FOURIER-FEYNMAN TRANSFORM AND FIRST VARIATION OF CYLINDER TYPE FUNCTIONS OVER WIENER PATHS IN ABSTRACT WIENER SPACE

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Abstract

In this paper, we express the analytic Feynman integral of the first variation of cylinder type functions F in terms of the analytic Feynman integral of the product of F with a linear factor over Wiener paths in abstract Wiener space. And then, we derive the Fourier-Feynman transform for the product of cylinder type functions which define the functions in a Banach algebra introduced by Yoo, with n linear factors.

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1 Introduction and preliminaries

Let $C_0[0, T]$ be the space of all continuous paths x on $[0, T]$ with $x(0) = 0$ which is known as the classical Wiener space. The concept of an L_1 analytic Fourier-Feynman transform for the functionals on this space was introduced by Brue in [1]. In [3], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform, and in [10] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theories for $1 \leq p \leq 2$ that extended the results in [1, 3] and gave various relationships between the L_1 and L_2 theories.

On the other hand, in [2], Cameron obtained the Wiener integral of the first variation of a functional F in terms of the Wiener integral of the product F with a linear factor. In [14], Park, Skoug and Storvick found the Fourier-Feynman transform of the product of a functional, with n linear factors, from

the Banach algebra $\mathcal{S}$ which was introduced by Cameron and Storvick in [4]. In [6], Chang, Song and Yoo expressed the analytic Feynman integral of the first variation of a functional F in terms of the analytic Feynman integral of the product of F with a linear factor on abstract Wiener space. And then, they derived the Fourier-Feynman transform for the product of a functional in the Fresnel class with n linear factors.

In [7], Cho introduced a Banach algebra $\mathcal{F}(C_0(\mathbb{B}); u)$ which is equivalent to the Fresnel class and in [16], Yoo did the Banach algebra $S''_{\mathbb{B}}$ which corresponds to the Banach algebra $\mathcal{S}''$ introduced by Cameron and Storvick in [4]. And also, in [8], Cho extended the results in [6] to the functionals in $\mathcal{F}(C_0(\mathbb{B}); u)$ over Wiener paths in abstract Wiener space.

In this paper, with cylinder type functions which defines the functionals in $S''_{\mathbb{B}}$, we extend the results in [6] over Wiener paths in abstract Wiener space. Further, for this type of functions F , we express the analytic Feynman integral of the first variation of F in terms of the analytic Feynman integral of the product of F with a linear factor over Wiener paths in abstract Wiener space. And then, we derive the Fourier-Feynman transform of the product of F with n linear factors.

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space([13]). Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that e_j 's are in $\mathbb{B}^*$, the dual space of real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ by

$$(h, y)^\sim = \begin{cases} \lim_{n \rightarrow \infty} \sum_{j=1}^n \langle h, e_j \rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot, \cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$ ([11]). Note that for each $h (\neq 0)$ in $\mathcal{H}$, $(h, \cdot)^\sim$ is a Gaussian random variable on $\mathbb{B}$ with mean zero, variance $|h|^2$; also $(h, y)^\sim$ is essentially independent of choice of the complete orthonormal set used in its definition and further, $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda(h, y)^\sim$ for all $\lambda \in \mathbb{R}$. It is well-known that if $\{h_1, h_2, \dots, h_n\}$ is an orthogonal set in $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$'s are independent. Moreover, if both h and y are in $\mathcal{H}$, then $(h, y)^\sim = \langle h, y \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product of h and y .

Let $C_0(\mathbb{B})$ denote the set of all continuous functions on $[0, T]$ into $\mathbb{B}$ which vanish at 0. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0 \leq t \leq T} \|x(t)\|_{\mathbb{B}}$. The minimal σ -field making the mapping $x \rightarrow x(t)$ measurable is $\mathcal{B}(C_0(\mathbb{B}))$, the Borel σ -field on $C_0(\mathbb{B})$. Further, Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean-zero Gaussian([15]). We will find a concrete form of $m_{\mathbb{B}}$. Let $\vec{t} = (t_1, t_2, \dots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \dots < t_k \leq T$. Let $T_{\vec{t}}: \mathbb{B}^k \rightarrow \mathbb{B}^k$

be given by

$$T_{\vec{t}}(x_1, x_2, \dots, x_k) = \left(\sqrt{t_1 - t_0}x_1, \sqrt{t_1 - t_0}x_1 + \sqrt{t_2 - t_1}x_2, \dots, \sum_{j=1}^k \sqrt{t_j - t_{j-1}}x_j \right).$$

We define a set function $\nu_{\vec{t}}$ on $\mathcal{B}(\mathbb{B}^k)$ by

$$\nu_{\vec{t}}(B) = \left(\prod_1^k m \right) (T_{\vec{t}}^{-1}(B))$$

for $B \in \mathcal{B}(\mathbb{B}^k)$. Then $\nu_{\vec{t}}$ is a Borel measure. Let $f_{\vec{t}} : C_0(\mathbb{B}) \rightarrow \mathbb{B}^k$ be the function defined by

$$f_{\vec{t}}(x) = (x(t_1), x(t_2), \dots, x(t_k)).$$

For Borel subsets $B_1, B_2, \dots, B_k$ of $\mathbb{B}$, $f_{\vec{t}}^{-1}(\prod_{j=1}^k B_j)$ is called the I -set with respect to $B_1, B_2, \dots, B_k$. Then the collection $\mathcal{I}$ of all I -sets is a semi-algebra. We define a set function $m_{\mathbb{B}}$ on $\mathcal{I}$ by

$$m_{\mathbb{B}} \left(f_{\vec{t}}^{-1} \left(\prod_{j=1}^k B_j \right) \right) = \nu_{\vec{t}} \left(\prod_{j=1}^k B_j \right).$$

Then $m_{\mathbb{B}}$ is well-defined and countably additive on $\mathcal{I}$. Using Carathéodory extension process, we have a Borel measure $m_{\mathbb{B}}$ on $\mathcal{B}(C_0(\mathbb{B}))$.

Now, we introduce Wiener integration theorem without proof. For the proof, see [15].

