A NOTE ON JORDAN^{*}– DERIVATIONS IN SEMIPRIME RINGS WITH INVOLUTION¹

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Abstract

In this paper we prove the following result. Let R be a 6–torsion free semiprime $*$ –ring and let $D : R \rightarrow R$ be an additive mapping satisfying the relation $D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x)$, for all pairs $x, y \in R$. In this case D is a Jordan $*$ –derivation.

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Throughout, R will represent an associative ring. Given an integer $n \geq 2$, a ring R is said to be n –torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all pairs $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$ –ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \rightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on 2–torsion free prime ring is a derivation. A brief proof of Herstein’s theorem can be found in [1]. Cusack [6] generalized Herstein’s theorem to 2–torsion free semiprime ring (see [2] for an alternative

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proof). An additive mapping $D : R \rightarrow R$, where R is a $*$ -ring, is called a $*$ -derivation in case $D(xy) = D(x)y^* + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan $*$ -derivation if $D(x^2) = D(x)x^* + xD(x)$ is fulfilled for all $x \in R$. By our knowledge $*$ -derivations and Jordan $*$ -derivations were first mentioned in [3]. Note that the mapping $x \rightarrow ax^* - xa$, where a is a fixed element in R , is a Jordan $*$ -derivation; such Jordan $*$ -derivations are said to be inner. One might expect that any Jordan $*$ -derivation on a 2-torsion free semiprime $*$ -ring is a $*$ -derivation, but this is not the case. It is easy to prove that there are no nonzero $*$ -derivations on noncommutative prime $*$ -rings (see [3] for the details). The study of Jordan $*$ -derivations has been motivated by the problem of the representability of quadratic forms by bilinear forms. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. Of course, any derivation is a Jordan triple derivation. Moreover, one can easily prove that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation (see [1] for the details). Brešar [4] has proved the following result.

THEOREM A Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ is a Jordan triple derivation. In this case D is a derivation.

In case we have a Jordan $*$ -derivation $D : R \rightarrow R$, where R is a 2-torsion free $*$ -ring, one can easily prove that

$$D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x), \quad (1)$$

holds for all pairs $x, y \in R$. It seems natural to ask under what additional assumptions the converse is true. More precisely, under what additional assumptions an additive mapping D which maps a $*$ -ring R into itself satisfying the relation (1) is a Jordan derivation. The equation (1) has been considered in [5, 8 – 11]. It is our aim in this paper to prove the following result which was inspired by Theorem A. Our approach differs from those used by Brešar in [4].

THEOREM 1. Let R be a 6-torsion free semiprime $*$ -ring and let $D : R \rightarrow R$ be an additive mapping satisfying the relation

$$D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x),$$

for all pairs $x, y \in R$. In this case D is a Jordan $*$ -derivation.

Let us point out that in case R has the identity element the proof of Theorem 1 can be proved immediately without assuming that R is semiprime. For the proof of Theorem 1 we shall need the following lemma.

LEMMA 1. Let R be a 2–torsion free semiprime * –ring. Suppose that the relation $ax^*b^* + bxa = 0$ holds for all $x \in R$ and some $a, b \in R$. In this case $ab = ba = 0$ is fulfilled. In case R is prime then either $a = 0$ or $b = 0$.

Proof. We have the relation

$$ax^*b^* + bxa = 0, x \in R. \quad (2)$$

Putting in the above relation ybx for x and applying (2) we obtain

$$\begin{aligned} 0 &= a(ybx)^*b^* + bybxa = (ax^*b^*)y^*b^* + bybxa = \\ &= -bx(ay^*b^*) + bybxa = bxbya + bybxa. \end{aligned}$$

We have therefore proved that

$$bxbya + bybxa = 0, x, y \in R. \quad (3)$$

In particular for $y = x$ the above relation reduces to

$$bxbxa = 0, x \in R$$

since we have assumed that R is 2–torsion free. Applying (2) we obtain from the above relation

$$bxa x^*b^* = 0, x \in R. \quad (4)$$

Now substituting in (3) xay for y and applying (2) and (4) we obtain

$$0 = bx(bxa)ya + bxaybxa =$$

$$-(bxa x^*b^*)ya + bxaybxa = bxaybxa.$$

We have therefore proved that $(bxa)y(bxa) = 0$ holds for all pairs $x, y \in R$ whence it follows

$$bxa = 0, x \in R. \quad (5)$$

From the above relation one obtains $(ab)x(ab) = 0$, for all $x \in R$ which gives $ab = 0$. Similarly, one obtains $ba = 0$. In case R is prime it follows from (5) that either $a = 0$ or $b = 0$. The proof is complete.

