Meromorphic Functions with Positive Coefficients

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Abstract

In this paper we consider the class $L(\alpha, \beta, \gamma, A, B, p)$ consisting of analytic and multivalent functions of order p with positive coefficients. The aim of the study is to present some results on coefficient estimates, convex linear combinations, some distortion theorems and radii of starlikeness and convexity for functions f in the class $L(\alpha, \beta, \gamma, A, B, p)$.

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1 Introduction

Let Σ_p denotes the class of functions of the form

$$f(z) = \frac{1}{z^{-p}} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \quad (p \in N) \quad (1.1)$$

which are normalized analytic p -valent in the punctured disk $E_0 = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma_p$ is said to be, the class of meromorphically starlike of order ρ if it satisfies the following:

$$\operatorname{Re} \left(\frac{-zf'(z)}{f(z)} \right) > \rho, \quad (z \in E)$$

for some ρ ($0 \leq \rho < p$). All such functions are denoted by $\Sigma^*(p, \rho)$. Similarly a function f in Σ_p is said to be meromorphically convex of order ρ , if it satisfies

the following:

$$\operatorname{Re}\left(-p - \frac{zf''(z)}{f'(z)}\right) > \rho, \quad (z \in E)$$

for some ρ ($0 \leq \rho < p$). All such functions are denoted by $\Sigma_c(p, \rho)$. The function f is in the class $\Sigma_c(p, \rho)$ if and only if $-zf'(z) \in \Sigma^*(p, \rho)$.

Many important properties and characteristics of various interesting subclasses of the class Σ_p of meromorphic p -valent functions, including the classes $\Sigma_p^*(\rho)$ and $\Sigma_p^C(\rho)$ defined above, were studied rather extensively by (among others) Uralgeddi[1], Aouf et.al([4][5], Kulkarni et.al [8], Mogra([6],[7]) and Srivastava et.al ([2],[3]).

First we define the class of functions $L(\alpha, \beta, \gamma, A, B, p)$ as follows:

Definition 1.1. A function f given by (1.1) is said to be a member of the class $L(\alpha, \beta, \gamma, A, B, p)$ if it satisfies

$$\left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| \leq \beta, \quad (1.2)$$

where $0 < \beta \leq 1$, $0 \leq \alpha < p$, $-1 < A \leq B \leq 1$ and $\frac{A}{B-A} \leq \gamma \leq 1$ for all $z \in E$.

2 Coefficient Inequalities

The following theorem gives a sufficient condition for a function f , analytic in E to be in $L(\alpha, \beta, \gamma, A, B, p)$.

Theorem 2.1 Let the function f be defined by (1.1). If

$$\sum_{n=1}^{\infty} (2p + n - 1) + \beta(B-A)\gamma(p + n - 1 - \alpha) - \beta A(n-1)a_{p+n-1} \quad (2.3)$$

$$\leq \beta(B-A)\gamma(p + \alpha) - 2\beta Ap,$$

where $0 < \beta \leq 1$, $0 \leq \alpha < p$, $-1 < A \leq B \leq 1$, and $\frac{A}{B-A} \leq \gamma \leq 1$ for all $z \in E$, then $f \in L(\alpha, \beta, \gamma, A, B, p)$.

Proof. Let us suppose that

$$\begin{aligned} \sum_{n=2}^{\infty} (2p + n - 1) + \beta(B-A)\gamma(p + n - 1 - \alpha) - \beta A(n-1)a_{p+n-1} \\ \leq \beta(B-A)\gamma(p + \alpha) - 2\beta Ap \end{aligned}$$

for $f \in \Sigma_p$). It suffices to show that

$$\left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| \leq \beta, (z \in E)$$