Theorem 1.1 (Wiener integration theorem) *Let $\vec{t} = (t_1, t_2, \dots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \dots < t_k \leq T$ and let $f : \mathbb{B}^k \rightarrow \mathbb{C}$ be a Borel measurable function. Then*

$$\begin{aligned} & \int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \dots, x(t_k)) \, dm_{\mathbb{B}}(x) \\ & \stackrel{*}{=} \int_{\mathbb{B}^k} (f \circ T_{\vec{t}})(x_1, x_2, \dots, x_k) \, d \left(\prod_{j=1}^k m \right) (x_1, x_2, \dots, x_k), \end{aligned}$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

A subset E of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$. A property is said to hold scale-invariant almost everywhere (in

abbreviation, s-a.e.) if it holds except for a scale-invariant null set. Let F be defined on $C_0(\mathbb{B})$. Suppose that $E[F(\lambda^{-\frac{1}{2}}\cdot)]$ exists for any $\lambda > 0$ and it has the analytic extension $J_\lambda^*(F)$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$. Then we call $J_\lambda^*(F)$ the analytic Wiener integral of F with parameter λ and it is denoted by

$$E^{anw_\lambda}[F] = J_\lambda^*(F).$$

Moreover, for a non-zero real q , if $E^{anw_\lambda}[F]$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman integral of F with parameter q and denoted by

$$\int_{C_0(\mathbb{B})}^{anf_q} F(x) dm_{\mathbb{B}}(x) = E^{anf_q}[F] = \lim_{\lambda \rightarrow -iq} E^{anw_\lambda}[F].$$

2 Feynman integrals of variations over Wiener paths in abstract Wiener space

In this section, we define the first variation of functionals over Wiener paths in abstract Wiener space. And then, we investigate its properties.

Definition 2.1 *Let F be a functional defined on $C_0(\mathbb{B})$ and let $w \in C_0(\mathbb{B})$. The derivative*

$$\frac{\partial}{\partial t} F(x + tw)|_{t=0}$$

for $x \in C_0(\mathbb{B})$ if it exists, is called the first variation of F at x in the direction of w and denoted by

$$\delta_w F(x) = \frac{\partial}{\partial t} F(x + tw)|_{t=0}.$$

Suppose that, for some $\epsilon > 0$, $\delta_w F(x + tw)$ exists for $|t| < \epsilon$. Then we have

$$\begin{aligned} \delta_w F(x + tw) &= \frac{\partial}{\partial \lambda} F(x + tw + \lambda w)|_{\lambda=0} \\ &= \frac{\partial}{\partial \mu} F(x + \mu w)|_{\mu=t} = \frac{\partial}{\partial t} F(x + tw). \end{aligned} \quad (1)$$

Let $\mathcal{H}$ be an infinite dimensional separable real Hilbert space and let $\Delta_n = \{(s_1, \dots, s_n) : 0 = s_0 < s_1 < \dots < s_n \leq T\}$ for $n \in \mathbb{N}$. For

$$\vec{s}_n = (s_1, \dots, s_n) \in \Delta_n \text{ and } \vec{h}_n = (h_1, \dots, h_n) \in \mathcal{H}^n, \quad (2)$$

let

$$F_{\vec{s}_n, \vec{h}_n}(x) = \exp \left\{ i \sum_{j=1}^n (h_j, x(s_j))^\sim \right\} \quad (3)$$

for $x \in C_0(\mathbb{B})$. Further, for $w \in C_0(\mathbb{B})$ with $w(s_j) \in \mathcal{H}(j = 1, 2, \dots, n)$, let

$$F_{w, \vec{s}_n, \vec{h}_n}(x) = F_{\vec{s}_n, \vec{h}_n}(x) \sum_{j=1}^n (w(s_j), x(s_j))^\sim \quad (4)$$

for $x \in C_0(\mathbb{B})$ where $F_{\vec{s}_n, \vec{h}_n}$ is given by (3).

Let $\mathcal{M}_n'' = \mathcal{M}_n''(\Delta_n \times \mathcal{H}^n)$ be the class of all complex Borel measures on $\Delta_n \times \mathcal{H}^n$ and let $\|\mu_n\| = \text{var} \mu_n$, the total variation of μ_n in $\mathcal{M}_n''$. Let $S_{n, \mathbb{B}}'' = S_{n, \mathbb{B}}''(\Delta_n \times \mathcal{H}^n)$ be the space of functionals of the form

$$F_n(x) = \int_{\Delta_n \times \mathcal{H}^n} F_{\vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \quad (5)$$

for s-a.e. $x \in C_0(\mathbb{B})$, where μ_{F_n} is in the class $\mathcal{M}_n''$ and $F_{\vec{s}_n, \vec{h}_n}$ is given by (3). Here we take $\|F_n\|_n'' = \inf\{\|\mu_{F_n}\|\}$, where the infimum is taken over all μ_{F_n} 's so that F_n and μ_{F_n} are related by (5). Let $\mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times \mathcal{H}^n)$ be the class of all sequences $\{\mu_n\}$ of measures such that each μ_n is in $\mathcal{M}_n''$ with $\sum_{n=1}^\infty \|\mu_n\| < \infty$. Let $S_{\mathbb{B}}'' = S_{\mathbb{B}}''(\sum \Delta_n \times \mathcal{H}^n)$ be the space of functions on $C_0(\mathbb{B})$ of the form

$$F(x) = \sum_{n=1}^\infty F_n(x) \quad (6)$$

where each F_n is in $S_{n, \mathbb{B}}''$ with $\sum_{n=1}^\infty \|F_n\|_n'' < \infty$. The norm of F is defined by $\|F\|'' = \inf\{\sum_{n=1}^\infty \|F_n\|_n''\}$, where the infimum is taken over all representations of F given by (6).

Now, we introduce a useful integral which appears in the proofs of several results. The proof follows from the fact that $(h, \cdot)^\sim$ is normally distributed with mean 0 and variance $|h|^2$ if $h \neq 0$.

Lemma 2.2 *Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space and let $h \in \mathcal{H}$. Suppose that α is pure imaginary or real. Then we have*

$$\int_{\mathbb{B}} \exp\{\alpha(h, x_1)^\sim\} dm(x_1) = \exp\left\{\frac{\alpha^2|h|^2}{2}\right\}.$$

Lemma 2.3 *Let $F_{\vec{s}_n, \vec{h}_n}$ be given by (3) and let w be in $C_0(\mathbb{B})$. Then $\delta_w F_{\vec{s}_n, \vec{h}_n}(x)$ exists and given by*

$$\delta_w F_{\vec{s}_n, \vec{h}_n}(x) = i \sum_{j=1}^n (h_j, w(s_j))^\sim F_{\vec{s}_n, \vec{h}_n}(x) \quad (7)$$

for x in $C_0(\mathbb{B})$.

