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Periodic Solutions of the Non-autonomous Liĕnard equations with deviating arguments

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Abstract

By using Mawhin's continuation theorem, the existence of periodic solutions for the non-autonomous Liĕnard equations with deviating arguments are studied.

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1 Introduction

The periodic solutions of the Liĕnard equations have been the subject of many investigations (see, e.g. [1-10]), while those of non-autonomous Liĕnard equations with a deviating argument are relatively scarce.

In [2],S. P.Lu and W.G.Ge, studied the exsitence of periodic solutions of the Liĕnard equations with a deviating argument of the form

$$x''(t) + f(t, x(t), x(t - \tau_0(t)))\, x'(t) + \beta(t)\, g(x(t - \tau_1(t))) = p(t), \qquad (1)$$

where f is real continuous function defined on R^3 with period $T\,(>0)$for t ;g is real continuous function defined on R, τ_0 , $\tau_{1,}\beta$ and p are real continuous functions defined on R with period T.

In this paper,we furthermore consider the exsitence of $T-$ periodic solutions of the non-autonomous Liĕnard equations with deviating arguments of the form

$$x''(t) + f(t, x(t), x(t - \tau_0(t)))\, x'(t) + g(t, x(t - \tau_1(t))) = p(t), \quad (2)$$

where f is real continuous function defined on R^3 with period $T\,(> 0)$for t ;g is real continuous function defined on $R^2$with period T for t ; τ_0 , $\tau_1$and p are real continuous functions defined on R with period T.

In this note, existence criteria for periodic solutions of (2) will be established. On (1), as corollaries of our results we also hope to extend the results in [2] . For this purpose, we will make use of a continuation theorem of Mawhin. Let X and Y be two Banach spaces and $L : \mathrm{Dom}L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$, and $\mathrm{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. It follows that $L\,|_{\mathrm{Dom}L\cap\mathrm{Ker}P}$: $(I - P)\,X \to \mathrm{Im}L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN\left(\bar{\Omega}\right)$ is bounded and$\overline{K_P\,(I - Q)\,N\left(\bar{\Omega}\right)}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$ there exist an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Theorem A (Mawhin's continuation theorem [1]). Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose

(i) for each $\lambda \in (0, 1)$, $x \in \partial\Omega, Lx \neq \lambda Nx$; and

(ii) for each $x \in \partial\Omega \cap \mathrm{Ker}L, QNx \neq 0$ and $\deg\left(JQN, \Omega \cap \mathrm{Ker}L, 0\right) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{dom}L$.

2 Existence Criteria

Our main results of these paper as follows.

Theorem 1.If

$$\sigma = \sup_{(t,x,y)\in R^3} |f(t, x, y)| < \frac{2}{T}, \quad (3)$$

$$\lim_{|x|\to+\infty} \sup \max_{0\le t\le T} \left|\frac{g(t, x)}{x}\right| \le \beta \quad (4)$$

and

$$\lim_{|x|\to+\infty} \min_{0\le t\le T} \mathrm{sgn}(x)\, g(t, x) = +\infty. \quad (5)$$

Then for $\beta < 2\left(\frac{2-\sigma T}{T^2}\right)$, (2) has a T-periodic solution.

Theorem 2.If

$$\sigma = \sup_{(t,x,y)\in[0,T]\times R^2} f(t, x, y) > 0, \quad (6)$$

furthermore suppose that (4) and (5) are satiatied,Then for $\beta < \frac{2\sigma}{T}$, (2) has a T-periodic solution.

Let $f(t,x,y) = f_1(t,x,y) + f_2(t,x,y)$,where $f_1(t,x,y)$ and $f_2(t,x,y)$are real continuous function defined on $R^2$with period T, then we have

Theorem 3.If

$$\sup_{(t,x,y)\in[0,T]\times R^2} f_1(t,x,y) \geqslant 0, \text{ and } \delta = \sup_{(t,x,y)\in[0,T]\times R^2} f_2(t,x,y) < \frac{1}{2T}, \tag{7}$$

furthermore suppose that (4) and (5) are satiatied,Then for $\beta < \frac{1-2\delta T}{T^2}$, (2) has a T-periodic solution.