Proof of Theorem 1. We have therefore the relation

$$D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x), x, y \in R. \quad (6)$$

The substitution xyx for y in the above relation gives

$$\begin{aligned} D(x^2yx^2) &= D(x)x^*y^*x^{*2} + xD(xyx)x^* + x^2yxD(x) = \\ &D(x)x^*y^*x^{*2} + xD(x)y^*x^{*2} + x^2D(y)x^{*2} + \\ &x^2yD(x)x^* + x^2yxD(x), x, y \in R.. \end{aligned}$$

We have therefore

$$\begin{aligned} D(x^2yx^2) &= D(x)x^*y^*x^{*2} + xD(x)y^*x^{*2} + \\ &x^2D(y)x^{*2} + x^2yD(x)x^* + x^2yxD(x), x, y \in R. \quad (7). \end{aligned}$$

On the other hand the substitution x^2 for x in (6) gives

$$D(x^2yx^2) = D(x^2)y^*x^2 + x^2D(y)x^{*2} + x^2yD(x^2), x, y \in R. \quad (8)$$

Subtracting (7) from (8) we obtain

$$A(x)y^*x^{*2} + x^2yA(x) = 0, x, y \in R, \quad (9)$$

where $A(x)$ stands for $D(x^2) - D(x)x^* - xD(x)$. From the relation above it follows according to Lemmal that

$$A(x)x^2 = 0, x \in R \quad (10)$$

and

$$x^2A(x) = 0, x \in R. \quad (11)$$

The substitution $x + y$ for x in the relation (10) gives

$$\begin{aligned} &A(x)y^2 + A(y)x^2 + B(x, y)x^2 + B(x, y)y^2 + \\ &A(x)(xy + yx) + A(y)(xy + yx) + B(x, y)(xy + yx) = 0, x, y \in R \quad (12) \end{aligned}$$

where $B(x, y)$ stands for $D(xy + yx) - D(x)y^* - D(y)x^* - xD(y) - yD(x)$. Putting $-x$ for x in the above relation and comparing the relation so obtained with the relation (12) we obtain since we have assumed that R is 2–torsion free

$$B(x, y)x^2 + B(x, y)y^2 + A(x)(xy + yx) + A(y)(xy + yx) = 0, x, y \in R \quad (12)$$

The substitution $2x$ for x in the above relation gives

$$4B(x, y)x^2 + B(x, y)y^2 + 4A(x)(xy + yx) + A(y)(xy + yx) = 0, x, y \in R \quad (13)$$

Subtracting the relation (12) from the relation (13) one obtains $3A(x)(xy + yx) + 3B(x, y)x^2 = 0, x, y \in R$ which gives since R is 3–torsion free

$$A(x)(xy + yx) + B(x, y)x^2 = 0, x, y \in R. \quad (14)$$

Right multiplication of the above relation by $A(x)x$ gives because of (11)

$$A(x)xyA(x)x + A(x)yxA(x)x = 0, x, y \in R. \quad (15)$$

Substituting in the above relation yx for y and multiplying the relation (15) from the left side by x we obtain $(xA(x)x)y(xA(x)x) = 0$, for all pairs $x, y \in R$ whence it follows $xA(x)x = 0, x \in R$. Now the relation (15) reduces to $(A(x)x)y(A(x)x) = 0$, for all pairs $x, y \in R$ which gives

$$A(x)x = 0, x \in R. \quad (16)$$

Now the relation (14) reduces to $A(x)yx + B(x, y)x^2 = 0, x, y \in R$. Right multiplication of this relation by $A(x)$ and left multiplication by x gives $(xA(x))y(xA(x)) = 0$, for all pairs $x, y \in R$ which gives

$$xA(x) = 0, x \in R. \quad (17)$$

From the relation (16) one obtains

$$A(x)y + B(x, y)x = 0, x, y \in R.$$

Right multiplication of the above relation by $A(x)$ gives because of (17) $A(x)yA(x) = 0$, for all pairs $x, y \in R$ which gives $A(x) = 0$, for all $x \in R$. In other words $D(x^2) = D(x)x^* + xD(x)$, for all $x \in R$ which means that D is a Jordan*–derivation. The proof of the theorem is complete.

We feel that Theorem 1 can be proved without the assumption that R is 3–torsion free but unfortunately we were unable to prove the result without this assumption.

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