3-Tuple domination number in complete grid graphs

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Abstract

In a graph G , a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is a k -tuple dominating set of G if each vertex of G is dominated by at least k vertices of S . A k -tuple dominating set S of a graph G is perfect if each vertex of G is dominated by exactly k vertices in S . The k -tuple domination number $\gamma_{\times k}(G)$ is the minimum cardinality among all k -tuple dominating sets of G . In this note we determine the 3-tuple domination number $\gamma_{\times 3}$ for the complete grid graphs $P_n \times P_m$ for $2 \leq n \leq 4$ and $m \geq 1$. We also study the existence and construction of perfect 3-tuple dominating sets in this graphs.

Keywords: complete grid graphs, domination number.

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1 Introduction

Let G be a graph with vertex set V . A vertex $v \in V$ is said to *dominate* all the vertices in its closed neighborhood $N[v]$. (For graph theory we follow the notation and terminology of [4].) A subset S of V is a *k -tuple dominating set* of G if each vertex of G is dominated by at least k vertices in S (see [5]). When $k = 1, 2$ the set S is called *dominating set* and *double dominating set* in G , respectively (see [5], [6]). When each vertex of G is dominated by exactly k element of S , then S is called a *perfect k -tuple dominating set* of G . The *k -tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality among all k -tuple

dominating sets of G . A *minimum k -tuple dominating set* of G is a k -tuple dominating set of cardinality $\gamma_{\times k}(G)$. The concept of domination in graph theory is a suitable model for many locating problems in operations research [10]. Dominating sets in graphs have been studied in the past ([2, 3, 8]). Perfect dominating sets and Perfect double dominating sets for certain graphs have also been investigated [1, 7, 11, 9]. In this paper we focus on 3-tuple domination number for the complete grid graphs $P_n \times P_m$ for $2 \leq n \leq 4$ and $m \geq 1$ and study the existence and construction of perfect 3-tuple dominating sets for grids, cylinders and torus. The paper is organized as follows: In the section 2 we determine 3-tuple domination number for the complete grid graphs $P_n \times P_m$ for $2 \leq n \leq 4$ and $m \geq 1$. In Section 3 we completely characterize all grids, cylinders and torus that possess a Perfect 3-tuple domination set and also specify the structure of the existing Perfect 3-tuple domination set.

2 3-tuple domination numbers for $P_n \times P_m$

Throughout this section $\mathfrak{T}$ is a 3-tuple dominating set in $P_n \times P_m$. We assume that the vertices of the i th copy of P_n in $P_n \times P_m$ are $u_1^i, u_2^i, \dots, u_n^i$ for $i = 1, 2, \dots, m$. (See Figure 1.)

Figure 1: The i th copy of P_n in $P_n \times P_m$

Definition 1 A sequence $(c_1, c_2, \dots, c_m)$ of nonnegative integers will be called a *column triple domination sequence* or simply *triple domination sequence* for $P_n \times P_m$ if there is a triple dominating set $\mathfrak{T}$ for $P_n \times P_m$ such that

$$c_j = |\mathcal{C}_j| \quad \text{for } j = 1, 2, \dots, m, \quad \text{where } \mathcal{C}_j = \mathfrak{T} \cap C_j.$$

The sequence $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ is called *(column) triple dominating sets sequence*.

Similarly, one can define *row triple domination sequence* and *row triple dominating sets sequence*.

Clearly $\gamma_{\times 3}(P_n \times P_m) = \gamma_{\times 3}(P_m \times P_n)$ and $\gamma_{\times 3}(P_n \times P_m)$ is the minimum of $c_1 + c_2 + \dots + c_m$ over all triple domination sequences $(c_1, c_2, \dots, c_m)$ for $P_n \times P_m$.

Every vertex in C_1 and C_m is dominated exactly by one vertex of C_2 and C_{m-1} , respectively. So, The set C_1 is a double dominating set for C_1 , and also C_m for C_m . Hence, we have

Proposition 2 For every $m, n \geq 1$, $c_1 \geq \lceil \frac{2n+2}{3} \rceil$ and $c_m \geq \lceil \frac{2n+2}{3} \rceil$.

Every vertex in C_i , $i = 2, 3, \dots, m-1$ is dominated exactly by two vertices of $V(P_n \times P_m) \setminus C_i$. So, The set C_i is a dominating set for C_i , and we have the following two results

Proposition 3 For every $m, n \geq 1$, and for all $i = 2, 3, \dots, m-1$, $c_i \geq \lceil \frac{n}{3} \rceil$.

Proposition 4 For some $i \in \{2, 3, \dots, m-1\}$, $c_i = \frac{n}{3}$ if and only if $c_{i-1} = c_{i+1} = n$.

Every vertex in C_i or C_{i+1} , $i = 2, 3, \dots, m-2$ is dominated only by one vertex of $V(P_n \times P_m) \setminus \{C_i \cup C_{i+1}\}$. So, The set $C_i \cup C_{i+1}$ is a double dominating set for $C_i \cup C_{i+1}$, and we have

Proposition 5 For every $m, n \geq 1$, and for all $i = 2, 3, \dots, m-2$, $c_i + c_{i+1} \geq n + 1$.

2.1 3-tuple domination numbers for $P_2 \times P_m$

Theorem 6 For all $m \geq 3$, $\gamma_{\times 3}(P_2 \times P_m) = \lceil \frac{3(m+1)}{2} \rceil$.