To prove our results ,we need preliminaries.Let Y be the Banach space of all real T-periodic continuous functions of the form $y = y(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|y\|_0 = \max_{0\le t\le T}|y(t)|$ .Let X be the Banach space of all real T-periodic continuous differentiable functions of the form $x = x(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max\{\|x\|_0, \|x'\|_0\}$. Define the mappings $L : X \cap C^{(2)}(R,R) \to Y$ and $N : X \to Y$ respectively by

$$Lx(t) = x''(t), \ t \in R. \tag{8}$$

and

$$Nx(t) = -f(t, x(t), x(t-\tau_0(t)))\, x'(t) - g(t, x(t-\tau_1(t))) + p(t). \tag{9}$$

Clearly,

$$\mathrm{Ker}L = R \tag{10}$$

and

$$\mathrm{Im}L = \left\{ y \in Y \mid \int_0^T y(t)\,dt = 0 \right\} \tag{11}$$

is closed in Y. Thus L is a Fredholm mapping of index zero.Let us define $P : X \to X$ and $Q : Y \to Y/\mathrm{Im}L$ respectively by

$$Px(t) = x(0), \ \ t \in R, \tag{12}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{T}\int_0^T y(t)\,dt, \ \ t \in R. \tag{13}$$

for $y = y(t) \in Y$ .It is easy to see that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$.It follows that $L\mid_{\mathrm{Dom}L\cap\mathrm{Ker}P}$: $(I-P)X \to \mathrm{Im}L$ has an inverse which will be denoted by K_P. furthermore for any $y = y(t) \in\mathrm{Im}L$ we have

$$K_P y(t) = -\frac{t}{T}\int_0^T (t-s)\,y(s)\,ds + \int_0^t (t-s)\,y(s)\,ds. \tag{14}$$

Let Ω is an open and bounded subset of X, from (9),(13) and (14),we can easy lead that $QN\left(\bar{\Omega}\right)$ is bounded and $\overline{K_P\left(I-Q\right)N\left(\bar{\Omega}\right)}$ is compact, thus the mapping N is L-compact on $\bar{\Omega}$.That is we have following result

Lemma 1.Let L, N, P and Q defined by (8),(9),(12),(13) respectively .Then L is a Fredholm mapping of index zero and N is L-compact on $\bar{\Omega}$,where Ω is any open and bounded subset of X.

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Let

$$x''\left(t\right)+\lambda f\left(t,x\left(t\right),x\left(t-\tau_0\left(t\right)\right)\right)x'\left(t\right)+\lambda g\left(t,x\left(t-\tau_1\left(t\right)\right)\right)=\lambda p\left(t\right), \quad (15)$$

where $\lambda\in\left(0,T\right)$.

Lemma 2.Suppose the condition (5) is satisfied,then there is a positive constant A^*,for any T-periodic solution $x\left(t\right)$ of (15),has a $\xi_x\in\left[0,T\right]$, such that

$$\left|x\left(\xi_x\right)\right|<A^*. \quad (16)$$

Proof.Let $x\left(t\right)$ be any T-periodic solution $x\left(t\right)$ of (15).By (5) we see that there is a positive constant A^*,such that when $|x|\geqslant A^*$,

$$\operatorname{sgn}\left(x\right)g\left(t,x\right)>\|p\|_0+1. \quad (17)$$

Let $x\left(t_1\right)=\max_{0\leq t\leq T}x\left(t\right)$ and $x\left(t_2\right)=\min_{0\leq t\leq T}x\left(t\right)$,where $t_1,t_2\in\left[0,T\right]$.It is easy to see that $x'\left(t_1\right)=0$ and $x'\left(t_1\right)\leq 0$. Further,by (15) we have

$$g\left(t_1,x\left(t_1-\tau_1\left(t_1\right)\right)\right)\geqslant p\left(t_1\right)\geqslant-\|p\|_0-1. \quad (18)$$

