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ON HELICES IN THE DOUBLY ISOTROPIC SPACE $I_3^{(2)}$

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Abstract

T. Ikawa obtained in [4] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \qquad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures k and τ of a time-like curve c on the Lorentzian manifold M are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [3] T. Ikawa's this result, i.e. $k(s)$ and $\tau(s)$ are variable, but $\frac{k(s)}{\tau(s)}$ is constant.

In [1] H.Balgetir, M.Bektas and M.Ergüt obtained a geometric characterization of Null Frenet curve with constant ratio of curvature and torsion (called null general helix).

In this paper, making use of method in [1,3,4] , we obtained characterizations of a curve with respect to the Frenet frame of Doubly Isotropic Space $I_3^{(2)}$.

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1 Introduction

Differential gometry of Doubly Isotropic Space $I_3^{(2)}$ has been studied in detail in [2],[5] and [6].

The absolute of this space consists of a plane w (the absolute plane), a line $f \subset w$ (the absolute line) and a point $U \in f$ (the absolute point). Projective transformations preserving the absolute (w, f, U) form a group which is called the similarity group in $I_3^{(2)}$. This group can be written in nonhomogeneous coordinates in the form

$$x' = a_1 x + a_2,$$
$$y' = b_1 x + b_2 y + b_3, \tag{1.1}$$
$$z' = c_1 x + c_2 y + c_3 z + c_4$$

where $a_i, b_j, c_k (i = 1, 2; j = 1, 2, 3; k = 1, 2, 3, 4)$ are real constants.

The subgroup of this group defined by $a_1 = b_2 = c_3 = 1$ is the isometry group of space $I_3^{(2)}$.

In this paper we shall use notations from [2, 6].

Let now $c : I \to I_3^{(2)}$, $I \subseteq R$ be a curve given in coordinates by

$$x = s,\ y = y(s),\ z = z(s), \tag{1.2}$$

where s is the invariant parameter which in the geometry of the space $I_3^{(2)}$ coincides with the abscisa of the curve's point. Derivatives with respect to s will be denoted by primes.

To any curve of the class $C^r (r \geq 3)$ the Frenet trihedron $\{T(s), N(s), B(s)\}$ is assigned where

$$T(s) = (1, y'(s), z'(s)),\ N(s) = (0, 1, \frac{z''(s)}{y''(s)}),\ B(s) = (0, 0, 1).$$

$T(s)$ is called the unit tangent vector, $N(s)$ the unit principal normal vector and $B(s)$ the unit binormal vector.

The curvature $k(s)$ and the torsion $\tau(s)$ of the curve c are defined by

$$k(s) = y''(s),\ \ \tau(s) = (\frac{z''(s)}{y''(s)})'.$$

For these vector fields the following Frenet formulas hold

$$\nabla_{T(s)} T(s) = k(s) N(s),$$
$$\nabla_{T(s)} N(s) = \tau(s) B(s), \tag{1.3}$$
$$\nabla_{T(s)} B(s) = 0.$$

2 The Characterizations of Curves on Doubly Isotropic Space.

Definition 2.1. Let c be a curve of Doubly Isotropic Space $I_3^{(2)}$ and $\{T(s), N(s), B(s)\}$ be the Frenet frame of Doubly Isotropic Space $I_3^{(2)}$ along c. If k and τ are positive constants along c, then c is called a circular helix with respect to the Frenet frame.

Definition 2.2. Let c be a curve of Doubly Isotropic Space $I_3^{(2)}$ and $\{T(s), N(s), B(s)\}$ be the Frenet frame of Doubly Isotropic Space $I_3^{(2)}$ along c. A curve c such that

$$\frac{k(s)}{\tau(s)} = const$$

is called a general helix with respect to Frenet frame.