Proof. By the definition of first variation, we have

$$\begin{aligned}\frac{\partial}{\partial t} F_{\vec{s}_n, \vec{h}_n}(x + tw)|_{t=0} &= \frac{\partial}{\partial t} \exp \left\{ i \sum_{j=1}^n (h_j, x(s_j) + tw(s_j))^\sim \right\} \Big|_{t=0} \\ &= i \sum_{j=1}^n (h_j, w(s_j))^\sim F_{\vec{s}_n, \vec{h}_n}(x)\end{aligned}$$

for x in $C_0(\mathbb{B})$, which is the desired result. $\square$

Theorem 2.4 *Let F_n be given by (5) and let $w \in C_0(\mathbb{B})$. Suppose that*

$$\int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^n (h_j, w(s_j))^\sim \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \quad (8)$$

where $\vec{s}_n$ and $\vec{h}_n$ are given by (2). Then $\delta_w F_n(x)$ exists for s -a.e. $x \in C_0(\mathbb{B})$ and given by

$$\delta_w F_n(x) = \int_{\Delta_n \times \mathcal{H}^n} \delta_w F_{\vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \quad (9)$$

where $\delta_w F_{\vec{s}_n, \vec{h}_n}$ is given by (7).

Proof. For s -a.e. $x \in C_0(\mathbb{B})$, we have

$$\begin{aligned}\delta_w F_n(x) &= \frac{\partial}{\partial t} \int_{\Delta_n \times \mathcal{H}^n} F_{\vec{s}_n, \vec{h}_n}(x + tw) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \Big|_{t=0} \\ &= \int_{\Delta_n \times \mathcal{H}^n} \frac{\partial}{\partial t} F_{\vec{s}_n, \vec{h}_n}(x + tw)|_{t=0} d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} \delta_w F_{\vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n)\end{aligned}$$

where the second equality follows from (8) and [9, Theorem 2.27]. Hence the result follows. $\square$

Theorem 2.5 *Let F be given by (6) and let $w \in C_0(\mathbb{B})$. Suppose that*

$$\sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^n (h_j, w(s_j))^\sim \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \quad (10)$$

where $\vec{s}_n$ and $\vec{h}_n$ are given by (2). Then $\delta_w F(x)$ exists for s-a.e. $x \in C_0(\mathbb{B})$ and it is given by

$$\delta_w F(x) = \sum_{n=1}^{\infty} \delta_w F_n(x)$$

where $\delta_w F_n$ is given by (9).

Proof. For s-a.e. x in $C_0(\mathbb{B})$ and t in $\mathbb{R}$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} |\delta_w F_n(x + tw)| \\ &= \sum_{n=1}^{\infty} \left| \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^{\sim} F_{\vec{s}_n, \vec{h}_n}(x + tw) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \right| \\ &\leq \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^n (h_j, w(s_j))^{\sim} \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \end{aligned}$$

by (10), where $F_{\vec{s}_n, \vec{h}_n}$ is given by (3) so that we have

$$\frac{\partial}{\partial t} F(x + tw) = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} F_n(x + tw) = \sum_{n=1}^{\infty} \frac{\partial}{\partial t} F_n(x + tw) = \sum_{n=1}^{\infty} \delta_w F_n(x + tw)$$

where the last equality follows from (1). Letting $t = 0$, we have the result. $\square$

From now on, we will regard s_0 as 0, unless otherwise specified.

Lemma 2.6 *Let $F_{\vec{s}_n, \vec{h}_n}$ be given by (3). Then, for $\lambda > 0$, we have*

$$\int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) = \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\}.$$

Proof. By Theorem 1.1, for $\lambda > 0$, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^n (h_j, x(s_j))^{\sim} \right\} dm_{\mathbb{B}}(x) \\ &= \int_{\mathbb{B}^n} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^n \left(h_j, \sum_{l=1}^j \sqrt{s_l - s_{l-1}} x_l \right)^{\sim} \right\} dm^n(\vec{x}) \\ &= \int_{\mathbb{B}^n} \exp \left\{ i\lambda^{-\frac{1}{2}} \sum_{l=1}^n \sqrt{s_l - s_{l-1}} \left(\sum_{j=l}^n h_j, x_l \right)^{\sim} \right\} dm^n(\vec{x}) \end{aligned} \quad (11)$$

where $\vec{x}_n = (x_1, \dots, x_n)$. Now, the result follows from Lemma 2.2. $\square$

Theorem 2.7 *Let F_n be given by (5) and let w be in $C_0(\mathbb{B})$. Let $q \in \mathbb{R} - \{0\}$ and suppose that (8) holds. Then we have*

$$\begin{aligned} & E^{anf_q}[\delta_w F_n] \\ &= \int_{C_0(\mathbb{B})}^{anf_q} \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^\sim F_{\vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) dm_{\mathbb{B}}(x) \\ &= \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^\sim \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

where $F_{\vec{s}_n, \vec{h}_n}$ is given by (3).

Proof. By Theorems 1.1, 2.4 and Fubini's theorem, for $\lambda > 0$, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} \delta_w F_n(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \\ &= \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^\sim \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^\sim \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

where the last equality follows from Lemma 2.6. Now, the result follows from Morera's and the dominated convergence theorems. $\square$

Theorem 2.8 *Let F be given by (6) and let $w \in C_0(\mathbb{B})$. Let $q \in \mathbb{R} - \{0\}$ and suppose that (10) holds. Then we have*

$$E^{anf_q}[\delta_w F] = \sum_{n=1}^{\infty} E^{anf_q}[\delta_w F_n]$$

where $E^{anf_q}[\delta_w F_n]$ is given as in Theorem 2.7.

Proof. By Theorem 2.4 and (10), for $\lambda > 0$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_{C_0(\mathbb{B})} |\delta_w F_n(\lambda^{-\frac{1}{2}} x)| dm_{\mathbb{B}}(x) \\ &= \sum_{n=1}^{\infty} \int_{C_0(\mathbb{B})} \left| \int_{\Delta_n \times \mathcal{H}^n} i \sum_{j=1}^n (h_j, w(s_j))^\sim F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \right| dm_{\mathbb{B}}(x) \\ &\leq \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{j=1}^n (h_j, w(s_j))^\sim \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \end{aligned}$$

so that we have the result by Theorem 2.7, Morera's and the dominated convergence theorems. $\square$

Theorem 2.9 *Let $w \in C_0(\mathbb{B})$ and let $F_{\vec{s}_n, \vec{h}_n}$, $F_{w, \vec{s}_n, \vec{h}_n}$ be given by (3), (4), respectively. Suppose that $w(s_j) \in \mathcal{H}$ for $j = 1, \dots, n$. Then we have the following integral formula*

$$\begin{aligned} & \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) \\ &= i \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) \quad (12) \end{aligned}$$

with the existences of the integrals of both sides of the equality.