From (17) and (18) we know that

$$x\left(t_1-\tau_1\left(t_1\right)\right)>-A^*. \quad (19)$$

Similarly,we have

$$x\left(t_2-\tau_1\left(t_2\right)\right)<A^*. \quad (20)$$

Note that $x\left(t\right)$ is continuous function,it is easy to see from (19) and (20) that there is $t_3\in R,\left(t_1-\tau_1\left(t_1\right)\geqslant t_3\geqslant t_2-\tau_1\left(t_2\right)\right.$ or $t_2-\tau_1\left(t_2\right)\geqslant t_3\geqslant t_1-\tau_1\left(t_1\right))$,such that $\left|x\left(t_3\right)\right|<A^*$.Since $x\left(t\right)$ with periodic T,thus there is $\xi\in\left[0,T\right]$,such that

$$\left|x\left(\xi\right)\right|<A^*. \quad (21)$$

The proof is complete.

Lemma 3.Suppose $x=x\left(t\right)\in X\cap C^{(2)}\left(R,R\right)$and $\xi\in\left[0,T\right]$. Then

$$\|x'\|_0\leq\frac{1}{2}\int_0^T\left|x''\left(s\right)\right|ds, \quad (22)$$

and

$$\|x\|_0 \le |x(\xi)| + \frac{T}{4}\int_0^T |x''(s)|\, ds. \tag{23}$$

Proof. $x = x(t) \in X \cap C^{(2)}(R, R)$and $\xi \in [0, T]$.Then for any $t \in [\xi, \xi + T]$,we have

$$x(t) = x(\xi) + \int_\xi^t x'(s)\, ds, \tag{24}$$

and

$$x(t) = x(\xi + T) + \int_{\xi+T}^t x'(s)\, ds = x(\xi) - \int_t^{\xi+T} x'(s)\, ds. \tag{25}$$

From (24) and (25),we see that for any $t \in [\xi, \xi + T]$,

$$x(t) = x(\xi) + \frac{1}{2}\left\{\int_\xi^t x'(s)\, ds - \int_t^{\xi+T} x'(s)\, ds\right\}. \tag{26}$$

it is following that

$$\|x\|_0 \le |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds. \tag{27}$$

Sometime, noting that $x(0) = x(T)$,so that there is $\xi_1 \in [0, T]$,such that $x'(\xi_1) = 0$.It is easy to show (22) holds.By (22) and (27) we see that

$$\begin{aligned} \|x\|_0 &\le |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\, ds \\ &\le |x(\xi)| + \frac{1}{2}\int_0^T \|x'\|_0\, ds \\ &= |x(\xi)| + \frac{T}{2}\|x'\|_0 \\ &\le |x(\xi)| + \frac{T}{4}\int_0^T |x''(s)|\, ds. \end{aligned} \tag{28}$$

The proof is complete.

The proof of Thereom 1: .First of all, we can prove that there exist constants M_0 and M_1,such that for any T-periodic solution $x(t)$ of (15)

$$\|x\|_0 \le M_0 \text{and } \|x'\|_0 \le M_1. \tag{29}$$

In view of Lemma 2,we can may find a positive constant A^*which is independent of λ and $\xi \in [0, T]$,such that

$$|x(\xi)| \le A^*. \tag{30}$$

Furthermore , by Lemma 3 we have

$$\begin{aligned} \|x\|_0 &\le |x(\xi)| + \frac{T}{4}\int_0^T |x''(s)|\,ds \\ &\le A^* + \frac{T}{4}\int_0^T |x''(s)|\,ds, \end{aligned} \tag{31}$$

and

$$\|x'\|_0 \le \frac{1}{2}\int_0^T |x''(s)|\,ds. \tag{32}$$

By (4) ,for constant $\varepsilon = \frac{1}{2}\left(2\left(\frac{2-\sigma T}{T^2}\right) - \beta\right)$,there is constant $A_1 \geqslant A^*$, such that for $|x(t-\tau_1(t))| \geqslant A_1$,

