Simple Groups With 2-Regular First Prime Graph Component

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Abstract

Let $G$ be a finite group. The prime graph of $G$ is the graph whose vertex set is the prime divisors of $|G|$, and two distinct primes $p$ and $q$ are joined by an edge if and only if $G$ contains an element of order $pq$. We denote by $\Gamma(G)$ the prime graph of $G$.

M. S. Lucido and A. R. Moghaddamfar in (Lucido and et. al. (2004), Groups with complete prime graph connected components, J. Group Theory, 31: 373-384) determined finite simple groups $G$, whose prime graph components are complete.

Let $\Gamma(G)$ be non-connected and $\Delta$ be a connected component of $\Gamma(G)$. It was proved that if the vertex set of $\Delta$ does not contain 2, then $\Delta$ is a clique. In this paper, we determine finite simple groups $G$ such that the connected component of $\Gamma(G)$ containing 2, is 2-regular.

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1 Introduction

If \( n \) is an integer, then we denote by \( \pi(n) \) the set of all prime divisors of \( n \).
If \( G \) is a finite group, then the set \( \pi(|G|) \) is denoted by \( \pi(G) \). Also the set of
order elements of \( G \) is denoted by \( \pi_e(G) \). Obviously \( \pi_e(G) \) is partially ordered
by divisibility. Therefore it is uniquely determined by \( \mu(G) \), the subset of its
maximal elements.

We construct the prime graph of \( G \) as follows: the prime graph \( \Gamma(G) \) of a
group \( G \) is the graph whose vertex set is \( \pi(G) \), and two distinct primes \( p \) and
\( q \) are joined by an edge (we write \( p \sim q \)) if and only if \( G \) contains an element
of order \( pq \). Let \( t(G) \) be the number of connected components of \( \Gamma(G) \) and let
\( \pi_1(G), \pi_2(G), \ldots, \pi_{t(G)}(G) \) be the connected components of \( \Gamma(G) \). Sometimes
we use the notation \( \pi_i \) instead of \( \pi_i(G) \). If \( 2 \in \pi(G) \), then we always suppose
\( 2 \in \pi_1 \). Denote by \( \mu_i = \mu_i(G) \) the set of all \( n \in \mu(G) \) such that each prime
divisor of \( n \) belongs to \( \pi_i \).

The concept of prime graph arose during the investigation of certain coho-
omological questions associated with integral representations of finite groups. It turns out that \( \Gamma(G) \) is not connected if and only if the augmentation ideal
of \( G \) is decomposable as a module \([6]\). Also non-connectedness of \( \Gamma(G) \) has
relations with the existence of isolated subgroups of \( G \). A proper subgroup \( H \)
of \( G \) is isolated if \( H \cap H^g = 1 \) or \( H \) for every \( g \in G \) and \( C_G(h) \leq H \) for all
\( h \in H \). It was proved in \([20]\) that \( G \) has a nilpotent isolated Hall \( \pi \)-subgroup
whenever \( G \) is non-solvable and \( \pi = \pi_i \) \((i > 1)\). In fact we have the following
equivalences:

**Theorem 1.1.** \( ([13]) \) If \( G \) is a finite group, then the following are equivalent:

(i) the augmentation ideal of \( G \) decomposes as a module,
(ii) the group \( G \) contains an isolated subgroup,
(iii) the prime graph of \( G \) has more than one component.

It is therefore interesting to discuss about the prime graph of finite groups. It has been proved that for every finite group \( G \) we have \( t(G) \leq 6 \) \([8, 13, 20]\)
and the diameter of \( \Gamma(G) \) is at most 5 \([14]\). Also Hagie in \([7]\) and the first
author in \([12]\) determined finite groups \( G \) satisfying \( \Gamma(G) = \Gamma(S) \), where \( S \) is
an almost sporadic simple group. It is proved that if \( G \) is a finite simple group
and \( \Gamma(G) \) is non-connected, then \( \pi_i(G) \), where \( i \geq 2 \), is a complete graph (or clique). Lucido and et. al. in \([17]\) determined finite simple groups \( G \)
whose prime graph components are complete. This is equivalent to \( \pi_1 \) is complete.
A graph is complete if every pair of elements is joined by an edge.