Theorem 2.1. Let c be a curve of Doubly Isotropic Space $I_3^{(2)}$. c is a general helix with respect to the Frenet frame $\{T(s), N(s), B(s)\}$ if and only if

$$\nabla_{T(s)}\nabla_{T(s)}\nabla_{T(s)}T(s) - K(s)\nabla_{T(s)}T(s) = 3k'(s)\nabla_{T(s)}N(s) \tag{2.1}$$

where $K(s) = \frac{k''(s)}{k(s)}$.

Proof. Suppose that c is general helix with respect to the Frenet frame $\{T(s), N(s), B(s)\}$. Then from (1.3) , we have

$$\nabla_{T(s)}\nabla_{T(s)}\nabla_{T(s)}T(s) = k''(s)N(s) + (2k'(s)\tau(s) + k(s)\tau'(s))B(s). \tag{2.2}$$

Now, since c is general helix with respect to the Frenet frame

$$\frac{k(s)}{\tau(s)} = const$$

and this upon the derivation gives rise to

$$k'(s)\tau(s) = k(s)\tau'(s). \tag{2.3}$$

If we substitute the equations (2.3),

$$N(s) = \frac{1}{k(s)}\nabla_{T(s)}T(s), \tag{2.4}$$

and

$$B(s) = \frac{1}{\tau(s)}\nabla_{T(s)}N(s) \tag{2.5}$$

in (2.2), we obtain (2.1).

Conversely let us assume that the equation (2.1) holds. We show that the curve c is a general helix. Differentiating covariantly (2.4) we obtain

$$\nabla_{T(s)}N(s) = -\frac{k'(s)}{k^2(s)}\nabla_{T(s)}T(s) + \frac{1}{k(s)}\nabla_{T(s)}\nabla_{T(s)}T(s) \tag{2.6}$$

and so

$$\nabla_{T(s)}\nabla_{T(s)}N(s) = \left(-\frac{k'(s)}{k^2(s)}\right)'\nabla_{T(s)}T(s) - 2\frac{k'(s)}{k^2(s)}\nabla_{T(s)}\nabla_{T(s)}T(s)$$

$$+\frac{1}{k(s)}\nabla_{T_{(s)}}\nabla_{T_{(s)}}\nabla_{T_{(s)}}T(s). \tag{2.7}$$

If we use (2.1) in (2.7) and make some calculations, we have

$$\nabla_{T_{(s)}}\nabla_{T_{(s)}}N(s)=\left[\left(-\frac{k'(s)}{k^2(s)}\right)'+\frac{K(s)}{k(s)}\right]\nabla_{T_{(s)}}T(s)-2\frac{(k'(s))^2}{k^2(s)}N(s)$$

$$+\frac{k'(s)\tau(s)}{k(s)}B(s). \tag{2.8}$$

Also we obtain

$$\nabla_{T_{(s)}}\nabla_{T_{(s)}}N(s)=\tau'(s)B(s) \tag{2.9}$$

since (2.8) and (2.9) are equal, routine calculations show that c is a general helix.

Theorem 2.2. Let c be a curve of Doubly Isotropic Space $I_3^{(2)}$. c is a general helix with respect to the Frenet frame $\{T(s),N(s),B(s)\}$, if and only if

$$\nabla_{T_{(s)}}\nabla_{T_{(s)}}\nabla_{T_{(s)}}T(s)-K(s)\nabla_{T_{(s)}}T(s)=3\lambda\tau'(s)\nabla_{T_{(s)}}N(s) \tag{2.10}$$

where $K(s)=\frac{k''(s)}{k(s)}$ and $\lambda=\frac{k(s)}{\tau(s)}=const.$

Proof. It is similar to the proof of Theorem 2.1.

Corollary 2.1. Let c be a curve of Doubly Isotropic Space $I_3^{(2)}$. c is a circular helix with respect to the Frenet frame $\{T(s),N(s),B(s)\}$, if and only if

$$T(s)=\frac{1}{2}s^2+s+d \tag{2.11}$$

where d is a constant

Proof. From the hypotesis of corollary 2.1 and since c is a circular helix, we can show easily (2.11).

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