2), q)$$

$$\leq \gamma_B(f(x_1), q) \lor \gamma_B(f(x_2), q)$$

$$= f^{-1}(\gamma_B)(x_1, q) \lor f^{-1}(\gamma_B)(x_2, q).$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic $Q$-fuzzy subalgebra of $X$.

Theorem 3.19. Let $f : X \to Y$ be a homomorphism from a BCK/BCI-algebra $X$ onto a BCK/BCI-algebra $Y$. If $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subalgebra of $X$, then the image $f(A) = (f_{\sup}(\mu_A), f_{\inf}(\gamma_A))$ of $A$ under $f$ is an intuitionistic $Q$-fuzzy subalgebra of $Y$.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy subalgebra of $X$ and let $y_1, y_2 \in Y$ and $q \in Q$. Noticing that

$$\{(x_1 \ast x_2, q) | (x_1, q) \in f^{-1}(y_1, q) \text{ and } (x_2, q) \in f^{-1}(y_2, q)\}$$

$$\subseteq \{(x, q) \in X \times Q | (x, q) \in f^{-1}(y_1 \ast y_2, q)\},$$

we have

$$f_{\sup}(\mu_A)(y_1 \ast y_2, q)$$

$$= \sup \{\mu_A(x, q) | (x, q) \in f^{-1}(y_1 \ast y_2, q)\}$$

$$\geq \sup \{\mu_A(x_1 \ast x_2, q) | (x_1, q) \in f^{-1}(y_1, q) \text{ and } (x_2, q) \in f^{-1}(y_2, q)\}$$

$$\geq \sup \{\mu_A(x_1, q) \land \mu_A(x_2, q) | (x_1, q) \in f^{-1}(y_1, q) \text{ and } (x_2, q) \in f^{-1}(y_2, q)\}$$

$$= \sup \{\mu_A(x_1, q) | (x_1, q) \in f^{-1}(y_1, q)\} \land \sup \{\mu_A(x_2, q) | (x_2, q) \in f^{-1}(y_2, q)\}$$

and

$$f_{\inf}(\gamma_A)(y_1 \ast y_2, q)$$

$$= \inf \{\gamma_A(x, q) | (x, q) \in f^{-1}(y_1 \ast y_2, q)\}$$

$$\leq \inf \{\gamma_A(x_1 \ast x_2, q) | (x_1, q) \in f^{-1}(y_1, q) \text{ and } (x_2, q) \in f^{-1}(y_2, q)\}$$

$$\leq \inf \{\gamma_A(x_1, q) \lor \gamma_A(x_2, q) | (x_1, q) \in f^{-1}(y_1, q) \text{ and } (x_2, q) \in f^{-1}(y_2, q)\}$$

$$= \inf \{\gamma_A(x_1, q) | (x_1, q) \in f^{-1}(y_1, q)\} \lor \inf \{\gamma_A(x_2, q) | (x_2, q) \in f^{-1}(y_2, q)\}$$

Hence $f(A) = (f_{\sup}(\mu_A), f_{\inf}(\gamma_A))$ is an intuitionistic $Q$-fuzzy subalgebra of $Y$.

References


