ON HELICES IN THE DOUBLY
ISOTROPIC SPACE $I_3^{(2)}$

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Abstract

T. Ikawa obtained in [4] the following characteristic ordinary differential equation

$$\nabla_X \nabla_X \nabla_X X - K \nabla_X X = 0, \quad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures $k$ and $\tau$ of a time-like curve $c$ on the Lorentzian manifold $M$ are constant.

N. Ekmeç and H. H. Hacisalihoğlu generalized in [3] T. Ikawa’s this result, i.e. $k(s)$ and $\tau(s)$ are variable, but $\frac{k(s)}{\tau(s)}$ is constant.

In [1] I. Balgetir, M. Bektaş and M. Ergüt obtained a geometric characterization of Null Frenet curve with constant ratio of curvature and torsion (called null general helix).

In this paper, making use of method in [1,3,4], we obtained characterizations of a curve with respect to the Frenet frame of Doubly Isotropic Space $I_3^{(2)}$.

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1 Introduction

Differential geometry of Doubly Isotropic Space $I_3^{(2)}$ has been studied in detail in [2],[5] and [6].

The absolute of this space consists of a plane $w$ (the absolute plane), a line $f \subset w$ (the absolute line) and a point $U \in f$ (the absolute point). Projective transformations preserving the absolute $(w,f,U)$ form a group which is called the similarity group in $I_3^{(2)}$. This group can be written in nonhomogeneous coordinates in the form
\[ x' = a_1 x + a_2, \]
\[ y' = b_1 x + b_2 y + b_3, \]  
\[ z' = c_1 x + c_2 y + c_3 z + c_4, \]  
(1.1)

where \( a_i, b_j, c_k (i = 1, 2; j = 1, 2, 3; k = 1, 2, 3, 4) \) are real constants.

The subgroup of this group defined by \( a_1 = b_2 = c_3 = 1 \) is the isometry group of space \( I_3^{(2)} \).

In this paper we shall use notations from [2, 6].

Let now \( c : I \to I_3^{(2)}, I \subseteq R \) be a curve given in coordinates by

\[ x = s, \quad y = y(s), \quad z = z(s), \]  
(1.2)

where \( s \) is the invariant parameter which in the geometry of the space \( I_3^{(2)} \) coincides with the abscissa of the curve’s point. Derivatives with respect to \( s \) will be denoted by primes.

To any curve of the class \( C^r (r \geq 3) \) the Frenet trihedron \( \{ T(s), N(s), B(s) \} \) is assigned where

\[ T(s) = (1, y'(s), z'(s)), \quad N(s) = (0, 1, \frac{z''(s)}{y''(s)}), \quad B(s) = (0, 0, 1). \]

\( T(s) \) is called the unit tangent vector, \( N(s) \) the unit principal normal vector and \( B(s) \) the unit binormal vector.

The curvature \( k(s) \) and the torsion \( \tau(s) \) of the curve \( c \) are defined by

\[ k(s) = y''(s), \quad \tau(s) = \left( \frac{z''(s)}{y''(s)} \right)' . \]

For these vector fields the following Frenet formulas hold

\[ \nabla_{T(s)} T(s) = k(s) N(s), \]
\[ \nabla_{T(s)} N(s) = \tau(s) B(s), \]  
\[ \nabla_{T(s)} B(s) = 0. \]  
(1.3)

2 The Characterizations of Curves on Doubly Isotropic Space.

Definition 2.1. Let \( c \) be a curve of Doubly Isotropic Space \( I_3^{(2)} \) and \( \{ T(s), N(s), B(s) \} \) be the Frenet frame of Doubly Isotropic Space \( I_3^{(2)} \) along \( c \). If \( k \) and \( \tau \) are positive constants along \( c \), then \( c \) is called a circular helix with respect to the Frenet frame.
Definition 2.2. Let $c$ be a curve of Doubly Isotropic Space $I^{(2)}_3$ and \{\(T(s), N(s), B(s)\)\} be the Frenet frame of Doubly Isotropic Space $I^{(2)}_3$ along $c$. A curve $c$ such that
\[
\frac{k(s)}{\tau(s)} = \text{const}
\]
is called a general helix with respect to Frenet frame.

