SOME RESULTS ON UNIVERSAL MODULES

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Abstract

In this work we are basically interested in some problems related to the universal modules of high order derivations introduced and developed by H. Osborn, R.G. Heyneman, M.E. Sweedler and Y. Nakai.

Firstly we have identify the generators of $Ker\theta$ where θ is a map from $\Omega_n(R)$ to $\Omega_1(R)$ and R is a finitely generated algebra.

Secondy we have characterized the homological dimension of $\Omega_n(R)$ for an affine domain presented by $R = \frac{k[x_1, \dots, x_s]}{(f)}$.

Finally we have given some examples.

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1 Introduction

Let R be a commutative algebra over an algebraically closed field k with characteristic zero. Let $\Omega_n(R)$ and $\delta_n : R \rightarrow \Omega_n(R)$ denote the universal module of n -th order derivations and the canonical n -th order k -derivation of R respectively.

The pair $(\delta_n, \Omega_n(R))$ has the universal mapping property that for any R -module N and any higher derivation $d : R \rightarrow N$ of order $\leq n$ there is a unique R -homomorphism $h : \Omega_n(R) \rightarrow N$ such that $d = h\delta_n$.

$\Omega_n(R)$ is generated by the set $\{\delta_n(r) : r \in R\}$. Hence if R is finitely generated k -algebra $\Omega_n(R)$ will be a finitely generated R -module.

Theorem 1.1 *Let $R = k[x_1, \dots, x_s]$ be an affine k -algebra. Then there exists a short exact sequence of R -modules*

$$0 \rightarrow \text{Ker}\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_1(R) \rightarrow 0$$

such that $\theta(\delta_n(f)) = \delta_1(f)$ for all $f \in R$ and $\text{Ker}\theta$ is generated by the set $\{\delta_n(x_1^{i_1} \dots x_s^{i_s}) - \sum \frac{\partial(x_1^{i_1} \dots x_s^{i_s})}{\partial x_i} \delta_n(x_i)\}$

Proof. By the universal property of $\Omega_n(R)$ there exists a unique R -module homomorphism $\theta : \Omega_n(R) \rightarrow \Omega_1(R)$ such that $\theta\delta_n = \delta_1$ and the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{\delta_1} & \Omega_1(R) \\ \delta_n \downarrow & & \downarrow 1_{\Omega_1(R)} \\ \Omega_n(R) & \xrightarrow{\theta} & \Omega_1(R) \end{array}$$

Since θ is onto

$$0 \rightarrow \text{Ker}\theta \rightarrow \Omega_n(R) \xrightarrow{\theta} \Omega_1(R) \rightarrow 0$$

is an exact sequence of R -modules.

Let $S = \{\delta_n(x_1^{i_1} \dots x_s^{i_s}) - \sum \frac{\partial(x_1^{i_1} \dots x_s^{i_s})}{\partial x_i} \delta_n(x_i)\}$.

Let N be a submodule of $\Omega_n(R)$ generated by S . Our aim is to show that $\text{Ker}\theta = N$.

Clearly $\text{Ker}\theta \supseteq N$.

Let p be the natural map from $\Omega_n(R)$ to $\Omega_n(R)/N$. Now $p\delta_n$ is a differential operator of order ≤ 1 .

$$\begin{aligned} [p\delta_n, x_i, x_j](1) &= ([p\delta_n, x_i]x_j - x_j[p\delta_n, x_i])(1) \\ &= p\delta_n(x_i x_j) - x_i p\delta_n(x_j) - x_j p\delta_n(x_i) + x_i x_j p\delta_n(1) \\ &= p(\delta_n(x_i x_j) - x_i \delta_n(x_j) - x_j \delta_n(x_i)) \\ &= 0 \end{aligned}$$

Since $\Omega_1(R)$ has the universal property, there exists a unique $\beta : \Omega_1(R) \rightarrow \Omega_n(R)/N$ such that $p\delta_n = \beta\delta_1$.

Let $t \in \text{Ker}\theta$. Then $\theta(t) = 0 \Rightarrow \beta\theta(t) = 0 \Rightarrow p(t) = 0 \Rightarrow t \in N$ and the proof follows. $\diamond$

Proposition 1.2 *Let $R = k[x_1, \dots, x_s]$ be an affine k -algebra. Then there exists a short exact sequence of R -modules*

$$0 \longrightarrow \text{Ker}\theta \longrightarrow \Omega_{n+1}(R) \xrightarrow{\theta} \Omega_n(R) \longrightarrow 0$$

and $\text{Ker}\theta$ is generated by $\{[\delta_{n+1}, r_0, r_1, \dots, r_n](1) \mid r_i \in R\}$.