$$|g(t, x(t-\tau_1(t)))| \le (\beta+\varepsilon)\,|x(t-\tau_1(t))|\,. \tag{33}$$

Let

$$C_0 = \max_{0\le t\le T, |x|\le A_1} |g(t,x)|\,, \tag{34}$$

$$E_1 = \{t \mid t\in[0,T]\,, |x(t-\tau_1(t))| < A_1\}\,, \tag{35}$$

and

$$E_2 = \{t \mid t\in[0,T]\,, |x(t-\tau_1(t))| \geqslant A_1\}\,. \tag{36}$$

In view of (3),(15),(31),(32),(33),(34), (35) and (36),we have

$$\begin{aligned} &\int_0^T |x''(s)|\,ds \\ \le\ & \{\int_0^T |f(t,x(t),x(t-\tau_0(t)))\,x'(t)|\,dt + \int_0^T |g(t,x(t-\tau_1(t)))|\,dt \\ & + \int_0^T |p(t)|\,dt\} \\ \le\ & \sigma T\|x'\|_0 + \int_{E_2} |g(t,x(t-\tau_1(t)))|\,dt \\ & + \int_{E_1} |g(t,x(t-\tau_1(t)))|\,dt + \int_0^T |p(t)|\,dt \\ \le\ & \sigma T\|x'\|_0 + (\beta+\varepsilon)\,T\|x\|_0 + C_0T + T\|P\|_0\,. \\ \le\ & \frac{1}{2}\sigma T\int_0^T |x''(s)|\,ds + (\beta+\varepsilon)\,T[A^* + \frac{T}{4}\int_0^T |x''(s)|\,ds] \\ & + C_0T + T\|P\|_0 \\ \le\ & (\frac{1}{2}\sigma T + \frac{1}{4}(\beta+\varepsilon)\,T^2)\int_0^T |x''(s)|\,ds + (\beta+\varepsilon)\,A^*T + C_0T + T\|P\|_0 \\ \le\ & \frac{1}{2}(\sigma T + \frac{1}{2}(\beta+\varepsilon)\,T^2)\int_0^T |x''(s)|\,ds + C, \end{aligned} \tag{37}$$

for some positive constant C.It is leads from (37) that

$$\int_0^T |x''(s)|\,ds \le \sigma_2, \tag{38}$$

where $\sigma_2 = \frac{C}{1-\sigma_1}, \sigma_1 = \frac{1}{2}\left\{\sigma T + \frac{1}{2}(\beta+\varepsilon)T^2\right\}$. It is easy to see from (31) ,(32) and (38) that

$$\|x\|_0 \le M_0 \text{and} \|x'\|_0 \le M_1, \tag{39}$$

where $M_0 = A^* + \frac{T}{4}\sigma_2$ and $M_1 = \frac{\sigma_2}{2}$.

Now,taking a positive numbers $\overline{D} > \max\{M_0, M_1\} + A^*$ and let

$$\Omega = \left\{x \in X \mid \|x\|_1 < \overline{D}\right\}. \tag{40}$$

From Lemma 1 ,we know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. In terms of valuation of bound of periodic solutions as above, we see that for any $\lambda \in (0,1)$ and any $x \in \partial\Omega, Lx \ne \lambda Nx$. Since for any $x \in \partial\Omega \cap \text{Ker}L, x = \overline{D}\ (> A^*)$ or $x = -\overline{D}$, then in view of (17),we have

$$\begin{aligned} QNx &= \frac{1}{T}\int_0^T \left(-f\left(t, x(t), x(t-\tau_0(t))\right)x'(t) - g\left(t, x(t-\tau_1(t))\right) + p(t)\right)ds \\ &= \frac{1}{T}\int_0^T \left(-g(t,x) + p(t)\right)ds \\ &\ne 0. \end{aligned}$$

In particular, we see that

$$\frac{1}{T}\int_0^T \left(-g\left(t,-\overline{D}\right) + p(t)\right)ds > 0, \text{and} \frac{1}{T}\int_0^T \left(-g\left(t,\overline{D}\right) + p(t)\right)ds < 0.$$

This show that $\deg(JQN, \Omega \cap \text{Ker}L, 0) \ne 0$.In view of Theorem A, There exists a $T-$periodic solution of (2).The proof is complete.

