A Simple Note about Matrices
Symmetric in Signs

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Abstract

This paper proves that any real matrix symmetric in signs, \{+, -, 0\}, has no pure imaginary eigenvalues, so that if it is also of full rank, then it is hyperbolic – information useful in dynamical systems.

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1 Introduction

This short note is concerned with the eigenvalues of real matrices with symmetric sign patterns, which will henceforth be denoted by $H$. The study of general "sign pattern matrices" - that is, matrices that are distinguished only by the signs of their entries, \{+, -, 0\} - has been in existence for decades (cf. [1, 3]), with a recently renewed interest due to the M-theory in quantum physics (see, e.g., [5]); major research interests on such "qualitative matrices" have been about the properties of positive matrices or the conditions of matrix invertibility (see, e.g., [4, 6]). Our \{H\} here actually contain positive matrices as a special case.

We will show that such matrices \{H\} can not have eigenvalues that lie on the imaginary axis of the complex-plane, except possibly at the point of origin. Our motivation here is that, if of full rank,\{H\} are then hyperbolic, which is a theme of interest in the study of dynamical systems (cf., e.g., [2], 107).

Section 2 below will derive the proposition, and Section 3 will draw a summary.
2 The Proposition

Definition 1 A matrix $H_{n \times n}$ will be termed (here in this paper by the author) "quasi-hyperbolic" if each of its eigenvalues $\lambda_j, j = 1, \ldots, n$, satisfies $\lambda_j^2 \notin (-\infty, 0)$; if in addition $\lambda_j \neq 0, \forall j = 1, \ldots, n$, then $H$ is (as defined in the textbook) hyperbolic.

Proposition 1 Assume that matrix $H_{n \times n}$ is symmetric in signs, i.e., $\text{sgn} h_{ij} = \text{sgn} h_{ji} \in \{1, -1, 0\} \ \forall i \forall j \in \{1, \ldots, n\}$. Then $H_{n \times n}$ is quasi-hyperbolic.

Proof. Let $H \equiv (h_{ij})_{n \times n}$ be as assumed. Let $\epsilon > 0$, and $\hat{I} \equiv \left(\delta_{ij}\right)_{n \times n}$ with $\delta_{ij} = 10^{-\kappa} \delta_{ij}, \kappa = \delta_{1j} \epsilon$. (Kronecker delta)

Consider $\tilde{H} \equiv \left(\tilde{h}_{ij}\right) := \hat{I}H\hat{I}^{-1}$, which shares the same set of eigenvalues as $H$. Let $b \equiv (b_i)_{n \times 1} \in \mathbb{R}^n$, and consider $e_1 \tilde{H}^2 b$, where $e_1 \equiv (\delta_{ij})_{1 \times n}$. We have:

$$e_1 \tilde{H}^2 b = \left( \sum_{j=1}^{n} h_{1j}h_{j1}, \ \hat{\delta}_{11} \sum_{j=1}^{n} h_{1j}h_{j2}, \ \cdots, \ \hat{\delta}_{11} \sum_{j=1}^{n} h_{1j}h_{jn} \right) b$$

$$= b_1 \sum_{j=1}^{n} h_{1j}h_{j1} + \hat{\delta}_{11} \left( \sum_{i=2}^{n} b_i \sum_{j=1}^{n} h_{1j}h_{ji} \right); \quad (2)$$

choose $\epsilon$ and thus $\hat{\delta}_{11}$ small; since $\sum_{j=1}^{n} h_{1j}h_{j1} \geq 0$, we have $\text{sgn} \left( e_1 \tilde{H}^2 b \right) \cdot \text{sgn} (b_1) \geq 0$; thus, $\tilde{H}^2 \equiv \hat{I}H^2\hat{I}^{-1}$ has no negative eigenvalues, and hence $H$ has no eigenvalues $\pm ri \ \forall r > 0$; i.e., $H$ is quasi-hyperbolic. \[\blacksquare\]

Example 1 As a simple illustration, consider the matrix

$$\begin{pmatrix} 1 & a \\ b & -1 \end{pmatrix}, \quad (3)$$

which has eigenvalues

$$\lambda = \pm \sqrt{1 + ab}; \quad (4)$$

thus, $\lambda^2 < 0$ implies that $ab < 0$.

Remark 1 Since $H$ has no pure imaginary eigenvalues, any continuous sign-preserving rank-preserving transformation of $H$ must leave all the eigenvalues of $H$ staying on the same side of the imaginary axis on the complex-plane. As such, if $H$ represents a linearized dynamical system around an equilibrium point, then its stability properties also remain invariant under such transformations. Two particular cases here are: (1) (small) perturbation of $H$; (2) multiplication of $H$ by a positive diagonal matrix - - a standard modeling of price dynamics in economics.
3 Summary

The study of qualitative matrices has been a special field in linear algebra for decades. Our paper here has contributed to the knowledge of a special class of matrices $\{H\}$; as many relations are mutual, $\{H\}$, being symmetric in signs, constitute a large class of matrix modeling in applications. Furthermore, once a matrix $H$ is ascertained to be of full rank, then it is hyperbolic - - some information that is very pertinent to dynamical systems.

4 References


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