Proof. Let $\mathfrak{T}$ be a 3-tuple dominating set in $P_2 \times P_m$. Since u_1^1, u_2^1 must be dominated by at least three vertices of $\mathfrak{T}$ we must have $c_1 = c_2 = 2$. Similarly, $c_n = c_{n-1} = 2$. By Proposition 5 we have $c_i + c_{i+1} \geq 3$ for $i = 2, 3, \dots, m-2$. Now, we have

$$2|\mathfrak{T}| = \sum_{i=1}^m 2c_i = (2c_1 + c_2) + \sum_{i=2}^{m-2} (c_i + c_{i+1}) + (c_{m-1} + 2c_m) \geq 3(m+1).$$

Thus we have $|\mathfrak{T}| \geq \lceil \frac{3(m+1)}{2} \rceil$.

Now we define a 3-tuple dominating set in $P_2 \times P_m$ with $\lceil \frac{3(m+1)}{2} \rceil$ vertices. Define

$$\begin{aligned} \mathfrak{T} = & \{u_1^i, u_2^i \mid i = 1, m-1, m, 2k \text{ and } k = 1, 2, \dots, \lfloor \frac{m}{2} \rfloor\} \cup \\ & \{u_1^i \mid i = 4k+3, k = 0, 1, \dots, \lfloor \frac{m}{4} \rfloor - 1\} \cup \\ & \begin{cases} \{u_2^i \mid i = 4k+1, k = 1, \dots, \lfloor \frac{m}{4} \rfloor - 1\} & \text{if } m \text{ is even} \\ \{u_2^i \mid i = 4k+1, k = 1, \dots, \lfloor \frac{m}{4} \rfloor\} & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

Then $\mathfrak{T}$ is a 3-tuple dominating set in $P_2 \times P_m$ with $\lceil \frac{3(m+1)}{2} \rceil$ vertices. This completes the proof. $\diamond$

2.2 3-tuple domination numbers for $P_3 \times P_m$

The proof of the following result is easy and we leave it for the reader.

Lemma 7 $3c_1 + c_2 \geq 11$ and $3c_n + c_{n-1} \geq 11$.

Lemma 8 If $c_i = 2$ for some $2 \leq i \leq m-2$, then $c_i + 2c_{i+1} + c_{i+2} \geq 9$.

Proof. We consider three cases.

case 1 $u_1^i, u_3^i \in \mathfrak{T}$. This forces $u_1^{i+1}, u_3^{i+1} \in \mathfrak{T}$, since u_1^i and u_3^i must be dominated by at least three vertices of $\mathfrak{T}$. If $c_{i+1} = 2$, then we must have $c_{i+2} = 3$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 9$. Let $c_{i+1} = 3$. By proposition 3 we have $c_{i+2} \geq 1$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 9$.

case 2 $u_1^i, u_2^i \in \mathfrak{T}$. This forces $u_2^{i+1}, u_3^{i+1}, u_3^{i+2} \in \mathfrak{T}$, since u_3^{i+1} must be dominated by at least three vertices of $\mathfrak{T}$. If $c_{i+1} = 3$ then obviously $c_i + 2c_{i+1} + c_{i+2} \geq 9$. Let $c_{i+1} = 2$. This forces $c_{i+2} = 3$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 9$.

case 3 $u_2^i, u_3^i \in \mathfrak{T}$. An argument similar to that described in case two shows that $c_i + 2c_{i+1} + c_{i+2} \geq 9$. This completes the proof. $\diamond$

Lemma 9 If $c_i + 2c_{i+1} + c_{i+2} = 8$ for some $1 \leq i \leq m-2$, then either $c_{i+1} + 2c_{i+2} + c_{i+3} \geq 10$ or $c_{i+1} + 2c_{i+2} + c_{i+3} = 9$ and $c_{i+2} + 2c_{i+3} + c_{i+4} \geq 10$ or $c_j + 2c_{j+1} + c_{j+2} = 9$ for $j = i+1, i+2$ and $c_{i+3} + 2c_{i+4} + c_{i+5} \geq 10$ or $c_j + 2c_{j+1} + c_{j+2} \geq 9$ for $j = i+1, i+2, i+3, i+4$.

Proof. Obviously, $c_{i+1} < 3$. By proposition 4 and lemma 8, we must have $c_i = 3$. We claim that $c_{i+2} = 3$. $c_{i+2} \neq 2$, since $2c_{i+1} \neq 3$. If $c_{i+2} = 1$ then $c_{i+1} = 3$ by proposition 4. This implies $c_i + 2c_{i+1} + c_{i+2} = 10$ which is a contradiction. Thus $c_{i+2} = 3$. This forces $c_{i+1} = 1$ and $u_2^{i+1} \in \mathfrak{T}$. Now, we must have $u_1^{i+3}, u_3^{i+3} \in \mathfrak{T}$. So $c_{i+3} \geq 2$. If $c_{i+3} = 3$ then obviously $c_{i+1} + 2c_{i+2} + c_{i+3} = 10$. Let $c_{i+3} = 2$. Then $c_{i+1} + 2c_{i+2} + c_{i+3} = 9$ and we must have $u_1^{i+4}, u_3^{i+4} \in \mathfrak{T}$. So $c_{i+4} \geq 2$. If $c_{i+4} = 3$, then obviously $c_{i+2} + 2c_{i+3} + c_{i+4} = 10$. Let $c_{i+4} = 2$. Then $c_{i+2} + 2c_{i+3} + c_{i+4} = 9$ and we must have $c_{i+5} = 3$ and $c_{i+3} + 2c_{i+4} + c_{i+5} = 9$. Now it is easy to see that $c_{i+4} + 2c_{i+5} + c_{i+6} \geq 9$. This completes the proof. $\diamond$

Theorem 10 For every $m > 1$, $\gamma_{\times 3}(P_3 \times P_m) = \lceil \frac{9m - \lceil \frac{m-2}{5} \rceil + 4}{4} \rceil$.