Proof. By (11) with $\lambda = 1$, we have

$$\int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) = \int_{\mathbb{B}^n} \exp \left\{ i \sum_{l=1}^n \sqrt{s_l - s_{l-1}} \left(\sum_{j=l}^n h_j, x_l \right)^{\sim} \right\} dm^n(\vec{x})$$

where $\vec{x} = (x_1, \dots, x_n)$. By Cameron-Martin translation theorem([12]) and Theorem 1.1, we have, for $t \in \mathbb{R}$,

$$\begin{aligned} & \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) \\ &= \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \int_{\mathbb{B}^n} \exp \left\{ i \sum_{l=1}^n \left(\sqrt{s_l - s_{l-1}} \sum_{j=l}^n h_j, \right. \right. \\ & \quad \left. \left. x_l + t \sqrt{s_l - s_{l-1}} \sum_{j=l}^n w(s_j) \right)^{\sim} \right\} \exp \left\{ -t \sum_{l=1}^n \sqrt{s_l - s_{l-1}} \left(\sum_{j=l}^n w(s_j), x_l \right)^{\sim} \right\} \\ & \quad dm^n(\vec{x}) \\ &= \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \int_{\mathbb{B}^n} \exp \left\{ i \sum_{l=1}^n \sqrt{s_l - s_{l-1}} \left(\sum_{j=l}^n h_j, \right. \right. \\ & \quad \left. \left. x_l \right)^{\sim} \right\} \exp \left\{ i t \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right\} \\ & \quad \times \exp \left\{ -t \sum_{l=1}^n \sqrt{s_l - s_{l-1}} \left(\sum_{j=l}^n w(s_j), x_l \right)^{\sim} \right\} dm^n(\vec{x}) \\ &= \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) \exp \left\{ -t \sum_{j=1}^n (w(s_j), \right. \\ & \quad \left. x(s_j))^{\sim} \right\} \exp \left\{ i t \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right\} dm_{\mathbb{B}}(x). \quad (13) \end{aligned}$$

Now, using similar method given as in the proof of Lemma 2.6, for $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} \exp \left\{ \alpha \sum_{j=1}^n (w(s_j), x(s_j))^\sim \right\} dm_{\mathbb{B}}(x) \\ &= \exp \left\{ \frac{\alpha^2}{2} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} < \infty \end{aligned} \quad (14)$$

by Lemma 2.2. Take $\epsilon > 0$, arbitrarily. Since $\sum_{j=1}^n |(w(s_j), x(s_j))^\sim| < \sum_{j=1}^n \exp\{|(w(s_j), x(s_j))^\sim|\}$, both $\exp\{-t \sum_{j=1}^n (w(s_j), x(s_j))^\sim\}$ and $\sum_{j=1}^n (w(s_j), x(s_j))^\sim \exp\{-t \sum_{j=1}^n (w(s_j), x(s_j))^\sim\}$ are integrable functions of x for $|t| < \epsilon$ by (14). Moreover, the integrals are independent of t . Differentiating both sides of (13) with respect to t , by [9, Theorem 2.27], we have

$$\begin{aligned} & 0 \\ &= \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) \left[-t \left[\sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right] \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \right. \right. \\ & \quad \times \left. \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \exp \left\{ -t \sum_{j=1}^n (w(s_j), x(s_j))^\sim \right\} \exp \left\{ it \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \right. \right. \\ & \quad \left. \left. \sum_{j=l}^n w(s_j) \right\rangle \right\} + \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \left[i \sum_{l=1}^n (s_l - s_{l-1}) \right. \\ & \quad \times \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle - \sum_{j=1}^n (w(s_j), x(s_j))^\sim \left. \right] \exp \left\{ -t \sum_{j=1}^n (w(s_j), x(s_j))^\sim \right\} \\ & \quad \times \exp \left\{ it \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right\} \Big] dm_{\mathbb{B}}(x). \end{aligned}$$

Letting $t = 0$, we have the result. $\square$

Remark 2.10 For $(s_1, \dots, s_n) \in \Delta_n$ and $(h_1, \dots, h_n) \in \mathcal{H}^n$, we have

$$\begin{aligned} & \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \\ &= \sum_{j=1}^n s_j \langle h_j, w(s_j) \rangle + \sum_{j=1}^{n-1} s_j \left\langle h_j, \sum_{l=j+1}^n w(s_l) \right\rangle + \sum_{j=1}^{n-1} s_j \left\langle \sum_{l=j+1}^n h_l, w(s_j) \right\rangle \end{aligned}$$

where $w \in C_0(\mathbb{B})$ with $w(s_j) \in \mathcal{H}$ for $j = 1, \dots, n$. Now, suppose that $\langle h_j, w(s_l) \rangle = 0$ if $j \neq l$. Then (12) is reduced to the following equation:

$$\int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) = i \left[\sum_{j=1}^n s_j \langle h_j, w(s_j) \rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x).$$

Let $0 < u \leq T$. In particular, if $n = 1$ and $s_1 = u$, then we have

$$\frac{1}{u} \int_{C_0(\mathbb{B})} F_{u,h}(x)(w(u), x(u))^\sim dm_{\mathbb{B}}(x) = \int_{C_0(\mathbb{B})} \delta_w F_{u,h}(x) dm_{\mathbb{B}}(x) \quad (15)$$

with $h \equiv h_1$.

Theorem 2.11 Suppose that the assumptions given as in Theorem 2.9 hold. Then, for $q \in \mathbb{R} - \{0\}$, we have

$$\begin{aligned} & \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})}^{anf_q} F_{\vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) \\ &= -q \int_{C_0(\mathbb{B})}^{anf_q} F_{w, \vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) \end{aligned}$$

with the existences of the analytic Feynman integrals of the both sides of the equality.