In this paper, we determine finite simple groups \( G \) such that \( \pi_1 \) is \( 2 \)-regular,
i.e. every element in \( \pi_1 \) is joined to exactly two elements of \( \pi_1 \). In fact we
prove the following result:
Main Theorem. Let $G$ be a finite simple group. Then the first connected component of $\Gamma(G)$ (i.e. $\pi_1(G)$) is $2$-regular if and only if $G$ is one of the following: $A_9$, $J_1$, $J_2$, $J_3$, $HS$, $PSL(2,q)$ where $4 \mid (q - 1)$ and $|\pi(q - 1)| = 3$, $PSL(2,q)$ where $4 \mid (q + 1)$ and $|\pi(q + 1)| = 3$; $PSp(4,q)$ where $q = 4, 5, 7, 8, 9, 17$; $PSp(6,2)$; $PSL(3,9)$, $PSL(4,3)$; $3D_4(2)$; $G_2(9)$; $O^+(8)$.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [3], for example. We use the results of J. S. Williams [20], N. Iiyori and H. Yamaki [8] and A. S. Kondrat’ev [13] about the prime graph of simple groups. We note that we will use the classification of finite simple groups, in the sequel.

2 Preliminary Results

Definition 2.1. A graph $P$ is called an $r$-regular graph, if every vertex of $P$ is joined to exactly $r$ vertices of $P$.

Lemma 2.1. ([20]) Let $G$ be a finite simple group whose prime graph $\Gamma(G)$ is not connected. Then $\pi_i$ is a clique, for $i \geq 2$.

Lemma 2.2. ([14, Lemma 5 and Proposition 7]) If $G$ is a finite simple group and $p \in \pi_1(G)$, then $d(2,p) \leq 2$. Also if $G$ is a finite group and $p \in \pi_1(G)$, then $d(2,p) \leq 3$.

Lemma 2.3. ([21]) Let $A_n$ be the alternating group on $n$ elements and $m = p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}$, where $p_1, p_2, \ldots, p_s$ are distinct primes and $a_1, a_2, \ldots, a_s$ and $s$ are natural numbers. Then $A_n$ has an element of order $m$ if and only if $p_1^{a_1} + p_2^{a_2} + \cdots + p_s^{a_s} \leq n$ for odd $m$ and $p_1^{a_1} + p_2^{a_2} + \cdots + p_s^{a_s} \leq n - 2$ for even $m$.

Lemma 2.4. ([17, Theorem 1]) Let $G$ be a finite simple group. Then all the connected components of $\Gamma(G)$ are cliques if and only if $G$ is one of the following: $A_5$, $A_6$, $A_7$, $A_9$, $A_{12}$, $A_{13}$; $M_{11}$, $M_{22}$, $J_1$, $J_2$, $J_3$, $HS$; $PSL(2,q)$ with $q > 2$, $Sz(q)$ with $q = 2^{2m+1}$, $PSp(4,q)$, $G_2(3^k)$, $PSL(3,q)$ where $q$ is a Mersenne prime, $PSU(3,q)$ where $q$ is a Fermat prime, $PSL(3,4)$, $PSU(3,9)$, $PSp(6,2)$, $PSU(4,3)$, $PSU(6,2)$, $O^+_8(2)$, $3D_4(2)$.

Lemma 2.5. ([17, Lemma 9]) If $G \neq A_{10}$ is a finite simple group and $\Gamma(G)$ is connected, then there exist three primes $r, s, t \in \pi(G)$ such that $\{rs, tr, ts\} \cap \pi_e(G) = \emptyset$. 

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [3], for example. We use the results of J. S. Williams [20], N. Iiyori and H. Yamaki [8] and A. S. Kondrat’ev [13] about the prime graph of simple groups. We note that we will use the classification of finite simple groups, in the sequel.
Lemma 2.6. ([19]) Let $a, b \in PSL(n, q)$ where $q = p^m$, $n \geq 4$, $|a| = p$ and \(ab = ba\). Then $\pi([b]) \subseteq \pi(SL(n - 2, q))$.