Proof. We have

$$\begin{aligned} 4|\mathfrak{T}| &= (3c_1 + c_2) + \sum_{i=1}^{m-2} (c_i + 2c_{i+1} + c_{i+2}) + (3c_m + c_{m-1}) \\ &\geq 22 + 8\lceil \frac{m-2}{5} \rceil + 9(m-2 - \lceil \frac{m-2}{5} \rceil) \\ &\geq 9m - \lceil \frac{m-2}{5} \rceil + 4 \end{aligned}$$

Since By lemma 9 at most one of the consecutive five terms $c_i + 2c_{i+1} + c_{i+2}$ is 8 and the other is grater than 8. Thus $\gamma_{\times 3}(P_3 \times P_m) \geq \lceil \frac{9m - \lceil \frac{m-2}{5} \rceil + 4}{4} \rceil$.

Now we define a 3-tuple dominating set of order $\lceil \frac{9m - \lceil \frac{m-2}{5} \rceil + 4}{4} \rceil$. Let $m = 5s + r$, $s, r \in \mathbb{Z}$ and $0 \leq r < m$. Define

$$\begin{aligned} \mathfrak{T} = & \{u_i^{5k+1}, u_i^{5k+4} \mid 0 \leq k \leq s-1, 1 \leq i \leq 3\} \cup \\ & \{u_i^{5k+2}, u_i^{5k+3}, u_2^{5l+5} \mid 0 \leq k \leq s-1, i = 1, 3, 0 \leq l \leq s-2\} \cup \\ & \left(\begin{array}{ll} \{u_1^m, u_2^m, u_3^m\} & \text{if } s \geq 1, r = 0 \\ \{u_1^{m-1}, u_3^{m-1}, u_1^m, u_2^m, u_3^m\} & \text{if } s \geq 1, r = 1 \\ \{u_2^{m-2}, u_i^{m-1}, u_i^m \mid 0 \leq i \leq 3\} & \text{if } r = 2 \\ \{u_2^{m-3}, u_1^{m-1}, u_3^{m-1}, u_i^{m-2}, u_i^m \mid 0 \leq i \leq 3\} & \text{if } r = 3 \\ \{u_2^{m-4}, u_j^{m-1}, u_j^{m-2}, u_i^{m-3}, u_i^m \mid 0 \leq i \leq 3, j = 1, 2\} & \text{if } r = 4. \end{array} \right) \end{aligned}$$

It is easy to see that in each case $\mathfrak{T}$ is a 3-tuple dominating set in $P_3 \times P_m$ with $\lceil \frac{9m - \lceil \frac{m-2}{5} \rceil + 4}{4} \rceil$ vertices. This completes the proof. $\diamond$

2.3 3-tuple domination numbers for $P_4 \times P_m$

Lemma 11 $c_i + 2c_{i+1} + c_{i+2} \geq 11$ for $1 \leq i \leq m-2$.

Proof. By proposition 3, $c_i \geq 2$ for each i . We consider three cases.

case 1 $c_i = 4$. If $c_{i+1} = 2$ then we have $c_{i+2} \geq 3$ by proposition 5. Then $c_i + 2c_{i+1} + c_{i+2} \geq 11$. Let $c_{i+1} \geq 3$. Then $c_i + 2c_{i+1} + c_{i+2} \geq 11$, since $c_{i+2} \geq 2$.

case 2 $c_i = 3$. We consider four subcases.

subcase 1 $u_1^i, u_2^i, u_3^i \in \mathfrak{T}$. Since u_1^{i+1}, u_4^{i+1} must be dominated by three vertices of $\mathfrak{T}$ we must have $u_3^{i+1}, u_4^{i+1} \in \mathfrak{T}$ and either u_1^{i+1} or $u_2^{i+1} \in \mathfrak{T}$. Therefore, $c_{i+1} \geq 3$ and also $c_{i+2} \geq 2$ by proposition 3. This implies $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

subcase 2 $u_1^i, u_2^i, u_4^i \in \mathfrak{T}$. If $c_{i+1} \geq 3$ then as above we have $c_i + 2c_{i+1} + c_{i+2} \geq 11$. Let $c_{i+1} = 2$. Since u_4^i must be dominated by three vertices of $\mathfrak{T}$ we must have $u_4^{i+1} \in \mathfrak{T}$. Since also u_3^{i+1} must be dominated by three vertices of $\mathfrak{T}$ we must have either u_2^{i+1} or $u_3^{i+1} \in \mathfrak{T}$. If $u_2^{i+1} \in \mathfrak{T}$ then we must have $c_{i+2} = 4$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 11$. If $u_3^{i+1} \in \mathfrak{T}$ then u_1^{i+1} is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction.

subcase 3 $u_1^i, u_3^i, u_4^i \in \mathfrak{T}$. An argument similar to that described in subcase 2 prove that $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

subcase 4 $u_2^i, u_3^i, u_4^i \in \mathfrak{T}$. As subcase 1 we can see $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

case 3 $c_i = 2$. We consider six subcases.

subcase 1 $u_1^i, u_4^i \in \mathfrak{T}$. This forces $c_{i+1} = 4$ and we have $c_i + 2c_{i+1} + c_{i+2} \geq 11$, since $c_{i+2} \geq 2$.

subcase 2 $u_1^i, u_3^i \in \mathfrak{T}$. This forces $u_1^{i+1}, u_3^{i+1}, u_4^{i+1} \in \mathfrak{T}$. If $c_{i+1} = 4$ then obviously $c_i + 2c_{i+1} + c_{i+2} \geq 11$. Let $c_{i+1} = 3$. This forces $u_1^{i+2}, u_2^{i+2}, u_4^{i+2} \in \mathfrak{T}$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

subcase 3 $u_1^i, u_2^i \in \mathfrak{T}$. This case is impossible since then u_4^i is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction.

subcase 4 $u_2^i, u_3^i \in \mathfrak{T}$. Since u_1^{i+1} and u_4^{i+1} must be dominated by three vertices of $\mathfrak{T}$, we must have $c_{i+1} = 4$ which implies $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

subcase 5 $u_2^i, u_4^i \in \mathfrak{T}$. As subcase 2 we have $c_i + 2c_{i+1} + c_{i+2} \geq 11$.

subcase 6 $u_3^i, u_4^i \in \mathfrak{T}$. This case is impossible (see subcase 3). This completes the proof. $\diamond$

Lemma 12 If $c_2 + 2c_3 + c_4 = 11$ then $4c_1 + 3c_2 + c_3 \geq 27$. Similarly, If $c_{n-2} + 2c_{n-1} + c_n = 11$ then $4c_n + 3c_{n-1} + c_{n-2} \geq 27$.