Proof. For $\lambda > 0$, with replacing h_j by $\lambda^{-\frac{1}{2}} h_j$ in (12), we have

$$\begin{aligned} & i\lambda^{-\frac{1}{2}} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) \sum_{j=1}^n (w(s_j), x(s_j))^\sim dm_{\mathbb{B}}(x) \\ &= \lambda^{\frac{1}{2}} \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \end{aligned}$$

by Theorem 2.9. Then we have

$$\begin{aligned} & \frac{i}{\lambda} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}} x) dm_{\mathbb{B}}(x) \end{aligned} \quad (16)$$

and, letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$, by Morera's and the dominated convergence theorems, we have the result. $\square$

Remark 2.12 Let $q \in \mathbb{R} - \{0\}$. Under the assumptions given as in Remark 2.10, we have

$$\frac{iu}{q} \int_{C_0(\mathbb{B})}^{anf_q} \delta_w F_{u,h}(x) dm_{\mathbb{B}}(x) = \int_{C_0(\mathbb{B})}^{anf_q} F_{u,h}(x)(w(u), x(u))^\sim dm_{\mathbb{B}}(x).$$

Lemma 2.13 Let $\mu_{F_n} \in \mathcal{M}_n''$ and let $w \in C_0(\mathbb{B})$ with $w(s) \in \mathcal{H}$ for all $s \in [0, T]$. Suppose that

$$\int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \quad (17)$$

where $\vec{s}_n$ and $\vec{h}_n$ are given by (2). Then we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathcal{H}^n} F_{w, \vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) dm_{\mathbb{B}}(x) \\ &= i \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathcal{H}^n} F_{\vec{s}_n, \vec{h}_n}(x) \sum_{l=1}^n \left[(s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \\ & \quad d\mu_{F_n}(\vec{s}_n, \vec{h}_n) dm_{\mathbb{B}}(x) \end{aligned} \quad (18)$$

where $F_{\vec{s}_n, \vec{h}_n}$, $F_{w, \vec{s}_n, \vec{h}_n}$ are given by (3), (4), respectively.

Proof. Let F_n be given by (5). Using similar method given as in the proof of Theorem 2.9, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} F_n(x) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathcal{H}^n} \exp \left\{ -\frac{1}{2} t^2 \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} \exp \left\{ it \sum_{l=1}^n (s_l - s_{l-1}) \right. \\ & \quad \times \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \left. \right\} \exp \left\{ -t \sum_{j=1}^n (w(s_j), x(s_j))^\sim \right\} F_{\vec{s}_n, \vec{h}_n}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ & \quad dm_{\mathbb{B}}(x) \end{aligned} \quad (19)$$

for $t \in \mathbb{R}$. Let $\|w\|_{C_0(\mathbb{B})} = \sup_{0 \leq s \leq T} \|w(s)\|_{\mathbb{B}}$. Then we have

$$\begin{aligned} & \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \leq T \sum_{l=1}^n \left| \sum_{j=l}^n \|w\|_{C_0(\mathbb{B})} \right|^2 \\ &= T \|w\|_{C_0(\mathbb{B})}^2 \sum_{l=1}^n l^2 = \frac{T}{6} \|w\|_{C_0(\mathbb{B})}^2 n(n+1)(2n+1) \equiv \beta \end{aligned}$$

and, for $\alpha \in \mathbb{R}$, we also have

$$\begin{aligned} & \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} \exp \left\{ \alpha \sum_{j=1}^n (w(s_j), x(s_j))^\sim \right\} dm_{\mathbb{B}}(x) d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} \exp \left\{ \frac{\alpha^2}{2} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n w(s_j) \right|^2 \right\} d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) \\ &\leq \exp \left\{ \frac{\alpha^2 \beta}{2} \right\} \|\mu_{F_n}\| < \infty \end{aligned}$$

so that

$$\begin{aligned} & \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} \left| \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right| \exp \left\{ \alpha \sum_{j=1}^n (w(s_j), \right. \\ & \left. x(s_j))^\sim \right\} dm_{\mathbb{B}}(x) d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) \\ & < \infty \end{aligned}$$

by (17). Now, differentiating both sides of (19) as a function of t for $|t| < \epsilon$ with $\epsilon > 0$, arbitrarily, and letting $t = 0$, by [9, Theorem 2.27], we have the result. $\square$

Remark 2.14 Suppose that the assumptions given as in Lemma 2.13 hold. For $E \in \mathcal{B}(\Delta_n \times \mathcal{H}^n)$, let

$$\mu_{F_n}^w(E) = \int_E i \sum_{l=1}^n \left[(s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] d\mu_{F_n}(\vec{s}_n, \vec{h}_n)$$

and let F_n^w be given by (5) with replacing μ_{F_n} by $\mu_{F_n}^w$. Then, $F_n^w \in S''_{n, \mathbb{B}}$ and another expression of (18) is given by

$$\begin{aligned} & \int_{C_0(\mathbb{B})} F_n^w(x) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} \int_{\Delta_n \times \mathcal{H}^n} F_{\vec{s}_n, \vec{h}_n}(x) \sum_{j=1}^n (w(s_j), x(s_j))^\sim d\mu_{F_n}(\vec{s}_n, \vec{h}_n) dm_{\mathbb{B}}(x). \end{aligned}$$

Theorem 2.15 Suppose that the assumptions given as in Lemma 2.13 hold. Then, for $q \in \mathbb{R} - \{0\}$, we have

$$\begin{aligned} & \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})}^{anf_q} F_{\vec{s}_n, \vec{h}_n}(x) \\ & dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= -q \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})}^{anf_q} F_{w, \vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

with the existences of the Feynman integrals of the both sides of the equality.

Proof. By (16) and (17), for $\lambda > 0$, we have

$$\begin{aligned} & \frac{i}{\lambda} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x) \\ & dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x) dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n). \end{aligned} \quad (20)$$

Letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$, we have the result by Morera's and the dominated convergence theorems. $\square$

Theorem 2.16 *Let $w \in C_0(\mathbb{B})$ with $w(s) \in \mathcal{H}$ for all $s \in [0, T]$. Suppose that*

$$\sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left| \sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty \quad (21)$$

where $\mu_{F_n} \in \mathcal{M}_n''$ for each $n \in \mathbb{N}$ and $\vec{s}_n, \vec{h}_n$ are given by (2). Then, for $q \in \mathbb{R} - \{0\}$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})}^{anf_q} F_{\vec{s}_n, \vec{h}_n}(x) \\ & dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= -q \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})}^{anf_q} F_{w, \vec{s}_n, \vec{h}_n}(x) dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

with the convergence of the series of both sides of the equality, where $F_{\vec{s}_n, \vec{h}_n}$, $F_{w, \vec{s}_n, \vec{h}_n}$ are given by (3), (4), respectively.

Proof. By (20), (21) and the dominated convergence theorem, for $\lambda > 0$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{i}{\lambda} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x) \\ & dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x) dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}_n, \vec{h}_n). \end{aligned}$$

Letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$, we have the result by Theorem 2.15, Morera's and the dominated convergence theorems. $\square$

3 First variation and Fourier-Feynman transform of cylinder type functions

For a given extended real number p with $1 < p \leq \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let G_n and G be measurable functions such that, for each $\gamma > 0$,

$$\lim_{n \rightarrow \infty} \int_{C_0(\mathbb{B})} |G_n(\gamma x) - G(\gamma x)|^{p'} dm_{\mathbb{B}}(x) = 0.$$

Then we write

$$\text{l.i.m.}_{n \rightarrow \infty} (w_s^{p'}) (G_n) \approx G$$

and call G the scale-invariant limit in the mean of order p' . A similar definition is understood when n is replaced by a continuously varying parameter.

