Latin Squares in a Cayley Table

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Abstract

Let G be a finite group of order $n \geq 2$ and T be the Cayley table of G . Obviously T is a Latin square. In this paper we study submatrices of T which are Latin squares. Let $L_m(G)$ where $m \leq n$ be the number of Latin squares of order m in T . We compute $L_m(G)$ and study some properties of such submatrices. Also we classify the Latin squares of order m in T and compute the number of each classes in some cases.

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A *Latin square of order n* is an $n \times n$ array in which n distinct symbols are arranged so that each symbol occurs once in each row and column.

Let G be a finite group. Visual display of G as a Cayley table is usually introduced in an elementary course on abstract algebra. Let T be the Cayley table of G . Elementary properties of G implies that T be a Latin square. Also it was proved that a Latin square is a Cayley table if the composite of two rows is some row in the table ([6]).

Let $(x_1, \dots, x_m)$ be an m -tuple of elements of G , where $m > 1$. Let $A = (a_{ij})$ be a square matrix of order m such that $a_{ij} = x_i x_j$. Obviously A is a

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diagonal submatrix of T . The investigation of how many different types of such matrices could exist and of their properties were problems solved by Freiman for $n = 2, 3$ (see [3]). If $K = (a, b)$ is a couple of elements of G - distinct or not - then the multiplication table

$$\begin{pmatrix} a^2 & ab \\ ba & b^2 \end{pmatrix}$$

of K is one of the following types

$$T_0 = \begin{pmatrix} A & A \\ A & A \end{pmatrix}, \quad T_1 = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad T_2 = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

$$T_3 = \begin{pmatrix} A & B \\ C & A \end{pmatrix}, \quad T_4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where equal (distinct) letters denote the same (different) elements of K^2 . Denote by $P_i(G) = p_i$ the number of couples K of elements of G of type T_i , where $i = 0, 1, 2, 3, 4$. Brailovsky and Herzog [2] have examined the case $n = 2$, showing that the number p_i of ordered couples of elements of G corresponding to the matrix A is a multiple of $|G|$ and determining, for $i = 1, 2, 4$, the properties of those groups for which $p_i = 0$. Then Bianchi and Gillio [1] extended these results and properties to the case $n \geq 3$. These results were used in determining of the bound on squares of two-sets by Slilaty and Vanderkam [7].

Since T has many other submatrices it is a natural question that:

For each $m \leq n$, how many submatrices of order m in T are Latin squares?

In the sequel we use the following notations:

$Inv(G) :=$ The set of involutions of G ,

$k :=$ The number of conjugacy classes of G ,

$k_i :=$ The number of conjugacy classes of involutions in G ,

$T :=$ The Cayley table of G ,

$L_m(G) :=$ The number of $m \times m$ submatrices of T which are Latin squares,

$LS(G) := \sum_{m|G|} L_m(G)$.

Notation. Let $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_m\}$. Let $L = (l_{ij})$, where $l_{ij} = b_i a_j$, ($1 \leq i, j \leq m$), i.e.

$$L := \begin{array}{c|cccc} & a_1 & a_2 & \cdots & a_m \\ \hline b_1 & b_1 a_1 & b_1 a_2 & \cdots & b_1 a_m \\ b_2 & b_2 a_1 & b_2 a_2 & \cdots & b_2 a_m \\ \vdots & \vdots & \vdots & & \vdots \\ b_m & b_m a_1 & b_m a_2 & \cdots & b_m a_m \end{array}$$

Then L is a $m \times m$ submatrix of T . Throughout this paper we denote this submatrix by $(b_1, \dots, b_m; a_1, \dots, a_m)$ or simply by $(B; A)$.

Let $r := \#(A \cup B)$. Obviously $m \leq r \leq 2m$. We use the notation $t_{m,r}(G)$ to denote the number of Latin squares of the form $L := (B; A)$ where $r := \#(A \cup B)$. Obviously

$$L_m(G) = \sum_{r=m}^{2m} t_{m,r}(G).$$

The aim of this paper is to compute $L_m(G)$, where $m \leq n$, and study the properties of these submatrices.