Proof. Let $c_2 + 2c_3 + c_4 = 11$. Obviously $c_1 = 4$ and $u_1^2, u_4^2 \in \mathfrak{T}$. If $c_2 = 2$ then we must have $c_3 = 4$. But then $c_2 + 2c_3 + c_4 \geq 12$, since $c_4 \geq 2$ by proposition 3, which is a contradiction. So let $c_2 \geq 3$. If $c_2 = 4$ then we must have $c_3 = 2$ and $c_4 = 3$, since $c_2 + 2c_3 + c_4 = 11$ and $c_4 \geq 2$. So we have $4c_1 + 3c_2 + c_3 \geq 27$. If $c_2 = 3$ Then obviously $4c_1 + 3c_2 + c_3 \geq 27$, since $c_3 \geq 2$ by proposition 3. This completes the proof. $\diamond$

Lemma 13 $4c_1 + 3c_2 + c_3 \geq 26$. Similarly, $4c_m + 3c_{m-1} + c_{m-2} \geq 26$.

Proof. It is easy to see that $c_1 = 4$ and $u_1^2, u_4^2 \in \mathfrak{T}$. If $c_2 \geq 3$ then obviously $4c_1 + 3c_2 + c_3 \geq 27$ by proposition 3. Let $c_2 = 2$. This forces $c_3 = 4$ and we have $4c_1 + 3c_2 + c_3 \geq 26$. This completes the proof. $\diamond$

Lemma 14 There is no i such that $c_j + 2c_{j+1} + c_{j+2} = 11$ for $i \leq j \leq i + 4$.

Proof. Let there exists an i such that $c_j + 2c_{j+1} + c_{j+2} = 11$ for $i \leq j \leq i + 4$. We consider three cases.

case 1 $c_i = 4$. This forces $c_{i+1} = 2$ and $c_{i+2} = 3$, since $c_i + 2c_{i+1} + c_{i+2} = 11$. If $u_1^{i+1}, u_4^{i+1} \in \mathfrak{T}$ or if $u_2^{i+1}, u_3^{i+1} \in \mathfrak{T}$ then we must have $c_{i+2} = 4$ which is a contradiction. Now, consider two subcases.

subcase 1 $u_1^{i+1}, u_3^{i+1} \in \mathfrak{T}$. This forces $u_1^{i+2}, u_3^{i+2}, u_4^{i+2} \in \mathfrak{T}$. Since $c_{i+1} + 2c_{i+2} + c_{i+3} = 11$ we must have $c_{i+3} = 3$ and $u_1^{i+3}, u_2^{i+3}, u_4^{i+3} \in \mathfrak{T}$. Now by assumption we must have $c_{i+4} = 2$ and $u_2^{i+4}, u_4^{i+4} \in \mathfrak{T}$. This implies $c_{i+5} = 4$. But by proposition 3 we have $c_{i+4} + 2c_{i+5} + c_{i+6} \geq 12$ which is a contradiction.

subcase 2 $u_2^{i+1}, u_4^{i+1} \in \mathfrak{T}$. An argument similar to that described in subcase 1 leads to a contradiction.

case 2 $c_i = 3$. By assumption we must have $c_{i+1} = 3$ and $c_{i+2} = 2$. This implies $c_{i+3} = 4$ since $c_{i+1} + 2c_{i+2} + c_{i+3} = 11$. This forces $c_{i+2} + 2c_{i+3} + c_{i+4} \geq 12$ by proposition 3, which is a contradiction.

case 3 $c_i = 2$. Then we have $2c_{i+1} + c_{i+2} = 9$ which forces $c_{i+1} = c_{i+2} = 3$. Now we must have $c_{i+3} = 2$ and $c_{i+4} = 4$. But then we have $c_{i+3} + 2c_{i+4} + c_{i+5} \geq 12$ by proposition 3, which is a contradiction. This completes the proof. $\diamond$

Theorem 15 $\gamma_{\times 3}(P_4 \times P_m) = \lceil \frac{11m + \lfloor \frac{m-4}{5} \rfloor + 9}{4} \rceil$.

Proof. We have

$$\begin{aligned} 4|\mathfrak{T}| &= \sum_{i=1}^m 4c_i \\ &= (4c_1 + 3c_2 + c_3) + \sum_{i=2}^{m-3} (c_i + 2c_{i+1} + c_{i+2}) + (4c_m + 3c_{m-1} + c_{m-2}). \end{aligned}$$

We consider two cases.

case 1 $c_2 + 2c_3 + c_4 = 11$. Then by lemma 12, we have $4c_1 + 3c_2 + c_3 \geq 27$. Now we have

$$\begin{aligned} 4|\mathfrak{T}| &\geq 53 + \sum_{i=2}^{m-3} (c_i + 2c_{i+1} + c_{i+2}) \\ &\geq 53 + 12\lfloor \frac{n-4}{5} \rfloor + 11(4\lfloor \frac{n-4}{5} \rfloor) + 11(n - 4 - 5\lfloor \frac{n-4}{5} \rfloor) \\ &= 11n + \lfloor \frac{n-4}{5} \rfloor + 9 \end{aligned}$$