The proof of Thereom 2:.From The proof of Thereom 1 we see that it is suffices to prove that there exist constants N_0 and N_1 ,such that for any T-periodic solution $x(t)$ of (15),

$$\|x\|_0 \le N_0 \text{and} \|x'\|_0 \le N_1. \tag{41}$$

First of all,as the proof of Thereom 1,we can see that has a constant A^*which is independent of $\lambda$and $\xi \in [0,T]$,such that

$$|x(\xi)| \le A^*, \tag{42}$$

$$\begin{aligned} \|x\|_0 &\le |x(\xi)| + \frac{1}{2}\int_0^T |x'(s)|\,ds \\ &\le A^* + \frac{1}{2}\int_0^T |x'(s)|\,ds, \end{aligned} \tag{43}$$

and for $\varepsilon = \frac{1}{2}\left(\frac{2\sigma}{T} - \beta\right)$, there is a constat $A_2 > 0$,such that for $|x(t-\tau_1(t))| > A_2$,

$$|g(t, x(t-\tau_1(t)))| \le (\beta+\varepsilon)|x(t-\tau_1(t))|. \tag{44}$$

and

$$\sigma \int_0^T |x'(s)|^2 ds \le -\int_0^T g(t, x(t-\tau_1(t)))x'(t)\,dt + \int_0^T p(t)x'(t)\,dt. \tag{45}$$

Let

$$C_1 = \max_{0\le t\le T, |x|\le A_3} |g(t,x)|, \tag{46}$$

$$E_1 = \{t \mid t\in[0,T], |x(t-\tau_1(t))| < A_2\}, \tag{47}$$

and

$$E_2 = \{t \mid t\in[0,T], |x(t-\tau_1(t))| \geqslant A_2\}. \tag{48}$$

From (43),(44),(45),(46) , (47) and (48)

$$\begin{aligned}
\sigma \int_0^T |x'(s)|^2 ds &\le \int_{E_1} |g(t, x(t-\tau_1(t)))|\,|x'(t)|\,dt \\
&+ \int_{E_2} |g(t, x(t-\tau_1(t)))|\,|x'(t)|\,dt + \int_0^T |p(t)|\,|x'(t)|\,dt \\
&\le (C_1 + \|p\|_0)\int_0^T |x'(s)|\,ds + (\beta+\varepsilon)\|x\|_0 \int_0^T |x'(s)|\,ds \\
&\le C_2 \int_0^T |x'(s)|\,ds + (\beta+\varepsilon)\left(A^* + \frac{1}{2}\int_0^T |x'(s)|\,ds\right)\int_0^T |x'(s)|\,ds \\
&\le \left(C_2 + \frac{(\beta+\varepsilon)A^*}{2}\right)\int_0^T |x'(s)|\,ds + \frac{(\beta+\varepsilon)}{2}\left(\int_0^T |x'(s)|\,ds\right)^2 \\
&\le \left(C_2 + \frac{(\beta+\varepsilon)A^*}{2}\right)T^{\frac{1}{2}}\left(\int_0^T |x'(s)|^2 ds\right)^{\frac{1}{2}} \\
&+\frac{(\beta+\varepsilon)T}{2}\left(\int_0^T |x'(s)|^2 ds\right).
\end{aligned} \tag{49}$$

where $C_2 = C_1 + \|p\|_0$. It is leads to $\varepsilon = \frac{1}{2}\left(\frac{2\sigma}{T} - \beta\right)$

$$\begin{aligned}
&\left\{\sigma - \frac{(\beta+\varepsilon)T}{2}\right\}\int_0^T \left|x^{'}(s)\right|^2 ds \\
\le\ &\left(C_2 + \frac{(\beta+\varepsilon)A^*}{2}\right)T^{\frac{1}{2}}\left(\int_0^T \left|x^{'}(s)\right|^2 ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{50}$$