Now:

$$\begin{aligned} & \left| \frac{z^{p+1}f'(z) + z^p f(z)p}{[(B-A)\gamma - A]z^{p+1}f'(z) - z^p f(z)[(B-A)\gamma\alpha - Ap]} \right| = \\ & \left| \frac{\sum_{n=1}^{\infty} (2p+n-1)a_{p+n-1}z^{2p+n-1}}{-(B-A)\gamma(p+\alpha) + 2Ap + \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1)a_{p+n-1}z^{2p+n-1}} \right|, \\ & \leq \frac{\sum_{n=1}^{\infty} (2p+n-1)|a_{p+n-1}|}{(B-A)\gamma(p+\alpha) - 2Ap - \sum_{n=1}^{\infty} (B-A)\gamma(p+n-1-\alpha) - A(n-1)|a_{p+n-1}|} < \beta. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{n=1}^{\infty} (2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)a_{p+n-1} \\ & \leq \beta(B-A)\gamma(p+\alpha) - 2\beta Ap. \end{aligned}$$

When $n = 1$, we have

$$a_p \leq \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{2p + \beta(B-A)\gamma(p-\alpha)}.$$

This is equivalent to our condition of the theorem, so that $f \in L(\alpha, \beta, \gamma, A, B, p)$.

The result is attained for a function f given by

$$\begin{aligned} f(z) &= z^{-p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} z^{p+n-1}, \\ & \quad (n \geq 1) \end{aligned} \tag{2.4}$$

Corollary 2.1. *Let the function f be defined by (1.1). If $f \in L(\alpha, \beta, \gamma, A, B, p)$, then*

$$a_{p+n-1} \leq \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)}, (n \geq 1) \tag{2.5}$$

The result is sharp for functions f given by (2.4).

3 Distortion Theorem

A distortion property for functions f to be in the class $L(\alpha, \beta, \gamma, A, B, p)$, is given as follows:

Theorem 3.1. *If the function f be defined by (1.1) is in the class $L(\alpha, \beta, \gamma, A, B, p)$, then for $0 < |z| = r < 1$, we have*

$$\begin{aligned} & \frac{1}{r^p} - \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{2p + \beta(B-A)\gamma(p-\alpha)} r^p \\ & \leq |f(z)| \leq \frac{1}{r^p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{2p + \beta(B-A)\gamma(p-\alpha)} r^p \end{aligned}$$

with equality holds for

$$f(z) = \frac{1}{z^p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{2p + \beta(B-A)\gamma(p-\alpha)} z^p, \quad (z = ir, r)$$

and

$$\begin{aligned} & \frac{p}{r^{p+1}} - \frac{p[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{2p + \beta(B-A)\gamma(p-\alpha)} r^{p-1} \\ & \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{p[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{2p + \beta(B-A)\gamma(p-\alpha)} r^{p-1}, \end{aligned}$$

with equality holds for function f given by

$$f(z) = \frac{1}{z^{p-1}} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta AB}{2p + \beta(B-A)\gamma(p-\alpha)} z^{p-1}, \quad (z = \pm ir, \pm r).$$

Proof. Since $f \in L(\alpha, \beta, \gamma, A, B, p)$, Theorem 2.1 yields

$$\sum_{n=1}^{\infty} a_{p+n-1} \leq \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)}, \quad (n \geq 1). \quad (3.6)$$

Thus, for $0 < |z| = r < 1$, and making use of (2.3) we have

$$\begin{aligned} |f(z)| & \leq |z|^{-p} + \sum_{n=1}^{\infty} a_{p+n-1} |z|^{p+n-1} \\ & \leq r^{-p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p) + \beta(B-A)\gamma(p-\alpha)} r^p \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z|^{-p} - \sum_{n=1}^{\infty} a_{p+n-1} |z|^{p+n-1} \\ &\geq r^{-p} - \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p) + \beta(B-A)\gamma(p-\alpha)} r^p. \end{aligned}$$

Also from Theorem 2.1, it follows that

$$\sum_{n=1}^{\infty} a_{p+n-1} \leq \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p) + \beta(B-A)\gamma(p-\alpha)}, \quad (3.7)$$

and hence

$$\begin{aligned} |f'(z)| &\leq p|z|^{-p-1} + \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}|z|^{p+n-2}, \\ &\leq pr^{-p-1} + \frac{p[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{2p + \beta(B-A)\gamma(p-\alpha)} r^{p-1}. \end{aligned}$$

Thus

$$\begin{aligned} |f'(z)| &\geq p|z|^{-p-1} - \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}|z|^{p+n-2}, \\ &\geq pr^{-p-1} - \frac{p[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{2p + \beta(B-A)\gamma(p-\alpha)} r^{p-1}. \end{aligned}$$

