The Characterization of the Matrix Sequences on the Sequence Space bv_p

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Abstract

In [7] Khan and Riazuddin characterized the class $(bv : C)$ of matrix sequences and Başar, Altay and Mursaleen [3] characterized the class $(bv_p : c)$ of infinite matrices. The main purpose of the present paper is to determine the necessary and sufficient conditions on a matrix sequence $\mathcal{A} = (A_v)$ in order that $\mathcal{A}$ belongs to the matrix class $(bv_p : C)$, where $1 < p \leq \infty$.

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1 Preliminaries, Background and Notation

By w , we denote the space of all real valued sequences. Any vector subspace of w is called a *sequence space*. We write ℓ_∞ , c , c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by bs , cs , ℓ_1 and ℓ_p , we denote the spaces of all bounded, convergent, absolutely and p -absolutely convergent series, respectively. We also denote by C and C_0 , the spaces of all convergent and null double sequences, respectively.

For the sequence spaces λ and μ define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \quad (1)$$

With the notation of (1), α -, β - and γ - duals of a sequence space λ , which are respectively denoted by λ^α , λ^β and λ^γ , are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs), \quad \lambda^\gamma = S(\lambda, bs).$$

A sequence $x = (x_k)$ is almost convergent to L if and only if

$$\lim_{n \rightarrow \infty} \frac{x_v + x_{v+1} + \cdots + x_{v+n-1}}{n} = L \quad \text{uniformly in } v = 1, 2, 3, \dots$$

Let λ, μ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} , where $n, k \in \mathbb{N}$. Then, the matrix A defines A -transformation from λ into μ , if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A -transform of x , exists and is in μ ; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞ . By $(\lambda : \mu)$, we denote the class of all such matrices. A sequence x is said to be A -summable to l if Ax converges to l which is called as the A -limit of x .

For a sequence space λ , the matrix domain λ_A of an infinite matrix A is defined by

$$\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}. \quad (2)$$

The idea of $\mathcal{A}$ -summability, explained below, was introduced by H. T. Bell in his doctoral work [5]. For $v = 1, 2, 3, \dots$, let $A_v = (a_{nk}(v))$ be a infinite matrix of real numbers and let $\mathcal{A}$ be a sequence of matrices (A_v) and $\lambda \subset w, \mu \subset W$. Then, the matrix sequences $\mathcal{A} = (A_v)$ defines a transformation from λ into μ if for every sequence $(x_k) \in \lambda$ the double sequence $\mathcal{A}x = ((\mathcal{A}x)_n^v)_{n,v=0}^\infty$, the $\mathcal{A}$ -transform of x exists and is in μ ; where $(\mathcal{A}x)_n^v = \sum_k a_{nk}(v)x_k$. For simplicity in notation, here and after we abbreviate the infinite sum notation $\sum_{k=0}^\infty$ as $\sum_k$. By $(\lambda : \mu)$, we denote the class of all such matrix sequences. A sequence x is said to be $\mathcal{A}$ -summable to L if $\lim_n (\mathcal{A}x)_n^v = L$ uniformly in v . We shall write throughout for brevity that

$$\tilde{a}_{nk}(v) = \sum_{j=k}^{\infty} a_{nj}(v),$$

$$a(n, k, v) = \sum_{i=1}^n a_{ik}(v),$$

for all $n, k, v \in \mathbb{N}$. If we choose $\mathcal{A} = (a_{nk}(v))$ as

$$a_{nk}(v) = \begin{cases} \frac{1}{n+1} & , \quad (v \leq k \leq v+n) \\ 0 & , \quad (\text{otherwise}) \end{cases}$$

in [5], although no ordinary limitation method can correspond to almost convergence defined in [8], it is shown that this is possible using matrix sequences.

Moreover, in the cases of $a_{nk}(v) = a_{nk}$ and $a_{nk}(v) = \delta_{nk}$, the Kronecker delta, for all $n, k, v \in \mathbb{N}$; the method $\mathcal{A}$ respectively corresponds to the usual summability method A and the identity method I which is equivalent to the ordinary convergence. If

$$a_{nk}(v) = \frac{1}{n+1} \sum_{j=0}^n a_{v+j,k}; \quad (k, n, v \in \mathbb{N})$$

then the method $\mathcal{A}$ corresponds to almost summability method given in [9]. Again, recently, in [6], $\mathcal{A} = (a_{nk}(v))$ is chosen to be

$$a_{nk}(v) = \begin{cases} \frac{1}{n+1} & , \quad (k = \sigma^j(v), 0 \leq j \leq k) \\ 0 & , \quad (\text{otherwise}) \end{cases}$$

and it is shown that it corresponds to σ -convergence. Therefore, the method $\mathcal{A}$ is more general and more comprehensive than the ordinary limitation method A .