Now, we define Fourier-Feynman transform of functions over Wiener paths in abstract Wiener space.

Definition 3.1 Let F be defined on $C_0(\mathbb{B})$ and for $\lambda \in \mathbb{C}_+$ let

$$T_\lambda(F)(y) = E^{anw_\lambda}[F(\cdot + y)]$$

for s -a.e. $y \in C_0(\mathbb{B})$ if it exists. For a non-zero real q , we define the L_1 Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$T_q^{(1)}(F)(y) = E^{anf_q}[F(\cdot + y)]$$

if it exists for s -a.e. $y \in C_0(\mathbb{B})$ and for $1 < p \leq \infty$ we define the L_p Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula

$$T_q^{(p)}(F) \approx \text{l.i.m.}_{\lambda \rightarrow -iq} (w_s^{p'}) (T_\lambda(F))$$

where λ approaches to $-iq$ through $\mathbb{C}_+$.

Lemma 3.2 Let $F_{\vec{s}_n, \vec{h}_n}$ be given by (3). Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then $T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})$ exists and it is given by

$$T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) = F_{\vec{s}_n, \vec{h}_n}(y) \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \quad (22)$$

for s -a.e. $y \in C_0(\mathbb{B})$. In particular, if $n = 1$, $h_1 \equiv h \in \mathcal{H}$ and $s_1 = u$ with $0 < u \leq T$, then (22) is reduced to the following equation

$$T_q^{(p)}(F_{u,h})(y) = F_{u,h}(y) \exp \left\{ -\frac{i u}{2q} |h|^2 \right\}$$

for s -a.e. $y \in C_0(\mathbb{B})$.

Proof. For $\lambda > 0$ and s -a.e. $y \in C_0(\mathbb{B})$, by Lemma 2.6, we have

$$\begin{aligned} T_\lambda(F_{\vec{s}_n, \vec{h}_n})(y) &= \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x + y) dm_{\mathbb{B}}(x) \\ &= F_{\vec{s}_n, \vec{h}_n}(y) \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x) dm_{\mathbb{B}}(x) \\ &= F_{\vec{s}_n, \vec{h}_n}(y) \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \end{aligned}$$

and the result holds for $\lambda \in \mathbb{C}_+$ by Morera's theorem. Let $T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})$ be given by (22). If $p = 1$, then $T_q^{(1)}(F_{\vec{s}_n, \vec{h}_n})$ is the genuine L_1 Fourier-Feynman transform of $F_{\vec{s}_n, \vec{h}_n}$ by the continuity of the exponential function. For $\gamma > 0$ and $\frac{1}{p} + \frac{1}{p'} = 1$ ($1 < p \leq \infty$), we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} |T_\lambda(F_{\vec{s}_n, \vec{h}_n})(\gamma y) - T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(\gamma y)|^{p'} dm_{\mathbb{B}}(y) \\ & \leq \int_{C_0(\mathbb{B})} \left| \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right. \\ & \quad \left. - \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right|^{p'} dm_{\mathbb{B}}(y) \end{aligned}$$

which tends to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Hence, the results follow. $\square$

The following theorem follows immediately from Lemmas 2.3 and 3.2.

Theorem 3.3 *Suppose that the assumptions given as in Lemma 3.2 hold. Then, for $w \in C_0(\mathbb{B})$, we have*

$$\begin{aligned} & T_q^{(p)}(\delta_w F_{\vec{s}_n, \vec{h}_n})(y) \\ & = \delta_w T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) \\ & = i F_{\vec{s}_n, \vec{h}_n}(y) \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \sum_{j=1}^n (h_j, w(s_j))^\sim \end{aligned}$$

for s -a.e. $y \in C_0(\mathbb{B})$.

Theorem 3.4 *Let F_n, F be given by (5), (6), respectively, let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, both $T_q^{(p)}(F_n)(y)$ and $T_q^{(p)}(F)(y)$ exist for s -a.e. $y \in C_0(\mathbb{B})$ and they are given by*

$$T_q^{(p)}(F_n)(y) = \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \quad (23)$$

where $T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})$ is given by (22). Moreover, for s -a.e. $y \in C_0(\mathbb{B})$, we have

$$T_q^{(p)}(F)(y) = \sum_{n=1}^{\infty} T_q^{(p)}(F_n)(y) \quad (24)$$

where $T_q^{(p)}(F_n)$ is given by (23).

Proof. The results follow from [5, Lemma 3.1 and Theorem 3.5] for $1 \leq p < \infty$. It remains to prove the results for the case $p = \infty$. Let $T_q^{(\infty)}(F_n)$, $T_q^{(\infty)}(F)$ be given by (23), (24), respectively. By [5, Lemma 3.1 and Theorem 3.5], for $\lambda \in \mathbb{C}_+$ and $\gamma > 0$, we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} |T_\lambda(F_n)(\gamma y) - T_q^{(\infty)}(F_n)(\gamma y)| dm_{\mathbb{B}}(y) \\ &= \int_{C_0(\mathbb{B})} \left| \int_{\Delta_n \times \mathcal{H}^n} F_{\vec{s}_n, \vec{h}_n}(\gamma y) \left[\exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right. \right. \\ & \quad \left. \left. - \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right] d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \right| dm_{\mathbb{B}}(y) \\ &\leq \int_{\Delta_n \times \mathcal{H}^n} \left| \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right. \\ & \quad \left. - \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) \end{aligned}$$

and

$$\begin{aligned} & \int_{C_0(\mathbb{B})} |T_\lambda(F)(\gamma y) - T_q^{(\infty)}(F)(\gamma y)| dm_{\mathbb{B}}(y) \\ &\leq \sum_{n=1}^{\infty} \int_{C_0(\mathbb{B})} |T_\lambda(F_n)(\gamma y) - T_q^{(\infty)}(F_n)(\gamma y)| dm_{\mathbb{B}}(y) \\ &\leq \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left| \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right. \\ & \quad \left. - \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} \right| d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n). \end{aligned}$$

By the dominated convergence theorem, letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$, the proof is completed. $\square$

Remark 3.5 Suppose that the assumptions given as in Theorem 3.4 hold. For $E \in \mathcal{B}(\Delta_n \times \mathcal{H}^n)$, let

$$\mu_{F_n}^q(E) = \int_E \exp \left\{ -\frac{i}{2q} \sum_{l=1}^n (s_l - s_{l-1}) \left| \sum_{j=l}^n h_j \right|^2 \right\} d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \quad (25)$$

and let F_n^q , F^q be given by (5), (6), respectively, with replacing μ_{F_n} by $\mu_{F_n}^q$. Then $T_q^{(p)}(F_n) = F_n^q \in S_{n, \mathbb{B}}''$ and $T_q^{(p)}(F) = F^q \in S_{\mathbb{B}}''$ so that $T_{-q}^{(p)}(T_q^{(p)}(F_n)) \approx F_n$

and $T_q^{(p)}(T_q^{(p)}(F)) \approx F$, that is, $T_q^{(p)}$ is the inverse transform of $T_q^{(p)}$ by Theorem 3.4. Thus we have the following theorem.