**Theorem 2.1.** Let $c$ be a curve of Doubly Isotropic Space $I^{(2)}_3$. $c$ is a general helix with respect to the Frenet frame \{\(T(s), N(s), B(s)\)\} if and only if
\[
\nabla_{T(s)} \nabla_{T(s)} \nabla_{T(s)} T(s) - K(s) \nabla_{T(s)} T(s) = 3k'(s) \nabla_{T(s)} N(s) \tag{2.1}
\]
where $K(s) = \frac{k''(s)}{k(s)}$.

Proof. Suppose that $c$ is general helix with respect to the Frenet frame \{\(T(s), N(s), B(s)\)\}. Then from (1.3), we have
\[
\nabla_{T(s)} \nabla_{T(s)} \nabla_{T(s)} T(s) = k''(s) N(s) + (2k'(s) \tau(s) + k(s) \tau'(s)) B(s) \tag{2.2}
\]
Now, since $c$ is general helix with respect to the Frenet frame
\[
\frac{k(s)}{\tau(s)} = \text{const}
\]
and this upon the derivation gives rise to
\[
k'(s) \tau(s) = k(s) \tau'(s). \tag{2.3}
\]
If we substitute the equations (2.3),
\[
N(s) = \frac{1}{k(s)} \nabla_{T(s)} T(s), \tag{2.4}
\]
and
\[
B(s) = \frac{1}{\tau(s)} \nabla_{T(s)} N(s) \tag{2.5}
\]
in (2.2), we obtain (2.1).

Conversely let us assume that the equation (2.1) holds. We show that the curve $c$ is a general helix. Differentiating covariantly (2.4) we obtain
\[
\nabla_{T(s)} \nabla_{T(s)} N(s) = -\frac{k'(s)}{k^2(s)} \nabla_{T(s)} T(s) + \frac{1}{k(s)} \nabla_{T(s)} \nabla_{T(s)} T(s) \tag{2.6}
\]
and so
\[
\nabla_{T(s)} \nabla_{T(s)} N(s) = \left( -\frac{k'(s)}{k^2(s)} \right)' \nabla_{T(s)} T(s) - 2\frac{k'(s)}{k^2(s)} \nabla_{T(s)} \nabla_{T(s)} T(s)
\]
If we use (2.1) in (2.7) and make some calculations, we have
\[ \nabla T(s) \nabla T(s) N(s) = \left[ \left( -\frac{k'(s)}{k^2(s)} \right)' + \frac{K(s)}{k(s)} \right] \nabla T(s) T(s) - 2 \frac{(k'(s))^2}{k^2(s)} N(s) + \frac{k'(s)\tau(s)}{k(s)} B(s). \] (2.8)

Also we obtain
\[ \nabla T(s) \nabla T(s) N(s) = \tau'(s) B(s) \] (2.9)
since (2.8) and (2.9) are equal, routine calculations show that \( c \) is a general helix.

**Theorem 2.2.** Let \( c \) be a curve of Doubly Isotropic Space \( I_3^{(2)} \). \( c \) is a general helix with respect to the Frenet frame \( \{ T(s), N(s), B(s) \} \), if and only if
\[ \nabla T(s) \nabla T(s) T(s) - K(s) \nabla T(s) T(s) = 3\lambda \tau'(s) \nabla T(s) N(s) \] (2.10)
where \( K(s) = \frac{k''(s)}{k(s)} \) and \( \lambda = \frac{k(s)}{\tau(s)} = \text{const.} \)

Proof. It is similar to the proof of Theorem 2.1.

**Corollary 2.1.** Let \( c \) be a curve of Doubly Isotropic Space \( I_3^{(2)} \). \( c \) is a circular helix with respect to the Frenet frame \( \{ T(s), N(s), B(s) \} \), if and only if
\[ T(s) = \frac{1}{2}s^2 + s + d \] (2.11)
where \( d \) is a constant.

Proof. From the hypothesis of corollary 2.1 and since \( c \) is a circular helix, we can show easily (2.11).

**References**


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