Hence completes the proof of Theorem 3.1

4 Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $L(\alpha, \beta, \gamma, A, B, p)$, is given by the following theorem:

Theorem 4.1. *If the function f be defined by (1.1) is in the class $L(\alpha, \beta, \gamma, A, B, p)$, then f is starlike of order ρ ($0 \leq \rho < p$), in the unit disk $|z| < r_1(\alpha, \beta, \gamma, A, B, p)$, where $r_1(\alpha, \beta, \lambda, p, \rho)$, is the largest value for which*

$$r_1 = \inf_{n \geq 1} \left(\frac{(p-\rho)[(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)]}{(\beta(B-A)\gamma(p+\alpha) - 2\beta Ap)(p+n+1-\rho)} \right)^{\frac{1}{2p+n-1}}.$$

The result is sharp for functions f given by (2.4).

Proof. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq (1 - \rho), \quad (4.8)$$

for $|z| \leq r_1$, we have

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} + 1 \right| &= \left| \frac{[z^{-p}(-p+1)] + \sum_{n=1}^{\infty} (p+n)a_{p+n-1}z^{2p+n-1}}{z^{-p}(1 + \sum_{n=1}^{\infty} a_{p+n-1}z^{2p+n-1})} \right| \\ &\leq \frac{(-p+1) + \sum_{n=1}^{\infty} (p+n) \frac{\beta(B-A)\gamma(p+\alpha)-2\beta Ap}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)} |z|^{2p+n-1}}{1 - \sum_{n=1}^{\infty} \frac{\beta(B-A)\gamma(p+\alpha)-2\beta Ap}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)} |z|^{2p+n-1}} \\ &\leq (1 - \rho). \end{aligned}$$

Hence (4.8) holds true if

$$\sum_{n=1}^{\infty} \frac{(p+n+1-\rho)[\beta(B-A)\gamma(p+\alpha)-2\beta Ap]}{(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)} |z|^{2p+n-1} \leq (p-\rho),$$

And it follows that

$$\begin{aligned} |z| &\leq \left(\frac{(p-\rho)[(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)]}{(p+n+1-\rho)[\beta(B-A)\gamma(p+\alpha)-2\beta Ap]} \right)^{\frac{1}{2p+n-1}}, \\ &\quad (n \geq 1) \end{aligned}$$

then,

$$r_1 = \inf_{n \geq 1} \left(\frac{(p-\rho)[(2p+n-1)+\beta(B-A)\gamma(p+n-1-\alpha)-\beta A(n-1)]}{(\beta(B-A)\gamma(p+\alpha)-2\beta Ap)(p+n+1-\rho)} \right)^{\frac{1}{2p+n-1}}.$$

as required.

Theorem 4.2. If the function f defined by (1.1) is in the class $L(\alpha, \beta, \gamma, A, B, p)$, then f is convex of order ρ ($0 \leq \rho < p$), in the unit disk $|z| < r_2(\alpha, \beta, \gamma, A, B, p)$, where $r_2(\alpha, \beta, \gamma, A, B, p)$, is the largest value for which

$$r_2 = \inf_{n \geq 1} \left(\frac{p(p-\rho)[(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)]}{(p+n-1)(p+n+1-\rho)(\beta(B-A)\gamma(p+\alpha) - 2\beta Ap)} \right)^{\frac{1}{2p+n-1}}.$$

The result is sharp for functions given by (2.4).

Proof. It suffices to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq (1-\rho). \quad (4.9)$$

For $|z| \leq r_2$, we have

$$\begin{aligned} \left| \frac{zf''(z)}{f'(z)} + 2 \right| &= \left| \frac{-(p-p^2) \sum_{n=1}^{\infty} (p+n)(p+n-1)a_{p+n-1}z^{2p+n-1}}{-p + \sum_{n=1}^{\infty} (p+n-1)a_{p+n-1}z^{2p+n-1}} \right| \\ &\leq \frac{(p-p^2) + \sum_{n=1}^{\infty} \frac{(p+n)(p+n-1)\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} |z|^{2p+n-1}}{p - \sum_{n=1}^{\infty} \frac{(p+n-1)[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} |z|^{2p+n-1}} \leq (1-\rho), \end{aligned}$$