Definition 2.2. By using the prime graph of $G$, the order of $G$ can be expressed as a product of coprime positive integers $m_i$, $i = 1, 2, \ldots, t(G)$ where $\pi(m_i) = \pi_i(G)$. These integers are called the order components of $G$. The set of order components of $G$ will be denoted by $OC(G)$. Also we call $m_2, \ldots, m_t(G)$ the odd order components of $G$.

The order components of non-abelian simple groups are listed in [9].

Also we need the following number theoretic lemmas:

Lemma 2.7. ([11]) The only solution of the equation $p^m - q^n = 1$; $p, q$ prime; and $m, n > 1$ is $3^2 - 2^3 = 1$.

Next lemma was introduced by Crescenzo and modified by Bugeaud:

Lemma 2.8. ([4, 11]) With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation

$$p^m - 2q^n = \pm 1; \ p, q \ \text{prime}; \ m, n > 1,$$

has exponents $m = n = 2$; i.e. it comes from a unit $p - q.2^{\frac{1}{2}}$ of the quadratic field $Q(2^{\frac{1}{2}})$ with the coefficients $p, q$ are prime.

Lemma 2.9. (Zsigmondy’s Theorem) ([22])

Let $p$ be a prime and $n$ be a positive integer. Then one of the following holds:

(i) there is a primitive prime $p'$ for $p^n - 1$, that is, $p' | (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$,

(ii) $p = 2$, $n = 1$ or 6,

(iii) $p$ is a Mersenne prime and $n = 2$.

We denote by $q_n$ one of the primitive prime divisors of $q^n - 1$. Therefore if $q_n$ divides $q^m - 1$, then $m \geq n$.

Lemma 2.10. ([16, Lemma 2]) Let $q = p^f$, where $p$ is a prime and $f$ a natural number. Then

(i) if $f$ is even, $3 \mid (2^f - 1)$, if $f$ is odd, $3 \mid (2^f + 1)$;

(ii) $|\pi(q^2 - 1)| \leq 2 \iff q = 2, 3, 4, 5, 7, 8, 9, 17$;

(iii) $|\pi((q^2 - 1)/(3, q - 1))| \leq 2 \iff q = 2, 3, 4, 5, 7, 8, 9, 16, 17, 25, 49, 97$ or $q = p$, $p - 1 = 3 \cdot 2^\alpha$, $p + 1 = 2t$, $\alpha \geq 2$ and $t$ an odd prime;
and $G$ power, then there exists a torus $T$ such that
\[ d(T) = p + 1 = 3 \cdot 2^n; \]

(v) $|\pi((q^2 - 1)/(3, q + 1))| \leq 2 \implies q = 2f$, $f$ a prime, or $q = 3$, $9$, or $q = p$ and $p + 1 = 3 \cdot 2^n$.

By using Lemma 2.3, it follows that $|\pi((q - 1)/(2, q - 1))| \leq 2 \implies q \in \{4, 9, 16, 81\}$ or $q = p^f$, $f = 1$ or an odd prime.

3 Proof of the Main Theorem

In this section we prove the main theorem. First we prove a few lemmas.

By using Lemma 2.2 and the definition of a $2$-regular graph, it follows that if $G$ is a finite group and $\pi_1(G)$ is $2$-regular, then $3 \leq |\pi_1(G)| \leq 5$, since $d(2, p) \leq 2$ for every $p \in \pi_1(G)$. If $G$ is an abelian simple group, then $G \simeq Z_p$ for some prime number $p$. Therefore $|\pi_1(G)| = 1$ and hence $\pi_1(G)$ is not $2$-regular.

The structure of the sporadic simple groups is described in [3]. Now easily we can prove the following lemma.

Lemma 3.1. Let $G$ be a sporadic simple group and $\pi_1(G)$ be $2$-regular. Then $G$ is $J_1$, $J_2$, $J_3$ or $HS$.