Theorem 1. Let $L = (b_1, \dots, b_m; a_1, \dots, a_m)$ be an $m \times m$ submatrix of the Cayley table of G . Then there exists a subgroup H of G , such that $|H| = m$ and there exist $\alpha, \beta \in G$ such that $\{a_1, \dots, a_m\} = \alpha H$ and $\{b_1, \dots, b_m\} = \beta \alpha H \alpha^{-1}$. Also the set of elements in each row or in each column of L is $\beta \alpha H$.

Proof. Let $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m\}$, $\bar{A} = (a_1, \dots, a_m)$ and $\bar{B} = (b_1, \dots, b_m)$. By the definition of L , the rows of L are $b_1 \bar{A}, b_2 \bar{A}, \dots, b_m \bar{A}$ and the columns of L are $\bar{B} a_1, \bar{B} a_2, \dots, \bar{B} a_m$. Also for every i and j , where $1 \leq i, j \leq m$ we have $b_i A = B a_j$, since L is a Latin square. Let $S := b_1 A$ and $H = a_1^{-1} A$. Now we prove that H is a subgroup of G . Since G is finite, it is enough to prove that H is closed.

Let $h_1 = a_1^{-1} a_i$ and $h_2 = a_1^{-1} a_j$, where $1 \leq i, j \leq m$. Then $(a_1^{-1} a_i)(a_1^{-1} a_j) = a_1^{-1} (a_i a_1^{-1} a_j)$. As we mentioned above, $b_1 A = B a_1$. Therefore there exists some t such that $b_1 a_i = b_t a_1$ and so $a_i a_1^{-1} = b_1^{-1} b_t$. Since $b_t A = b_1 A$, it follows that there exist some f such that $1 \leq f \leq m$ and $b_t a_j = b_1 a_f$. Therefore $a_i a_1^{-1} a_j = b_1^{-1} b_t a_j = b_1^{-1} b_1 a_f = a_f \in A$ and so H is closed. Hence H is a subgroup of G and $A = a_1 H$. Since $b_1 A = B a_1$, it follows that $B = b_1 a_1 H a_1^{-1}$. Therefore $S = b_1 a_1 H$ is the set of elements in L . $\diamond$

Corollary 1. If $m \nmid |G|$, then $L_m(G) = 0$. Hence if $|G|/2 < m < |G|$, then $L_m(G) = 0$.

Theorem 2. Let G be a finite group and p be a prime divisor of $|G|$. Then

$$L_p(G) = \frac{t|G|^2}{p^2(p-1)},$$

where t is the number of elements of order p in G .

Proof. By Theorem 3.1, it follows that if $L = (B; A)$ is a Latin square of order p , then there exists a subgroup H of order p such that $A = \alpha H$ and $B = \beta \alpha H \alpha^{-1}$, where $\alpha, \beta \in G$. Therefore A is a left coset of H and B is a right coset of H^α . Hence

$$L_p(G) = |G : H| \times |G : H^\alpha| \times s,$$

where s is the number of subgroups of order p . As $|H| = |H^\alpha| = p$, it follows that

$$L_p(G) = \frac{|G|^2}{p^2} \times s.$$

Also every $p - 1$ elements of order p determine a unique subgroup of order p in G . Therefore

$$L_p(G) = \frac{|G|^2 t}{p^2(p-1)},$$

where t is the number of elements of order p in G . $\diamond$

Example 1. If $S_3 = \langle a, b \rangle$, where $a^3 = 1$ and $b^2 = 1$, then the 3×3 Latin squares are $(1, a, a^2; 1, a, a^2)$, $(1, a, a^2; b, ab, a^2b)$, $(b, ab, a^2b; 1, a, a^2)$ and $(b, ab, a^2b; b, ab, a^2b)$. Hence $L_3(S_3) = 4$. Similarly $L_2(S_3) = 27$.

If $Z_6 = \langle x \rangle$, where $x^6 = 1$, then the 3×3 Latin submatrices are $(1, x^2, x^4; 1, x^2, x^4)$, $(x, x^3, x^5; x, x^3, x^5)$, $(x, x^3, x^5; 1, x^2, x^4)$ and $(1, x^2, x^4; x, x^3, x^5)$. Therefore $L_3(Z_6) = 4$ but $L_2(Z_6) = 9$.