Since By lemma 14 at most one of the consecutive five terms $c_i + 2c_{i+1} + c_{i+2}$ is greater than 11 and the other is greater than 10.

case 2 $c_2 + 2c_3 + c_4 = 12$. Then it is easy to see that

$$\begin{aligned} 4|\mathfrak{T}| &\geq 26 + \sum_{i=2}^{m-3} (c_i + 2c_{i+1} + c_{i+2}) + (4c_m + 3c_{m-1} + c_{m-2}) \\ &\geq 26 + (12\lfloor \frac{n-4}{5} \rfloor + 11(4\lfloor \frac{n-4}{5} \rfloor) + 11(n - 4 - 5\lfloor \frac{n-4}{5} \rfloor) + 27) \\ &= 11n + \lfloor \frac{n-4}{5} \rfloor + 9 \end{aligned}$$

Thus $|\mathfrak{T}| \geq \lceil \frac{11m + \lfloor \frac{m-4}{5} \rfloor + 9}{4} \rceil$.

Now we define a 3-tuple dominating set of order $\lceil \frac{9m - \lceil \frac{m-2}{5} \rceil + 4}{4} \rceil$. Let $m = 5s + r$, $s, r \in \mathbb{Z}$ and $0 \leq r < m$. Define

$$\begin{aligned} \mathfrak{T} = & \{u_i^j \mid 0 \leq i \leq 4, j = 1, n, 5k + 3 \text{ and } 0 \leq k \leq s - 1\} \cup \{u_1^2, u_4^2\} \cup \\ & \{u_1^{5k+4}, u_3^{5k+4} \mid 0 \leq k \leq s - 1\} \cup \\ & \{u_1^{5k}, u_3^{5k}, u_4^{5k} \mid 1 \leq k \leq s - 1\} \cup \\ & \{u_1^{5k+1}, u_2^{5k+1}, u_4^{5k+1} \mid 1 \leq k \leq s - 1\} \cup \\ & \{u_2^{5k+2}, u_4^{5k+2} \mid 1 \leq k \leq s - 1\} \cup \\ & \begin{cases} \{u_1^{m-1}, u_4^{m-1}\} & \text{if } s \geq 1, r = 0 \\ \{u_1^{m-2}, u_3^{m-2}, u_1^{m-1}, u_3^{m-1}, u_4^{m-1}\} & \text{if } s \geq 1, r = 1 \\ \{u_1^{m-3}, u_4^{m-3}, u_1^{m-1}, u_4^{m-1}, u_i^{m-2} \mid 1 \leq i \leq 4\} & \text{if } r = 2 \\ \{u_1^{m-4}, u_3^{m-4}, u_1^{m-3}, u_3^{m-3}, u_4^{m-3}\} \cup \\ \{u_1^{m-2}, u_2^{m-2}, u_4^{m-2}, u_2^{m-1}, u_4^{m-1}\} & \text{if } r = 3 \\ \{u_i^{m-5}, u_i^{m-3}, u_i^{m-1} \mid i = 1, 4\} \cup \\ \{u_i^{m-4}, u_i^{m-2} \mid 1 \leq i \leq 4\} & \text{if } r = 4. \end{cases} \end{aligned}$$

It is easy to see that in each case $\mathfrak{T}$ is a 3-tuple dominating set in $P_4 \times P_m$ with $\lceil \frac{11m + \lceil \frac{m-4}{5} \rceil + 9}{4} \rceil$ vertices. This completes the proof. $\diamond$

3 perfect 3-tuple dominating sets

In this section we focus on perfect 3-tuple dominating sets for grids, cylinders and torus. In subsection 3.1 we completely characterize all grids that possess a perfect 3-tuple dominating sets and also specify the structure of the existing perfect 3-tuple dominating sets. In subsection 3.2 we determined which cylinders contain a perfect 3-tuple dominating sets and characterize the structure of their perfect 3-tuple dominating sets. Subsection 3.3 is devoted to determine all torus which contain a perfect 3-tuple dominating sets and to characterize the structure of their perfect 3-tuple dominating sets.

3.1 perfect 3-tuple dominating sets for grids

In this subsection we characterize all grids $P_n \times P_m$ that have perfect 3-tuple dominating set and determine the construction of their existing perfect 3-tuple dominating sets. Throughout this section $\mathfrak{T}$ is a perfect 3-tuple dominating set in $P_n \times P_m$ when it does exist. We assume that the vertices of the i th copy of P_n in $P_n \times P_m$ are $u_1^i, u_2^i, u_3^i, \dots, u_n^i$ for $i = 1, 2, \dots, m$. (See Figure 1.) We also assume that $\mathfrak{T}$ has precisely c_i vertices in the i th copy of P_n in $P_n \times P_m$.

Lemma 16 $P_2 \times P_m$ has a perfect 3-tuple dominating set if and only if $m = 2$.

Proof. Suppose that $\mathfrak{T}$ is a perfect 3-tuple dominating set of $P_2 \times P_m$. Since u_1^1 and u_2^1 must be dominated by three vertices of $\mathfrak{T}$ we must have $u_1^1, u_2^1, u_1^2, u_2^2 \in \mathfrak{T}$. If $m = 2$ then $\mathfrak{T}$ is a perfect 3-tuple dominating set of $P_2 \times P_2$. Let $m \geq 3$. Then we must have $c_3 = 0$. But then u_1^3 is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction. This completes the proof. $\diamond$

Lemma 17 Let $m \geq 3$. Then $P_3 \times P_m$ has a perfect 3-tuple dominating set if and only if $m = 4$.