2 Main Results

In this section, the space bv_p of sequences of p -bounded variation was defined and studied by Başar and Altay in [2], where

$$bv_p = \left\{ x = (x_k) \in w : \sum_k |x_k - x_{k-1}|^p < \infty \right\}, (1 \leq p < \infty).$$

It was proved that bv_p is a BK -space which is linearly isomorphic to the space ℓ_p and the inclusion $bv_p \supset \ell_p$ strictly holds. The α -, β - and γ -duals of the space bv_p are determined together with the fact that bv_2 is the only Hilbert space among the spaces bv_p . The continuous dual of the space bv_p is determined and is given in the following lemma which is required in the proof of our next theorem:

Lemma 2.1 [1, Theorem 3.2] Define the spaces d_1 and d_q consisting of all sequences $a = (a_k)$ normed by

$$\|a\|_{d_1} = \sup_{k, n \in \mathbb{N}} \left| \sum_{j=k}^n a_j \right| < \infty$$

and

$$\|a\|_{d_q} = \left(\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^n a_j \right|^p \right)^{1/q} < \infty, \quad (1 < q < \infty).$$

Then, $\{bv_1\}^*$ and $\{bv_p\}^*$ are isometrically isomorphic to d_1 and d_q , respectively.

The basis of the space bv_p is also constructed and given by the following lemma:

Lemma 2.2 [2, Theorem 4.1] Define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ of the elements of the space bv_p for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)} = \begin{cases} 0, & (n < k) \\ 1, & (n \geq k) \end{cases} ; (k, n \in \mathbb{N}). \quad (3)$$

Then the sequence $\{b_n^{(k)}\}_{n \in \mathbb{N}}$ is a basis for the space bv_p and any $x \in bv_p$ has a unique representation of the form

$$x = \sum_k \lambda_k b^{(k)}, \quad (4)$$

where $\lambda_k = x_k - x_{k-1}$ for all $k \in \mathbb{N}$.

More recently, the paranormed sequence spaces $bv(u, p)$ and $bv_\infty(u, p)$ have been introduced and α -, β -, γ -duals, the Schauder basis and the characterizations of related matrix mappings have been given by Başar, Altay and Mursaleen in [3].

The main purpose of this paper is to characterize the class $(bv_p : C)$. Before giving the main theorem, we state the following lemma given by Başar and Altay [2]:

Lemma 2.3 [2, Theorem 2.2] The space bv_p of sequences of p -bounded variation is linearly isomorphic to the space ℓ_p , i.e., $bv_p \cong \ell_p$; where $1 \leq p \leq \infty$.

Theorem 2.4 Let $1 < p < \infty$. Then $\mathcal{A} \in (bv_p : C)$ if and only if

$$\sup_{m, v \in \mathbb{N}} \sum_k \left| \sum_{j=k}^m a_{nj}(v) \right|^q < \infty; \quad (n \in \mathbb{N}) \quad (5)$$

$$\sup_{n, v \in \mathbb{N}} \sum_k |\tilde{a}_{nk}(v)|^q < \infty, \quad (6)$$

$$\lim_n \tilde{a}_{nk}(v) = \alpha_k \text{ (uniformly in } v). \quad (7)$$

Proof: Let $\mathcal{A} \in (bv_p : C)$ and $1 < p < \infty$. Then $\mathcal{A}x$ exists for every $x \in bv_p$ and this implies that $\{a_{nk}(v)\}_{k, v \in \mathbb{N}} \in bv_p^\beta$ for each $n, v \in \mathbb{N}$ which shows the necessity of (5).