Lemma 3.2. If $G$ is an alternating group on $n \geq 5$ elements and $\pi_1(G)$ is $2$-regular, then $n = 9$.

Proof. By using Lemma 2.3, it follows that if $n \geq 10$, then $2 \sim 3 \sim 5 \sim 2$ and $7 \in \pi_1(G)$, which implies that $\pi_1(G)$ is not $2$-regular. If $n = 8$, then $2 \sim 3 \sim 5$ but $2 \not\sim 5$. If $n = 5, 6$ or $7$, then $|\pi_1(G)| \leq 2$ and so $\pi_1(G)$ is not $2$-regular. If $n = 9$, then $|\pi_1(G)| = 3$ and $\pi_1(G)$ is $2$-regular. $\diamond$

Lemma 3.3. Let $G$ be a finite simple group. If $\Gamma(G)$ is connected, then $\Gamma(G)$ is not $2$-regular.

Proof. We know that $\Gamma(A_{10})$ is not $2$-regular, by Lemma 3.2. Let $G \neq A_{10}$ be a finite simple group and $\Gamma(G)$ be $2$-regular. Now by using Lemma 2.5, it follows that there exist $r, s, t \in \pi(G)$ such that $\{tr, ts, rs\} \cap \pi_e(G) = \emptyset$. Also $3 \leq |\pi_1(G)| \leq 5$, which is a contradiction, since every element of $\pi_1(G)$ is connected to exactly two elements of $\pi_1(G)$ and so $G$ is not $2$-regular. $\diamond$

Now we remind a well known result about simple groups of Lie type (see [1, 2]). Let $G = ^dL_n(q)$ be a simple group of Lie type of rank $n$ over the field $F = GF(q)$, where $q = p^n$. By using the table of order components of simple groups (see [9]), it follows that $\pi(q - 1) \subseteq \pi_1(G)$, where $G$ is a finite simple group of Lie type and $G \neq PSL(2, q)$ and $G \neq Sz(q)$. So if $G \neq PSL(2, q)$ and $G \neq Sz(q)$ is a finite simple group of Lie type and $q$ is an odd prime power, then there exists a torus $T$ of $G$, an involution $x \in T$ and a unipotent
subgroup \( H \) of \( G \), such that \( x \) centralizes an element of order \( p \) of \( H \) (see [1]). Therefore if \( q = p^a \) is odd, then \( p \sim 2 \) in \( \Gamma(G) \).

We recall that every element of \( \pi_1(G) - \{ p \} \) divides the order of an abelian maximal torus \( T \) of \( G \) (see [2]). If \( t \in \pi_1(G) - \{ p \} \) and \( t \mid |T| \), then there exists an involution \( x \in G \), such that \( k = (|T|, |C_G(x)|) \neq 1 \). If \( p' \) is a prime divisor of \( k \), then \( tp' \mid |T| \) and \( p' \mid |C_G(x)| \), which implies that \( t \sim p' \) and \( 2 \sim p' \) in \( \Gamma(G) \).

By using the above discussion, we can conclude that:

**Lemma 3.4.** Let \( G = dL_n(q) \) be a simple group of Lie type of rank \( n \) over the field \( F = GF(q) \), where \( q = p^a \). Also let \( G \neq PSL(2,q) \) and \( G \neq Sz(q) \). If \( p \) is odd, then \( 2 \sim p \) in \( \Gamma(G) \). If \( p = 2 \) and there exists a maximal torus \( T \) of \( G \) such that \( |T| = s^m \), where \( s \in \pi_1(G) \), then \( 2 \sim s \) in \( \Gamma(G) \).

**Lemma 3.5.** Let \( G \) be a finite simple group of Lie type. Then \( \pi_1(G) \) is 2–regular if and only if \( G \) is one of the following groups: \( PSL(2,q) \), where \( 4 \mid (q - 1) \) and \( |\pi(q - 1)| = 3 \); \( PSL(2,q) \), where \( 4 \mid (q + 1) \) and \( |\pi(q + 1)| = 3 \); \( PSp(4,q) \), where \( q = 4, 5, 7, 8, 9, 17; PSp(6,2), PSU(4,3), PSU(3,9), 3D_4(2), G_2(9), O_8^+(2) \).

