International Mathematical Forum, 1, 2006, no. 18, 893-895

A Simple Note about Matrices Symmetric in Signs

Gregory L. Light

Providence College, Providence
Rhode Island 02918 U.S.A.
glight@providence.edu

Abstract

This paper proves that any real matrix symmetric in signs, {+, -, 0}, has no pure imaginary eigenvalues, so that if it is also of full rank, then it is hyperbolic – information useful in dynamical systems.

Mathematics Subject Classification: 15A18, 37C75, 37D10

Keywords: Sign pattern matrix, hyperbolicity, imaginary eigenvalues

1 Introduction

This short note is concerned with the eigenvalues of real matrices with symmetric sign patterns, which will henceforth be denoted by H. The study of general "sign pattern matrices" - - that is, matrices that are distinguished only by the signs of their entries, $\{+, -, 0\}$ - - has been in existence for decades $(cf. [1, 3])$, with a recently renewed interest due to the M-theory in quantum physics (see, e.g., [5]); major research interests on such "qualitative matrices" have been about the properties of positive matrices or the conditions of matrix invertibility $(see, e.g., [4, 6])$. Our $\{H\}$ here actually contain positive matrices as a special case.

We will show that such matrices $\{H\}$ can not have eigenvalues that lie on the imaginary axis of the complex-plane, except possibly at the point of origin. Our motivation here is that, if of full rank,$\{H\}$ are then hyperbolic, which is a theme of interest in the study of dynamical systems $(cf., e.g., [2], 107)$.

Section 2 below will derive the proposition, and Section 3 will draw a summary.

2 The Proposition

Definition 1 *A matrix $H_{n\times n}$ will be termed (here in this paper by the author) "quasi-hyperbolic" if each of its eigenvalues $\lambda_j, j = 1, \cdots, n$, satisfies $\lambda_j^2 \notin (-\infty, 0)$; if in addition $\lambda_j \neq 0, \forall j = 1, \cdots, n$, then H is (as defined in the textbook) hyperbolic.*

Proposition 1 *Assume that matrix $H_{n\times n}$ is symmetric in signs, i.e., $sgnh_{ij} = sgnh_{ji} \in \{1, -1, 0\}\ \forall i \forall j \in \{1, \cdots, n\}$. Then $H_{n\times n}$ is quasi-hyperbolic.*

Proof. Let $H \equiv (h_{ij})_{n\times n}$ be as assumed. Let $\epsilon > 0$, and $\hat{I} \equiv \left(\hat{\delta}_{ij}\right)_{n\times n}$ with

$$\hat{\delta}_{ij} := 10^{-\kappa}\delta_{ij},\ \kappa \equiv \frac{\delta_{1j}}{\epsilon}.(Kronecker\ delta) \tag{1}$$

Consider $\tilde{H} \equiv \left(\tilde{h}_{ij}\right) := \hat{I}H\hat{I}^{-1}$, which shares the same set of eigenvalues as H. Let $\mathbf{b} \equiv (b_i)_{n\times 1} \in \mathbb{R}^n$, and consider $\mathbf{e}_1\tilde{H}^2\mathbf{b}$, where $\mathbf{e}_1 \equiv (\delta_{1j})_{1\times n}$. We have:

$$\begin{aligned} \mathbf{e}_1\tilde{H}^2\mathbf{b} &= \left(\sum_{j=1}^n h_{1j}h_{j1},\quad \hat{\delta}_{11}\sum_{j=1}^n h_{1j}h_{j2},\quad \cdots,\quad \hat{\delta}_{11}\sum_{j=1}^n h_{1j}h_{jn} \right)\mathbf{b} \\ &= b_1\sum_{j=1}^n h_{1j}h_{j1} + \hat{\delta}_{11}\left(\sum_{i=2}^n b_i\sum_{j=1}^n h_{1j}h_{ji}\right); \end{aligned} \tag{2}$$

choose ϵ and thus $\hat{\delta}_{11}$ small; since $\sum_{j=1}^n h_{1j}h_{j1} \geq 0$, we have $sgn\left(\mathbf{e}_1\tilde{H}^2\mathbf{b}\right) \cdot sgn\,(b_1) \geq 0$; thus, $\tilde{H}^2 \equiv \hat{I}H^2\hat{I}^{-1}$ has no negative eigenvalues, and hence H has no eigenvalues $\pm ri\ \forall r > 0$; i.e., H is quasi-hyperbolic. ■

Example 1 *As a simple illustration, consider the matrix*

$$\begin{pmatrix} 1 & a \\ b & -1 \end{pmatrix}, \tag{3}$$

which has eigenvalues

$$\lambda = \pm\sqrt{1 + ab}; \tag{4}$$

thus, $\lambda^2 < 0$ implies that $ab < 0$.

Remark 1 *Since H has no pure imaginary eigenvalues, any continuous sign-preserving rank-preserving transformation of H must leave all the eigenvalues of H staying on the same side of the imaginary axis on the complex-plane. As such, if H represents a linearized dynamical system around an equilibrium point, then its stability properties also remain invariant under such transformations. Two particular cases here are:* (1) *(small) perturbation of H;* (2) *multiplication of H by a positive diagonal matrix - - a standard modeling of price dynamics in economics.*

3 Summary

The study of qualitative matrices has been a special field in linear algebra for decades. Our paper here has contributed to the knowledge of a special class of matrices $\{H\}$; as many relations are mutual, $\{H\}$, being symmetric in signs, constitute a large class of matrix modeling in applications. Furthermore, once a matrix H is ascertained to be of full rank, then it is hyperbolic - - some information that is very pertinent to dynamical systems.

4 References

[1] A. Brieden and S. Cokus, On the hardness of efficiently approximating maximal non-L submatrices, Linear Algebra and its Applications, 377 (2004), 195-205.

[2] J.K. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, 1980.

[3] F.J. Hall, Z. Li and D. Wang, Symmetric sign pattern matrices that require unique inertia, Linear Algebra and its Applications, 338 (1) (2001), 153-169.

[4] C.R. Johnson, T.J. Laffey and R. Loewy, The real and the symmetric nonnegative inverse eigenvalue problems are different, Proc. Amer. Math. Soc. 124 (1996), 3647-3651.

[5] K. Rietsch, Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties, J. Amer. Math. Soc. 16 (2003), 363-392.

[6] J.-Y. Shao and H.-Y. Shan, Matrices with signed generalized inverses, Linear Algebra and its Applications, 322 (1) (2001), 105-127.

Received: September 23, 2005