Theorem 3.6 *Suppose that the assumptions given as in Theorem 3.4 hold. Let $w \in C_0(\mathbb{B})$ and suppose that (10) holds. Then, for s-a.e. $y \in C_0(\mathbb{B})$, we have*

$$T_q^{(p)}(\delta_w F_n)(y) = \delta_w(T_q^{(p)}(F_n))(y) \quad (26)$$

and

$$T_q^{(p)}(\delta_w F)(y) = \delta_w(T_q^{(p)}(F))(y) \quad (27)$$

with the existences of both sides of the equalities.

Proof. For any Borel subset E of $\Delta_n \times \mathcal{H}^n$, let

$$\mu_{F_n}^w(E) = \int_E i \sum_{j=1}^n (h_j, w(s_j))^\sim d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \quad (28)$$

where $\vec{s}_n$ and $\vec{h}_n$ are given by (2). Then we have $\mu_{F_n}^w \in \mathcal{M}_n''$ so that $\delta_w F_n \in S_{n, \mathbb{B}}''$ by Theorem 2.4. Thus, for s-a.e. $y \in C_0(\mathbb{B})$, we have

$$\begin{aligned} & T_q^{(p)}(\delta_w F_n)(y) \\ &= \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}^w(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(\delta_w F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} \delta_w T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= \int_{\Delta_n \times \mathcal{H}^n} \delta_w F_{\vec{s}_n, \vec{h}_n}(y) d\mu_{F_n}^q(\vec{s}_n, \vec{h}_n) \\ &= \delta_w(T_q^{(p)}(F_n))(y) \end{aligned}$$

by (9), (23), (28) and Theorem 3.3, where $\mu_{F_n}^q$ is given by (25). This completes the proof of (26). (27) follows immediately from (26). $\square$

Remark 3.7 *For the result in (26), the condition (10) can be replaced by the condition (8).*

Theorem 3.8 Suppose that the assumptions given as in Theorem 2.11 hold. Then, for $1 \leq p \leq \infty$, we have

$$\begin{aligned} & T_q^{(p)}(F_{w, \vec{s}_n, \vec{h}_n})(y) \\ &= -\frac{1}{q} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) \\ & \quad + \left[\sum_{j=1}^n (w(s_j), y(s_j))^\sim \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Proof. For $\lambda > 0$, multiplying $F_{\vec{s}_n, \vec{h}_n}(y)$ to the both sides of (16), we have

$$\begin{aligned} & \frac{i}{\lambda} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x + y) dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})} F_{w, \vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x + y) dm_{\mathbb{B}}(x) - \left[\sum_{j=1}^n (w(s_j), y(s_j))^\sim \right] \\ & \quad \times \int_{C_0(\mathbb{B})} F_{\vec{s}_n, \vec{h}_n}(\lambda^{-\frac{1}{2}}x + y) dm_{\mathbb{B}}(x) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$. Now, $T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})$ exists by Lemma 3.2 and letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$, we have the result. $\square$

Remark 3.9 Let $1 \leq p \leq \infty$. Under the assumptions given as in Remark 2.12, we have

$$\begin{aligned} & T_q^{(p)}(F_{w, u, h})(y) \\ &= \frac{i u}{q} T_q^{(p)}(\delta_w F_{u, h})(y) + (w(u), y(u))^\sim T_q^{(p)}(F_{u, h})(y) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Theorem 3.10 Suppose that the assumptions given as in Theorem 2.15 hold. Then, for $1 \leq p \leq \infty$, we have

$$\begin{aligned} & \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(F_{w, \vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= -\frac{1}{q} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ & \quad + \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{j=1}^n (w(s_j), y(s_j))^\sim \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Proof. By (17) and (22), the first integral of the right hand side of the equality exists. Using the process given as in the proof of Lemma 2.13, by (22), we have

$$\int_{\Delta_n \times \mathcal{H}^n} \int_{C_0(\mathbb{B})} \left| T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) \sum_{j=1}^n (w(s_j), y(s_j))^\sim \right| dm_{\mathbb{B}}(y) d\|\mu_{F_n}\|(\vec{s}_n, \vec{h}_n) < \infty$$

so that the second integral of the right hand side of the equality exists for s -a.e. y in $C_0(\mathbb{B})$. Now, integrating both sides of the equation given as in Theorem 3.8 with respect to μ_{F_n} , we have the result. $\square$

Now, by (21), (22) and Theorem 3.10, we have the following theorem.

Theorem 3.11 *Suppose that the assumptions given as in Theorem 2.16 hold. Let $1 \leq p \leq \infty$ and suppose that $\sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(F_{w, \vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n)$ exists for s -a.e. $y \in C_0(\mathbb{B})$. Then we have*

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} T_q^{(p)}(F_{w, \vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \\ &= -\frac{1}{q} \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{l=1}^n (s_l - s_{l-1}) \left\langle \sum_{j=l}^n h_j, \sum_{j=l}^n w(s_j) \right\rangle \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) \\ & \quad d\mu_{F_n}(\vec{s}_n, \vec{h}_n) + \sum_{n=1}^{\infty} \int_{\Delta_n \times \mathcal{H}^n} \left[\sum_{j=1}^n (w(s_j), y(s_j))^\sim \right] T_q^{(p)}(F_{\vec{s}_n, \vec{h}_n})(y) d\mu_{F_n}(\vec{s}_n, \vec{h}_n) \end{aligned}$$

for s -a.e. $y \in C_0(\mathbb{B})$.