Proof. Let $\mathfrak{T}$ be a perfect 3-tuple dominating set in $P_3 \times P_m$. Since u_1^1 and u_3^1 must be dominated by three vertices of $\mathfrak{T}$ we must have $u_1^1, u_2^1, u_3^1, u_1^2, u_3^2 \in \mathfrak{T}$. This forces $u_2^2, u_2^3 \notin \mathfrak{T}$ which implies $u_1^3, u_3^3, u_1^4, u_2^4, u_3^4 \in \mathfrak{T}$. If $m = 4$ then obviously $\mathfrak{T}$ is a perfect 3-tuple dominating set of $P_3 \times P_4$. Let $m \geq 5$. Then we must have $c_5 = 0$. But then u_1^5 is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction. This completes the proof. $\diamond$

Lemma 18 If $m, n \geq 4$ then $P_n \times P_m$ does not have a perfect 3-tuple dominating set.

Proof. Let $\mathfrak{T}$ be a perfect 3-tuple dominating set in $P_n \times P_m$. It is clear that $u_1^1, u_2^1, u_1^2 \in \mathfrak{T}$. If $u_3^1 \notin \mathfrak{T}$ then we must have $u_2^2, u_3^2 \in \mathfrak{T}$ which is a contradiction. Let $u_3^1 \in \mathfrak{T}$. This forces $u_2^2 \notin \mathfrak{T}$ and $u_1^3 \in \mathfrak{T}$. We consider two cases.

case 1 $u_3^2 \in \mathfrak{T}$. This forces $u_4^1 \notin \mathfrak{T}$ which implies $u_4^2 \in \mathfrak{T}$. This forces $u_3^3 \notin \mathfrak{T}$. But then u_3^2 is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction.

case 2 $u_3^2 \notin \mathfrak{T}$. This implies $u_2^3 \in \mathfrak{T}$. In a similar vein, we obtain a contradiction. This completes the proof. $\diamond$

Now we state the main result of this subsection.

Theorem 19 $P_n \times P_m$ has a perfect 3-tuple dominating set if and only if

$$\begin{cases} n = m = 2 \\ n = 3 \text{ and } m = 4. \end{cases}$$

3.2 perfect 3-tuple dominating sets for cylinders

In this subsection we characterize cylinders which contain a perfect 3-tuple dominating set and determined the structure of their 3-tuple dominating sets. We assume that $u_1^i, u_2^i, \dots, u_n^i$ are the vertices of the i th copy of C_n in $C_n \times P_m$ for $i = 1, 2, \dots, m$. We also assume that $\mathfrak{T}$ is a 3-tuple dominating set in $C_n \times P_n$ and has precisely c_i vertices in the i th copy of C_n in $C_n \times P_m$. Throughout this section $n \geq 3$.

The proof of the following result is straightforward and we leave it for the reader.

Lemma 20 $C_n \times P_1$ has a perfect 3-tuple dominating set for each $n \geq 3$.

Lemma 21 $C_n \times P_2$ has a perfect 3-tuple dominating set if and only if n is multiply of 4.

Proof. Suppose that $\mathfrak{T}$ is a perfect 3-tuple dominating set of $C_n \times P_2$. Without loss of generality we can assume $u_1^1 \notin \mathfrak{T}$. This forces $u_1^2, u_3^1, u_2^1, u_2^2 \in \mathfrak{T}$. Since $\mathfrak{T}$ is a perfect 3-tuple dominating set we must have $u_3^2 \notin \mathfrak{T}$. With a similar argument we have $u_1^5 \notin \mathfrak{T}$ and $u_1^4 \in \mathfrak{T}$. Now, an induction argument shows that $u_1^{4k+1} \notin \mathfrak{T}$ and $u_1^{4k+2}, u_1^{4k+3}, u_1^{4k+4} \in \mathfrak{T}$ for each $k \geq 0$. This implies $4|n$. Conversely, Let $4|n$. Define

$$\mathfrak{T} = \{u_1^{4k+2}, u_1^{4k+3}, u_1^{4k+4}, u_2^{4k+1}, u_2^{4k+2}, u_2^{4k+4} \mid 0 \leq k \leq \frac{n}{4} - 1\}.$$

It is easy to see that $\mathfrak{T}$ is a perfect 3-tuple dominating set in $C_n \times P_2$. This completes the proof. $\diamond$

Lemma 22 $C_n \times P_m$ has no perfect 3-tuple dominating set for any two integers $n, m \geq 3$.

Proof. Let $\mathfrak{T}$ be a perfect 3-tuple dominating set in $C_n \times P_m$. Without loss of generality, we can assume $u_1^1 \notin \mathfrak{T}$. This forces $u_1^n, u_1^2, u_2^n, u_2^1, u_2^2 \in \mathfrak{T}$ which implies $u_3^n, u_3^1, u_3^2 \notin \mathfrak{T}$. But then u_3^1 is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction. This completes the proof. $\diamond$

Now we state the main result of this subsection.

Theorem 23 $C_n \times P_m$ has a perfect 3-tuple dominating set if and only if

$$\begin{cases} n \geq 3 \text{ and } m = 1 \\ n \equiv 0 \pmod{4} \text{ and } m = 2. \end{cases}$$

3.3 perfect 3-tuple dominating sets for torus

In this subsection we characterize all torus $C_n \times C_m$, $m, n \geq 3$, that have perfect 3-tuple dominating set and determine the construction of their existing perfect 3-tuple dominating sets. We assume that $u_1^i, u_2^i, \dots, u_n^i$ are the vertices of the i th copy of C_n in $C_n \times C_m$ for $i = 1, 2, \dots, m$. We also assume that $\mathfrak{T}$ has precisely c_i vertices in the i th copy of C_n in $C_n \times C_m$. Throughout this section $n, m \geq 3$.

Lemma 24 $C_3 \times C_m$ has no perfect 3-tuple dominating set for any positive integer m .