Example 2. By using the above theorem it follows that $L_2(Q_8) = 16$, $L_2(D_8) = 80$, $L_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = 112$, $L_3(\mathbb{Z}_9) = 9$, $L_3(\mathbb{Z}_3 \times \mathbb{Z}_3) = 36$.

Theorem 3. Let G be a finite group. Then

- (i) $L_2(G) = |G|^2 |Inv(G)|/4$,
- (ii) $t_{2,2}(G) = |G|k_i/2$,
- (iii) $t_{2,3}(G) = |G|(|Inv(G)| - k_i)/2$,
- (iv) $t_{2,4}(G) = |G| |Inv(G)|(|G| - 2)/4$.

Proof. A Latin square of order 2 in the Cayley table of G is of the form $(a, b; c, d)$, where $a \neq b$, $c \neq d$, $ac = bd$ and $bc = ad$. By Theorem 1, it follows that $\{a, b\} = \{\alpha, \alpha x\}$ and $\{c, d\} = \{\beta, \beta \alpha x \alpha^{-1}\}$, where x is an involution of G . Also by using Theorem 2, we have $L_2(G) = |G|^2 |Inv(G)|/4$.

By the definition, $t_{2,2}(G)$ is the number of Latin squares of type $(a, b; a, b)$ where a and b are different elements of G , i.e. $a \neq b$, $ab = ba$ and $a^2 = b^2$. If $x = ba^{-1}$, then x is an involution of G . Also $b = xa$ and so $ab = axa$. As $ab = ba$, it follows that $b = ax$. Therefore $a \in C_G(x)$ and $b = xa$.

Conversely if x is an involution and $a \in C_G(x)$, then a unique Latin square of type $(a, ax; a, ax)$ is obtained and hence

$$t_{2,2}(G) = \frac{1}{2} \sum_{x \in Inv(G)} |C_G(x)| = k_i |G|/2,$$

where k_i denotes the number of conjugacy classes of involutions in G .

Similarly $t_{2,3}(G)$ is the number of Latin squares of the form $(a, b; a, c)$ where a , b and c are different elements of G such that $ac = ba$ and $a^2 = bc$. As we

suppose that $b \neq c$, it follows that $b, c \in G \setminus C_G(a)$. If $x = a^{-1}b$, then $a = xc$ and $c = xa$. Hence $x \in \text{Inv}(G)$ and obviously $a \notin C_G(a)$. Conversely let x be an involution of G . If $a \in G \setminus C_G(x)$, then a Latin square of type $(a, ax; a, xa)$ is obtained. therefore

$$t_{2,3}(G) = \frac{1}{2} \sum_{x \in \text{Inv}(G)} (|G| - |C_G(x)|) = (|\text{Inv}(G)| - k_i).$$

We know that $L_2(G) = \sum_{r=2}^4 t_{2,r}(G)$, and hence

$$t_{2,4}(G) = L_2(G) - t_{2,2}(G) - t_{2,3}(G) = |G| |\text{Inv}(G)| (|G| - 2)/4,$$

which is the number of Latin squares of type $(a, b; c, d)$ where a, b, c and d are different elements of G . $\diamond$

Corollary 2. Let G be a finite group. Then

- (i) $L_2(G) = 0 \Leftrightarrow G$ is of odd order,
- (ii) $t_{2,2}(G) = 0 \Leftrightarrow G$ is of odd order,
- (iii) $t_{2,3}(G) = 0 \Leftrightarrow \text{Inv}(G) \subset Z(G)$,
- (iv) $t_{2,4}(G) = 0 \Leftrightarrow$ either $G \cong \mathbb{Z}_2$ or G is of odd order.

Proof. (i), (ii) and (iv) are immediate consequences of the above theorem. If $k_i = |\text{Inv}(G)|$, then every element of $\text{Inv}(G)$ is its conjugacy class. Hence every involution commutes with every element of G ; i.e. every involution belongs to $Z(G)$. Therefore $\text{Inv}(G) \subset Z(G)$. $\diamond$

Theorem 3. Let G be a finite group and m be a divisor of $|G|$ Then

$$L_m(G) = s|G|^2/m^2,$$

where s is the number of subgroups of order m .

Proof. Similar to the proof of Theorem 2, the result follows and we omit the proof for convenience. $\diamond$

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