4 Transformation of cylinder type functions with linear factors

Let $0 < u \leq T$ be fixed, but arbitrarily, let F_0 be defined on $C_0(\mathbb{B})$ and for any given fixed $n \in \mathbb{N}$ let

$$F_j(x) = F_0(x) \prod_{l=1}^j (w_l(u), x(u))^\sim \quad (29)$$

for s -a.e. $x \in C_0(\mathbb{B})$ where $w_l \in C_0(\mathbb{B})$ with $w_l(u) \in \mathcal{H}$ for $l = 1, \dots, n$.

Our first theorem gives a recurrence relation in which we express the transform of F_j in terms of the transforms and variation of F_{j-1} . The proof follows immediately from the definition of Fourier-Feynman transform([8]).

Theorem 4.1 For $1 \leq p \leq \infty$, for a non-zero real q and for $j = 1, \dots, n$, assume that both $T_q^{(p)}(\delta_{w_j} F_{j-1})(y)$ and $T_q^{(p)}(F_{j-1})(y)$ can be expressible as analytic Feynman integrals for s-a.e. $y \in C_0(\mathbb{B})$. Moreover, $F_{j-1}(\cdot + y)$ satisfies the equation given as in Remark 2.12 with replacing $w, F_{u,h}$ by $w_j, F_{j-1}(\cdot + y)$, respectively. Then $T_q^{(p)}(F_j)(y)$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by the recurrence relation

$$T_q^{(p)}(F_j)(y) = \frac{i u}{q} T_q^{(p)}(\delta_{w_j} F_{j-1})(y) + (w_j(u), y(u))^\sim T_q^{(p)}(F_{j-1})(y).$$

The following theorem is an immediate result of Theorem 4.1([8]).

Theorem 4.2 For some $\epsilon > 0$, the assumptions given as in Theorem 4.1 hold with replacing F_{j-1} by $F_{j-1}(\cdot + t w_{j+1})$ for $|t| < \epsilon$ and $j = 1, \dots, n-1$. Moreover, assume that both

$$T_q^{(p)}(\delta_{w_{j+1}} F_j)(y) = \delta_{w_{j+1}}(T_q^{(p)}(F_j))(y)$$

and

$$T_q^{(p)}(\delta_{w_{j+1}} F_{j-1})(y) = \delta_{w_{j+1}}(T_q^{(p)}(F_{j-1}))(y)$$

exist for s-a.e. $y \in C_0(\mathbb{B})$. Then, for s-a.e. $y \in C_0(\mathbb{B})$, we have the recurrence relation

$$\begin{aligned} T_q^{(p)}(\delta_{w_{j+1}} F_j)(y) &= \frac{i u}{q} \delta_{w_{j+1}}(T_q^{(p)}(\delta_{w_j} F_{j-1}))(y) + \langle w_j(u), w_{j+1}(u) \rangle \\ &\quad \times T_q^{(p)}(F_{j-1})(y) + (w_j(u), y(u))^\sim T_q^{(p)}(\delta_{w_{j+1}} F_{j-1})(y). \end{aligned}$$

Now, let $h \in \mathcal{H}$ and F_0 be given by

$$F_0(x) = \exp\{i(h, x(u))^\sim\}$$

for s-a.e. $x \in C_0(\mathbb{B})$. Let F_n be given by (29). Then we have

$$T_q^{(p)}(F_1)(y) = \frac{i u}{q} \delta_{w_1}(T_q^{(p)}(F_0))(y) + (w_1(u), y(u))^\sim T_q^{(p)}(F_0)(y)$$

for s-a.e. $y \in C_0(\mathbb{B})$, by (15), Remarks 2.12, 3.9 and Theorem 3.3. Also, we have

$$\delta_{w_1}(T_q^{(p)}(F_0))(y) = i \langle h, w_1(u) \rangle \exp\left\{-\frac{i u}{2q} |h|^2\right\} F_0(y)$$

and hence

$$\delta_{w_2}(\delta_{w_1}(T_q^{(p)}(F_0)))(y) = -\langle h, w_1(u) \rangle \langle h, w_2(u) \rangle \exp\left\{-\frac{i u}{2q} |h|^2\right\} F_0(y)$$

for s -a.e. $y \in C_0(\mathbb{B})$. Thus, by Theorem 4.2, we have

$$\begin{aligned} T_q^{(p)}(\delta_{w_2} F_1)(y) &= \frac{i u}{q} \delta_{w_1}(T_q^{(p)}(w_2 F_0))(y) + \langle w_1(u), w_2(u) \rangle T_q^{(p)}(F_0)(y) \\ &\quad + (w_1(u), y(u)) \sim T_q^{(p)}(\delta_{w_2} F_0)(y) \\ &= \delta_{w_2}(T_q^{(p)}(F_1))(y) \end{aligned}$$

for s -a.e. $y \in C_0(\mathbb{B})$. By Theorem 4.1 with $j = 2$, we see that

$$T_q^{(p)}(F_2)(y) = \frac{i u}{q} \delta_{w_2}(T_q^{(p)}(F_1))(y) + (w_2(u), y(u)) \sim T_q^{(p)}(F_1)(y)$$

for s -a.e. $y \in C_0(\mathbb{B})$. Continuing this process, we have

$$\delta_{w_n}(\cdots \delta_{w_1}(T_q^{(p)}(F_0)) \cdots)(y) = \left(\prod_{l=1}^n i \langle h, w_l(u) \rangle \right) \exp \left\{ -\frac{i u}{2q} |h|^2 \right\} F_0(y)$$

for s -a.e. $y \in C_0(\mathbb{B})$. Hence $\delta_{w_n}(T_q^{(p)}(F_{n-1}))(y)$ exists for s -a.e. $y \in C_0(\mathbb{B})$ so that we obtain, by Theorem 4.1,

$$T_q^{(p)}(F_n)(y) = \frac{i u}{q} \delta_{w_n}(T_q^{(p)}(F_{n-1}))(y) + (w_n(u), y(u)) \sim T_q^{(p)}(F_{n-1})(y)$$

for s -a.e. $y \in C_0(\mathbb{B})$. Thus we have the following theorem.

Theorem 4.3 *Under the above assumptions, for $k = 1, \dots, n$, we have*

$$\begin{aligned} T_q^{(p)}(F_k)(y) &= \frac{i u}{q} \sum_{j=0}^{k-1} \left[\delta_{w_{j+1}}(T_q^{(p)}(F_j))(y) \left(\prod_{l=j+2}^k (w_l(u), y(u)) \sim \right) \right] \\ &\quad + T_q^{(p)}(F_0)(y) \left(\prod_{l=1}^k (w_l(u), y(u)) \sim \right) \end{aligned}$$

for s -a.e. $y \in C_0(\mathbb{B})$.

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