Proof. By the universal property of $\Omega_{n+1}(R)$ there exists a unique R -module homomorphism $\theta : \Omega_{n+1}(R) \longrightarrow \Omega_n(R)$ such that $\theta(d_{n+1}) = d_n$ and the following diagram commutes.

$$\begin{array}{ccc} R & \xrightarrow{d_n} & \Omega_n(R) \\ d_{n+1} \downarrow & & \downarrow 1_{\Omega_n(R)} \\ \Omega_{n+1}(R) & \xrightarrow{\theta} & \Omega_n(R) \end{array}$$

Since θ is onto

$$0 \rightarrow \text{Ker}\theta \rightarrow \Omega_{n+1}(R) \xrightarrow{\theta} \Omega_n(R) \rightarrow 0$$

is an exact sequence of R -modules.

Rest of proof can be seen as in the Theorem 1.1. $\diamond$

2 Homological Dimension of Universal modules

Let I be an ideal of a ring R . Consider the universal n -th order derivative operators $(d_n, \Omega_n(R))$ and $(\delta_n, \Omega_n(R/I))$ for R and R/I respectively. Since the composite of the two maps

$$R \xrightarrow{\pi} R/I \xrightarrow{\delta_n} \Omega_n(R/I)$$

is an n -th order derivative operator, by universality there exists a unique surjective map $\beta : \Omega_n(R) \longrightarrow \Omega_n(R/I)$ such that $\beta d_n = \delta_n \pi$. Here π is the natural map. Now we can define an R/I -module homomorphism $\phi : R/I \otimes_R \Omega_n(R) \longrightarrow \Omega_n(R/I)$ by $\phi = 1 \otimes \beta$. Hence we have

$$0 \longrightarrow \frac{(N + I\Omega_n(R))}{I\Omega_n(R)} \longrightarrow \frac{\Omega_n(R)}{I\Omega_n(R)} \xrightarrow{\phi} \Omega_n(R/I) \longrightarrow 0$$

an exact sequence of R/I -modules. Here N is the submodule of $\Omega_n(R)$ generated by the elements of the form $d_n(r)$, $r \in I$. We identified $R/I \otimes_R \Omega_n(R)$ with the quotient $\Omega_n(R)/(I\Omega_n(R))$.

We suppose that R be a polynomial ring $k[x_1, \dots, x_s]$. Then $\Omega_n(R)$ is a free R -module of rank $\binom{n+s}{s} - 1$ with a basis $\{d_n(x^\alpha) \mid 0 \leq |\alpha| \leq n\}$

where $x^\alpha = x_1^{\alpha_1}, \dots, x_s^{\alpha_s}$ and $|\alpha| = \alpha_1 + \dots + \alpha_s$.

Let S be a multiplicatively closed subset of R . From the fact that $\Omega_n(R) \otimes_R R_S \simeq \Omega_n(R_S)$, it follows that $\Omega_n(R_S)$ is free of rank $\binom{n+s}{s} - 1$. Furthermore if I is an ideal of R generated by $f_1, \dots, f_p$ then we have the following Lemma.

Lemma 2.1 $\frac{(N + I\Omega_n(R))}{I\Omega_n(R)}$ is generated by the set

$$\{d_n(x_1^{\alpha_1} \dots x_t^{\alpha_t} f_j) + I\Omega_n(R) \mid 0 \leq \alpha_1 + \dots + \alpha_t \leq n-1, j = 1, \dots, p\}$$

Proof. [1] $\diamond$

Corollary 2.2 $\Omega_n(R/I)$ is generated by the set

$$\{d_n(x^\alpha + I) \mid |\alpha| \leq n\}$$

where $d_n : R/I \rightarrow \Omega_n(R/I)$ is the universal differential operator.

Proof. [1] $\diamond$

Let R be a polynomial ring $k[x_1, \dots, x_s]$ with an ideal I of R generated by $f \in R$.

Let $S = R/I$. By Corollary 2.2. $\Omega_n(S) \simeq F/N$ where F is a free S -module on $\{d_n(x^\alpha) \mid |\alpha| \leq n\}$ and N is a submodule of F generated by $\{d_n(x^\alpha f) \mid |\alpha| < n\}$. Therefore we have

$$0 \longrightarrow N \longrightarrow F \longrightarrow \Omega_n(S) \longrightarrow 0$$

the short exact sequence of S -modules.