Proof. Let $\mathfrak{T}$ be a perfect 3-tuple dominating set in $C_3 \times C_m$. It is easy to see that $c_i \geq 1$ for each i . We claim that $c_i = 2$ for $i = 1, 2, \dots, n$. Let $c_i \neq 2$ for some i . If $c_i = 3$ then we must have $c_{i+1} = 0$. But then the vertices in i th copy of C_3 is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction. If $c_i = 1$ then we must have $c_{i+1} = 3$ which leads to a contradiction. Thus $c_i = 2$ for each i . Without loss of generality we can assume $u_1^1, u_2^1 \in \mathfrak{T}$. This implies u_3^n or $u_3^2 \in \mathfrak{T}$. Without loss of generality we can assume $u_3^2 \in \mathfrak{T}$. Since $c_2 = 2$ we must have u_2^1 or $u_2^2 \in \mathfrak{T}$. But then we must have $c_3 = 1$ which is a contradiction. This completes the proof. $\diamond$

Lemma 25 $C_4 \times C_m$ has no perfect 3-tuple dominating set for any integer $m \geq 4$.

Proof. By Lemma 24 we can assume $m \geq 4$. Let $\mathfrak{T}$ be a perfect double dominating set in $C_4 \times C_m$. It is easy to see that $2 \leq c_i \leq 3$ for each i . If $c_i = 2$ for some i then obviously $c_{i+1} = c_{i-1} = 1$ which is a contradiction. So $c_i = 3$ for each i . Without loss of generality we can assume $u_1^1, u_2^1, u_3^1 \in \mathfrak{T}$. Since u_4^1 must be dominated by exactly three vertices of $\mathfrak{T}$ we must have u_4^n or $u_4^2 \in \mathfrak{T}$. Without loss of generality we can assume $u_4^2 \in \mathfrak{T}$. This forces $u_4^n, u_2^n \notin \mathfrak{T}$ and $c_n \leq 2$ which is a contradiction. This completes the proof. $\diamond$

Lemma 26 Let $m, n \geq 5$ and $\mathfrak{T}$ be a perfect 3-tuple dominating set for $C_n \times C_m$. Then there is no i, j such that $u_i^j, u_i^{j+1}, u_i^{j+2}, u_i^{j+3} \in \mathfrak{T}$ or $u_i^j, u_{i+1}^j, u_{i+2}^j, u_{i+3}^j \in \mathfrak{T}$.

Proof. Let $u_i^j, u_i^{j+1}, u_i^{j+2}, u_i^{j+3} \in \mathfrak{T}$ for some i, j . This forces $u_{i-1}^{j+1}, u_{i-1}^{j+2}, u_{i+1}^{j+1}, u_{i+1}^{j+2} \notin \mathfrak{T}$ which implies $u_{i-1}^{j-1}, u_{i+1}^{j-1} \in \mathfrak{T}$. Then u_i^j is dominated by at least four vertices of $\mathfrak{T}$ which is a contradiction. $\diamond$

Lemma 27 Let $m, n \geq 5$ and $\mathfrak{T}$ be a perfect 3-tuple dominating set for $C_n \times C_m$. Then there is no i, j such that $u_i^j, u_i^{j+1}, u_{i+1}^j, u_{i+1}^{j+1} \in \mathfrak{T}$.

Proof. Let $u_i^j, u_i^{j+1}, u_{i+1}^j, u_{i+1}^{j+1} \in \mathfrak{T}$ for some i, j . This forces $u_i^{j+2}, u_i^{j-1}, u_{i+1}^{j+2}, u_{i+1}^{j-1}, u_{i+2}^j, u_{i+2}^{j+1}, u_{i-1}^j, u_{i-1}^{j+1} \notin \mathfrak{T}$. Then we must have $u_{i-2}^{j-1}, u_{i-2}^j, u_{i-2}^{j+1}, u_{i-2}^{j+2} \in \mathfrak{T}$. This leads to a contradiction by lemma 26. $\diamond$

Lemma 28 Let $m, n \geq 5$ and $\mathfrak{T}$ be a perfect 3-tuple dominating set for $C_n \times C_m$. Then there is no i, j such that $u_i^j, u_i^{j+1}, u_i^{j+2}, u_{i+1}^j, u_{i+1}^{j+2} \in \mathfrak{T}$ or $u_i^j, u_i^{j+1}, u_i^{j+2}, u_{i-1}^j, u_{i-1}^{j+2} \in \mathfrak{T}$. Similarly, there is no i, j such that $u_i^j, u_{i+1}^j, u_{i+2}^j, u_i^{j+1}, u_{i+2}^{j+1} \in \mathfrak{T}$ or $u_i^j, u_{i+1}^j, u_{i+2}^j, u_i^{j-1}, u_{i+2}^{j-1} \in \mathfrak{T}$.

Proof. Let $u_i^j, u_i^{j+1}, u_i^{j+2}, u_{i+1}^j, u_{i+1}^{j+2} \in \mathfrak{T}$ for some i, j . This forces $u_{i-1}^j, u_{i-1}^{j+1}, u_{i-1}^{j+2} \notin \mathfrak{T}$. But then u_{i-1}^{j+1} is dominated by at most two vertices of $\mathfrak{T}$ which is a contradiction. $\diamond$

Lemma 29 Let $m, n \geq 5$. If $C_n \times C_m$ has a perfect 3-tuple dominating set then both n and m are multiples of 5.

Proof. Let $\mathfrak{T}$ be a perfect 3-tuple dominating set in $C_n \times C_m$. Without loss of generality, we can assume $u_4^1, u_4^2 \in \mathfrak{T}$. By lemma 26 we must have u_4^m or $u_4^3 \notin \mathfrak{T}$. Consider three cases.