Let us consider the following relation between the sequences $x = (x_k)$ and $y = (y_k)$:

$$y_k = (\Delta x)_k = \begin{cases} x_0, & k = 0; \\ x_k - x_{k-1}, & k \geq 1. \end{cases} \quad (8)$$

By applying the Abel's partial sum to the m^{th} partial sum of the series $\sum_k a_{nk}(v)x_k$ and letting $m \rightarrow \infty$, one can see that

$$\begin{aligned} \sum_k a_{nk}(v)x_k &= \sum_k a_{nk}(v) \left(\sum_{j=0}^k \Delta x_j \right) \\ &= \sum_j \sum_{k=j}^{\infty} a_{nk}(v) \Delta x_j = \sum_j \tilde{a}_{nk}(v) y_j; \quad (n, v \in \mathbb{N}). \end{aligned}$$

That is, we obtain

$$\sum_k a_{nk}(v)x_k = \sum_j \tilde{a}_{nk}(v)y_j. \quad (9)$$

Taking supremum over n, v and applying Hölder's inequality we obtain from (9) that

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}(v)x_k| \leq \sup_{n \in \mathbb{N}} \left(\sum_k |\tilde{a}_{nk}(v)|^q \right)^{1/q} \left(\sum_k |y_k|^p \right)^{1/p} < \infty.$$

This implies the necessity of (6).

To prove the necessity of (7) we define, for every fixed $k \in \mathbb{N}$, the sequence of the elements of bv_p as

$$b_n^{(k)} = \begin{cases} 0, & (n < k); \\ 1, & (n \geq k). \end{cases} \quad (10)$$

Because of the $\mathcal{A}$ -transform of every $x \in bv_p$ exists and lies in C by the hypothesis, $\mathcal{A}b_n^{(k)} = \{\tilde{a}_{nk}(v)\}_{n \in \mathbb{N}}$ is also in C for every fixed $k \in \mathbb{N}$, which shows the necessity of (7).

Conversely suppose that the conditions (5)–(7) hold, and take any $x = (x_k)$ in the space bv_p . Then, $\mathcal{A}x$ exists. We observe for every $m, n \in \mathbb{N}$ that

$$\sum_{j=0}^m \left| \sum_{k=j}^m a_{kj}(v)x_k \right| \leq \sup_{i,v \in \mathbb{N}} \sum_j |\tilde{a}_{ij}(v)y_j|$$

which leads us to the following fact, by letting $m, n \rightarrow \infty$ in (7) and using (5),

$$\sum_j \left| \sum_{k=j}^{\infty} \alpha_k \right| < \infty. \quad (11)$$

Hence, $(\alpha_k) \in bv_p$ which implies that the series $\sum \alpha_k x_k$ is convergent for every $x \in bv_p$.

Let us now consider the equality obtained from (9) with $a_{nk}(v) - \alpha_k$ instead of $a_{nk}(v)$:

$$\sum_k [a_{nk}(v) - \alpha_k] x_k = \sum_k b_{nk}(v) y_k, \quad (n, v \in \mathbb{N}); \quad (12)$$

where $\mathcal{B} = (b_{nk}(v))$ with $b_{nk}(v) = \sum_{j=k}^{\infty} a_{nj}(v) - \alpha_k$ for all $n, k, v \in \mathbb{N}$. Therefore we have at this stage from [10] with $\alpha_k = 0$ for all $k \in \mathbb{N}$ that the matrix $\mathcal{B}$ belongs to the class $(\ell_p : C_0)$. Thus we see by (12) that

$$\lim_{n \rightarrow \infty} \sum_k (a_{nk}(v) - \alpha_k) x_k = 0 \quad (13)$$

which means that $\mathcal{A}x \in C$ whenever $x \in bv_p$ and this is what we wished to prove.

For the case $p = \infty$ the conditions for $\mathcal{A}x \in (bv_{\infty}, C)$ are (6), (7) and

$$\sum_k \left| \sum_{j=k}^{\infty} a_{nj}(v) \right| < \infty. \quad (14)$$

The proof is similar to the above proof.

It is trivial that the sequence spaces $bv(u, p)$ and $bv_{\infty}(u, p)$ may be reduced to bv_p and bv_{∞} in the case $p_k = p$ and $u_k = 1 (k = 0, 1, 2, \dots)$ respectively. If we take $p = 1$ then Theorem 2.4 reduces to Theorem 2.1 of [7]. In the case $p_k = p$ and $u_k = 1 (k = 0, 1, 2, \dots)$ if we take $\mathcal{A} = A$ for every v then Theorem 2.4 reduces to Theorem 4.2 of [3]. Hence, Theorem 2.4 is more general than the corresponding results of [7] and [3].

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