Thus

$$\begin{aligned}\int_0^T |x'(s)|^2 ds &\leq \frac{1}{T}\left\{\frac{4C_2T+2(\beta+\varepsilon)A^*T}{2\sigma-\beta T}\right\}^2 \\ &= \frac{1}{T}\left\{\frac{4C_2T+2\sigma A^*+\beta A^*T}{2\sigma-\beta T}\right\}^2. \qquad (51)\end{aligned}$$

In view of (43) and (51),we see that

$$\begin{aligned}\|x\|_0 &\leq A^*+\frac{1}{2}\int_0^T |x'(s)| ds \\ &\leq A^*+\frac{T^{\frac{1}{2}}}{2}\left(\int_0^T |x'(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq N_0, \qquad (52)\end{aligned}$$

where

$$N_0 = A^*+\frac{1}{2}\frac{4C_2T+2\sigma A^*+\beta A^*T}{2\sigma-\beta T}.$$

Let

$$f_{N_0} = \sup_{t\in[0,T],|x|\leq N_0,|y|\leq N_0} f(t,x,y) \qquad (53)$$

and

$$g_{N_0} = \sup_{t\in[0,T],|x|\leq N_0} |g(t,x)|. \qquad (54)$$

From(15), (22) , (52) ,(53) and (54),we have

$$\begin{aligned}\|x'\|_0 &\leq \frac{1}{2}\int_0^T |x''(s)| ds \\ &\leq \frac{1}{2}\{f_{N_0}\int_0^T |x'(s)| ds + g_{N_0}T+T\|p\|_0\} \\ &\leq \frac{1}{2}f_{N_0}T^{\frac{1}{2}}\left(\int_0^T |x'(s)|^2 ds\right)^{\frac{1}{2}} + \frac{g_{N_0}T+T\|p\|_0}{2} \\ &\leq N_1, \qquad (55)\end{aligned}$$

where

$$N_1 = \frac{1}{2}f_{N_0}\frac{4C_2T+2\sigma A^*+\beta A^*T}{4\sigma-2\beta T}+\frac{\beta g_{N_0}T+T\|p\|_0}{2}. \qquad (56)$$

The proof is complete.

The proof of Thereom 3:.Let $x(t)$ be a T-periodic solution of (15) ,In a similar to The proof of Thereom 1 ,we see that there are positive A and $t_0 \in [0,T]$such that for any T-periodic solution $x(t)$of (15),

$$|x(t_0)| \leq A, \qquad (57)$$

and

$$\|x\|_0 \le A + \frac{1}{2}\int_0^T \left|x^{'}(s)\right| ds. \tag{58}$$

Since $x(0) = x(T)$, so that there is $t_1 \in (0, T)$ such that $x^{'}(t_1) = 0$. Getting $\varepsilon = \frac{1}{2}(\frac{1-2\delta T}{T^2} - \beta)$,by (4) there is a constat $A_3 > A$,such that for $|x(t - \tau_1(t))| > A_3$,

$$|g(t, x(t - \tau_1(t)))| \le (\beta + \varepsilon)|x(t - \tau_1(t))|. \tag{59}$$

Let

$$C_3 = \max_{0 \le t \le T, |x| \le A_3} |g(t, x)| \tag{60}$$

From (15) we have

$$x'(t)x''(t) + \lambda f(t, x(t), x(t - \tau_0(t)))(x'(t))^2 + \lambda g(t, x(t - \tau_1(t)))x'(t) = \lambda p(t)x'(t). \tag{61}$$