Case 1 $u_4^m \notin \mathfrak{T}$ and $u_4^3 \in \mathfrak{T}$. This implies $u_3^2, u_5^2 \notin \mathfrak{T}$. By lemma 28 we must have $u_3^1, u_5^3 \in \mathfrak{T}$ or $u_3^3, u_5^1 \in \mathfrak{T}$. without loss of generality we can assume $u_3^1, u_5^3 \in \mathfrak{T}$. This forces $u_3^3, u_5^1, u_4^4 \notin \mathfrak{T}$ and $u_2^2, u_2^3, u_2^4, u_3^4, u_3^5, u_2^6, u_6^1, u_6^m, u_5^m \in \mathfrak{T}$. Thus, $u_2^1, u_3^3 \notin \mathfrak{T}$ by lemma 28. This forces $u_1^2, u_5^4, u_7^2 \in \mathfrak{T}$. This forces $u_1^3, u_1^4, u_2^5, u_4^5, u_7^m, u_7^1 \notin \mathfrak{T}$ and $u_1^5, u_6^1, u_3^6, u_4^6, u_5^5, u_8^1, u_8^m \in \mathfrak{T}$. But then $u_2^6, u_7^3, u_5^6, u_6^4, u_8^2 \notin \mathfrak{T}$ and $u_2^7, u_4^7, u_6^5, u_3^7, u_7^4, u_9^1 \in \mathfrak{T}$. This implies $u_3^8, u_7^5, u_8^3 \notin \mathfrak{T}$ and $u_2^9, u_3^9, u_8^4, u_6^8, u_4^8, u_3^9, u_2^8 \in \mathfrak{T}$. Now, we must have $u_9^4, u_5^7 \notin \mathfrak{T}$ and $u_{10}^3, u_5^8 \in \mathfrak{T}$. So far we have proved if $u_4^1, u_2^2, u_4^3, u_3^1, u_5^3 \in \mathfrak{T}$ then $u_6^6, u_7^4, u_8^8, u_6^1, u_3^8 \in \mathfrak{T}$ and $u_9^1, u_2^9, u_3^9, u_8^1, u_{10}^3 \in \mathfrak{T}$. A simple induction argument shows that

$$u_{5k+3}^1, u_{5k+4}^1 \in \mathfrak{T} \text{ and } u_4^{5k+1}, u_4^{5k+2}, u_4^{5k+3} \in \mathfrak{T}$$

for every non-negative integer k . (Note that the superscript $5k+a$ in u_i^{5k+a} is the unique integer $r \in \{1, 2, \dots, m\}$ such that $5k+a \equiv r \pmod{m}$ and the subscript $5k+b$ in u_{5k+b}^j is the unique integer $s \in \{1, 2, \dots, n\}$ such that $5k+b \equiv s \pmod{n}$.) Now since $u_1^{5k+3} \in \mathfrak{T}$ for every non-negative integer k it follows that 5 divides m , otherwise $u_i^i \in \mathfrak{T}$ for every $i = 1, 2, \dots, m$ which is impossible. Similarly, 5 divides n since $u_{5k+1}^4 \in \mathfrak{T}$ for every non-negative integer k .

Case 2 $u_4^m \in \mathfrak{T}$ and $u_4^3 \notin \mathfrak{T}$. In a similar fashion with case 1 we can prove that 5 divides n and m .

Case 3 $u_4^m, u_4^3 \notin \mathfrak{T}$. This forces $u_3^1, u_5^2 \in \mathfrak{T}$ or $u_3^2, u_5^1 \in \mathfrak{T}$ by lemma 27. Without loss of generality we can assume $u_3^1, u_5^2 \in \mathfrak{T}$. This forces $u_3^3, u_5^1 \notin \mathfrak{T}$ and $u_3^3, u_3^4, u_2^3 \in \mathfrak{T}$. This implies $u_2^2 \notin \mathfrak{T}$ and $u_2^1, u_1^2 \in \mathfrak{T}$. This forces $u_3^m \notin \mathfrak{T}$ and $u_2^m \in \mathfrak{T}$. Now by relabeling the vertices of $C_n \times C_m$ this case can be converted to Case 1. So 5 divides m and n . $\diamond$

Lemma 30 Let 5 divides m and n . Then

$$\mathfrak{T} = \{u_{5k+2j-1}^j, u_{5k+2j}^j, u_{k+2j+1}^j \mid 0 \leq j \leq m, 0 \leq k < \frac{n}{5} - 1\}$$

where addition in subscript is taken module n , is a perfect 3-tuple dominating set in $C_n \times C_m$.

Proof. It is straightforward to check that each vertex of $\mathfrak{T}$ is dominated by exactly two vertex of $\mathfrak{T}$.

Now let $u_i^j \in V \setminus \mathfrak{T}$. Then $i \neq 5k+2j-1, 5k+2j, 5k+2j+1$ for each nonnegative integer k , by definition of $\mathfrak{T}$. It is easy to see that

$$\begin{aligned} \text{if } i = 5k + 2j + 2 \text{ then } u_i^j &\in N[u_i^{j-1}] \cap N[u_i^{j+1}] \cap N[u_{i-1}^j]; \\ \text{if } i = 5k + 2j + 3 \text{ then } u_i^j &\in N[u_i^{j-1}] \cap N[u_i^{j+1}] \cap N[u_{i+1}^j]. \end{aligned}$$

It is also easy to see that in each case u_i^j is not dominated by another vertex of $\mathfrak{T}$. So u_i^j is dominated by exactly three vertices of $\mathfrak{T}$. Thus, $\mathfrak{T}$ is a perfect 3-tuple dominating set in $C_n \times C_m$. $\diamond$

Now we state the main result of this subsection.

Theorem 31 Let $n, m \geq 3$. Then $C_n \times C_m$ has a perfect 3-tuple dominating set if and only if 5 divides both m and n .

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