Theorem 2.3 Let S be a regular local ring of dimension $s-1$ and m a maximal ideal of S . Then $N \not\subseteq mF$.

Proof. Let S be a regular local ring. Then $\Omega_n(S)$ is a free S -module of rank $\binom{n+s-1}{s-1} - 1$. Therefore $\frac{\Omega_n(S)}{m\Omega_n(S)}$ is a vectorspace of dimension $\binom{n+s-1}{s-1} - 1$ over S/m . If $N \subseteq mF$ then

$$\frac{\Omega_n(S)}{m\Omega_n(S)} \simeq \frac{F/N}{m(F/N)} \simeq \frac{F/N}{(mF+N)/N} \simeq \frac{F}{mF}$$

$\frac{F}{mF}$ which has dimension $\binom{n+s}{s} - 1$ as an S/m vectorspace. This contradiction shows that $\Omega_n(S)$ cannot be projective. Hence $N \not\subseteq mF$. $\diamond$

Corollary 2.4 *Let S be a regular ring and m a maximal ideal of S . Then $N \not\subseteq mF$.*

Theorem 2.5 *Let $S = \frac{k[x_1, \dots, x_s]}{(f)}$ be an affine domain. Then homological dimension of $\Omega_n(S)$ is less than or equal to one.*

Proof. [2] $\diamond$

Theorem 2.6 *Let $S = \frac{k[x_1, \dots, x_s]}{(f)}$ be an affine domain and m a maximal ideal of S . If $N \subseteq mF$ then homological dimension of $\Omega_n(S)$ is one.*

Proof. From Theorem 2.5. homological dimension of $\Omega_n(S)$ is less than or equal to one.

We need to see that $hd(\Omega_n(S)) \geq 1$. Suppose that $\Omega_n(S)$ is a projective S -module. We can assume that S is a local ring and so $\Omega_n(S)$ is free of rank $\binom{n+s-1}{s-1} - 1$ as the dimension of S is $s-1$. If m is the maximal ideal of S then $\frac{\Omega_n(S)}{m\Omega_n(S)}$ is a vectorspace of dimension $\binom{n+s-1}{s-1} - 1$ over S/m .

If $N \subseteq mF$ then

$$\frac{\Omega_n(S)}{m\Omega_n(S)} \simeq \frac{F/N}{m(F/N)} \simeq \frac{F/N}{(mF+N)/N} \simeq \frac{F}{mF}$$

which has dimension $\binom{n+s}{s} - 1$ as an S/m vectorspace. This contradiction shows that $\Omega_n(S)$ cannot be projective. So $hd(\Omega_n(S)) = 1$ $\diamond$

Example 2.7 *Let $R = \frac{k[x, y]}{(y^2 - x^3)}$ be a quotient polynomial ring. For the short exact sequence of R -modules*

$$0 \longrightarrow \text{Ker}\theta \longrightarrow \Omega_3(R) \xrightarrow{\theta} \Omega_1(R) \longrightarrow 0$$

$\Omega_1(R)$ and $\Omega_3(R)$ are generated by $\{\delta_1(x), \delta_1(y)\}$ and $\{\delta_3(x), \delta_3(y), \delta_3(xy), \delta_3(x^2), \delta_3(x^2y), \delta_3(x^3)\}$ respectively.

Hence $\text{Ker}\theta$ is generated by

$$\{\delta_3(xy) - x\delta_3(y) - y\delta_3(x), \delta_3(x^2) - 2x\delta_3(x), \delta_3(x^2y) - 2xy\delta_3(x) - x^2\delta_3(y), \delta_3(x^3) - 3x^2\delta_3(y)\}.$$

Example 2.8 Let $R = \frac{k[x, y]}{(x^2 + y^2 - 1)}$ be a quotient polynomial ring. R is a regular ring. Let m be a maximal ideal in R . Let F be a free R -module generated by $\{\delta_1(x), \delta_1(y)\}$ and N be a submodule of F generated by $\{x\delta_1(x) + y\delta_1(y)\}$. We know that $\Omega_1(R) \simeq F/N$.

Then $N \not\subseteq mF$.

Example 2.9 $S = \frac{k[x, y]}{(y^2 - x^3)}$ affine domain with dimension one. One may calculate that projective dimension of $\Omega_n(S)$ is one.

Example 2.10 $S = \frac{k[x, y, z]}{(z^2 - xy)}$ affine domain with dimension two. One may calculate that projective dimension of $\Omega_n(S)$ is one.

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