In view of (59),(60) and (61), for $t \in [t_1, t_1 + T]$ we have

$$\begin{aligned}
\frac{1}{2}(x'(t))^2 &= -\lambda \int_{t_1}^t f_2(s, x(s), x(s - \tau_0(s)))(x'(s))^2 ds \\
&\quad -\lambda \int_{t_1}^t g(s, x(s - \tau_1(s)))x'(s) ds + \lambda \int_{t_1}^t p(s)x^{'}(s) ds \\
&\le \int_{t_1}^{t_1+T} |f_2(s, x(s), x(s - \tau_0(s)))| |x'(s)|^2 ds \\
&\quad + \int_{t_1}^{t_1+T} |g(s, x(s - \tau_1(s)))| |x'(s)| ds + \int_{t_1}^{t_1+T} |p(s)| |x'(s)| ds \\
&= \int_0^T |f_2(s, x(s), x(s - \tau_0(s)))| |x'(s)|^2 ds + \int_0^T |g(s, x(s - \tau_1(s)))| |x'(s)| ds \\
&\quad + \int_0^T |p(s)| |x'(s)| ds \\
&\le \delta \int_0^T |x'(s)|^2 ds + (\beta + \varepsilon)T\|x\|_0\|x'\|_0 + C_3T\|x'\|_0 + T\|p\|_0\|x'\|_0 \\
&\le \delta T\|x'\|_0^2 + (\beta + \varepsilon)T\left\{A + \frac{1}{2}\int_0^T |x'(s)| ds\right\}\|x'\|_0 + C_3T\|x'\|_0 + T\|p\|_0\|x'\|_0 \\
&\le \delta T\|x'\|_0^2 + \frac{(\beta + \varepsilon)T^2}{2}\|x'\|_0^2 + \{(\beta + \varepsilon)TA + C_3T + T\|p\|_0\}\|x'\|_0 \\
&\le \left(\delta T + \frac{(\beta + \varepsilon)T^2}{2}\right)\|x'\|_0^2 + \{(\beta + \varepsilon)TA + C_3T + T\|p\|_0\}\|x'\|_0
\end{aligned} \tag{62}$$

From (58) and (62) ,we see that

$$\|x'\|_0^2 \leq \left\{2\delta T + (\beta+\varepsilon) T^2\right\} \|x'\|_0^2 + 2\{(\beta+\varepsilon) TA + C_3 T + T\|p\|_0\} \|x'\|_0 , \quad (63)$$

It is following that

$$\|x'\|_0 \leq W_1, \quad (64)$$

and

$$\begin{aligned} \|x\|_0 &\leq A + \frac{1}{2}\int_0^T |x'(s)|\, ds \\ &\leq W_2 \end{aligned} \quad (65)$$

where

$$W_1 = \frac{2\{(\beta+\varepsilon) TA + C_3 T + T\|p\|_0\}}{1 - 2\delta T - (\beta+\varepsilon) T^2}, \quad (66)$$

and

$$W_2 = A + \frac{1}{2} T W_1. \quad (67)$$

We can prove the remainder parts by the same way of Thereom 1.The proof is complete.

EXAMPLE.Consider a Liĕnard equation of the form

$$x''(t) + \{2 + \sin t + (x(t))^2 + (x(t-\cos t))^2\} x'(t-\sin t)$$

$$+ \frac{2x(t-\sin t)^{\frac{1}{3}} + \cos t}{7} x(t-\sin t)^{\frac{2}{3}} = 2\cos t - \frac{6}{7}\sin t. \quad (68)$$

Since $\beta = 2/7$ and $\sigma = 1$,so that $\beta = 2/7 < 2\sigma/2\pi = 1/\pi$, thus from Thereom 2 we see that (68) has a 2π-periodic solution. .Furthermore, this solution is nontrivial since $y(t) \equiv 0$ is not a solution.of (68).

We remark that (68) can not be expressed as the form of (1),that is, the ruselts of [2] can not used to (68) , thus our results in this paper are new.Also,It is easy to see from our results in this paper that the conditions $f_1 < \frac{1}{T}$ and $r < \frac{1-f_1T}{\beta_1 T^2}$ in Thereom 1 of [4] can be relaced by the weaker conditions $f_1 < \frac{2}{T}$ and $r < 2\left(\frac{2-f_1T}{\beta_1 T^2}\right)$,the condition $r < \frac{\sigma}{\beta_1 T}$ in Thereom 2 of [4] can be relaced by the weaker condition $r < \frac{2\sigma}{\beta_1 T}$and the condition $r < \frac{1-2\delta T}{2\beta_1 T^2}$in Thereom 3 of [4] can be relaced by the weaker condition $r < \frac{1-2\delta T}{\beta_1 T^2}$,.

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