Hence (4.9) holds true if

$$\sum_{n=1}^{\infty} \frac{(p+n+1-\rho)(p+n-1)[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} |z|^{2p+n-1} \leq p(p-\rho),$$

And it follows that

$$|z| \leq \left(\frac{p(p-\rho)[(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)]}{(p+n+1-\rho)(p+n-1)[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap]} \right)^{\frac{1}{2p+n-1}},$$

$$(n \geq 1).$$

Then

$$r_2 = \inf_{n \geq 1} \left(\frac{p(p-\rho)[(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)]}{(p+n-1)(p+n+1-\rho)(\beta(B-A)\gamma(p+\alpha) - 2\beta Ap)} \right)^{\frac{1}{2p+n-1}}$$

as required.

5 Convex linear Combination

Our next result involves a linear combination of function of the type (1.1).

Theorem 5.1. Let

$$f_1 = z^{-p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{2p + \beta(B-A)\gamma(p-\alpha)} z^p, \quad (5.10)$$

and

$$f_n(z) = z^{-p} + \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} z^{p+n-1}, \quad (5.11)$$

$$(n \geq 1),$$

then $f \in L(\alpha, \beta, \gamma, A, B, p)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad (5.12)$$

$$\text{where } \lambda_n \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1$$

Proof. $\Leftarrow$ From (5.10), (5.11) and (5.12), it is easily seen that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z^{-p} + \sum_{n=1}^{\infty} \frac{[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap] \lambda_n}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} z^{p+n-1}. \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} \frac{[\beta(B-A)\gamma(p+\alpha) - 2\beta Ap] \lambda_n}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} = \sum_{n=1}^{\infty} \lambda_n \leq 1,$$

it follows from Theorem 2.1 that the function $f \in L(\alpha, \beta, \gamma, A, B, p)$.

$\Leftarrow$ Conversely, let us suppose that $f \in L(\alpha, \beta, \gamma, A, B, p)$. Since

$$a_{p+n-1} \leq \frac{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap}{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)} \quad (n \geq 1)$$

Setting

$$\lambda_n = \frac{(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)}{\beta(B-A)\gamma(p+\alpha) - 2\beta Ap} a_{p+n-1} \quad (n \geq 1)$$

it follows that

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

this complete the proof of theorem. And, finally we prove

Theorem 5.2. *The class $L(\alpha, \beta, \gamma, A, B, p)$ is closed under convex linear combinations.*

Proof. Suppose that the functions f_1 and f_2 defined by,

$$f_i(z) = z^{-p} + \sum_{n=1}^{\infty} a_{p+n-1,i} z^{p+n-1} \quad (i = 1, 2; z \in E) \quad (5.13)$$

be in the class $L(\alpha, \beta, \gamma, A, B, p)$.

Setting $f(z) = \mu f_1(z) + (1 - \mu)f_2(z) \quad (0 \leq \mu \leq 1)$,

from (5.13) we obtain

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} \left\{ \mu a_{p+n-1,1} + (1 - \mu)a_{p+n-1,2} \right\} z^{p+n-1}, \quad (z \in E)$$

In view of theorem 2.1, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} [(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)] \{ \mu a_{p+n-1,1} + (1-\mu)a_{p+n-1,2} \} \\ &= \mu \sum_{n=1}^{\infty} [(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)] a_{p+n-1,1} \\ &+ (1-\mu) \sum_{n=1}^{\infty} [(2p+n-1) + \beta(B-A)\gamma(p+n-1-\alpha) - \beta A(n-1)] a_{p+n-1,2} \\ &\leq \mu [\beta(B-A)\gamma(p+\alpha) - 2\beta Ap] + (1-\mu) [\beta(B-A)\gamma(p+\alpha) - 2\beta Ap] \\ &= \beta(B-A)\gamma(p+\alpha) - 2\beta Ap. \end{aligned}$$

which show that $f \in L(\alpha, \beta, \gamma, A, B, p)$.

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