**Proof.** In the sequel we use the classification of finite simple groups and for the order components of non-abelian simple groups we refer to the tables in [9]. Let \( G \) be a finite simple group and \( \pi_1(G) \) be 2–regular.

Now in the following cases we consider the simple groups with non-connected prime graph.

**Case 1.** Let \( G \cong PSL(n,q) \), where \( q = p^a \). We note that the orders of maximal tori of \( PSL(n,q) \) are

\[
\frac{\prod_{i=1}^{k} (q^{r_i} - 1)}{(q - 1)(n, q - 1)} \quad (r_1, \ldots, r_k) \in Par(n).
\]

(1)

First let \( n = 2 \). We know that for \( q > 2 \), every component of the prime graph of \( G \) is complete, since

\[
\mu(PSL(2,q)) = \left\{ p, \frac{q - 1}{d}, \frac{q + 1}{d} \right\},
\]

where \( d = (q - 1, 2) \). Hence \( \pi_1(PSL(2,q)) \) is 2–regular if and only if \( |\pi_1(G)| = 3 \). If \( q = 2^a \), then \( \pi_1(G) = \{ 2 \} \) and hence \( \pi_1(G) \) is not 2–regular. If \( 4 \mid (q - 1) \), then \( m_1 = q - 1 \) and hence \( \pi_1(G) \) is 2–regular if and only if \( |\pi(q - 1)| = 3 \). If \( 4 \mid (q + 1) \), then similarly we get the result.

For \( n = 3 \) we have the following results ([17]). If \( d = (q - 1, 3) \) and \( q \) is odd, then

\[
\mu(PSL(3,q)) = \begin{cases} 
\{ q - 1, p(q - 1)/3, (q^2 - 1)/3, (q^2 + q + 1)/3 \} & \text{if } d = 3 \\
\{ p(q - 1), q^2 - 1, q^2 + q + 1 \} & \text{if } d = 1
\end{cases}
\]

(2)
and if \( q = 2^a \), then

\[
\mu(PSL(3, q)) = \begin{cases} 
\{4, q-1, 2(q-1)/3, (q^2 - 1)/3, (q^2 + q + 1)/3\} & \text{if } d = 3 \\
\{4, 2(q-1), q^2 - 1, q^2 + q + 1\} & \text{if } d = 1 \\
\end{cases} 
\]

(3)

By using (1) it follows that there exists a maximal torus \( T \) of order \((q^2 - 1)/(3, q - 1)\). Every maximal torus is abelian and hence if \( \pi_1(G) \) is 2–regular, then \(|\pi((q^2 - 1)/(3, q - 1))| \leq 2\), since \( p \in \pi_1(G) \) and \((p, q^2 - 1) = 1\). Therefore \( q = 2, 3, 4, 5, 7, 8, 9, 16, 17, 25, 31, 49 \) or \( q = p \) where \( p - 1 = 3 \cdot 2^\beta \), \( p + 1 = 2t^\gamma \) and \( t \) is an odd prime number and \( \beta \geq 2 \), by Lemma 2.10. In each case by easy calculation we can compute \( \mu(G) \) and it follows that \( \pi_1(PSL(3, q)) \) is not 2–regular. For example let \( q = p \) where \( p - 1 = 3 \cdot 2^\beta \), \( p + 1 = 2t^\gamma \) and \( t \) is an odd prime number and \( \beta \geq 2 \). Then \( d = 3 \) and so

\[
\mu(G) = \{3 \cdot 2^\beta, p \cdot 2^\beta, 2^{\beta+1}t^\gamma, 3 \cdot 2^{2\beta} + 2^{\beta+1} + 2^\beta + 1\}.
\]

Therefore \( 2 \sim 3 \), \( 2 \sim p \) and \( 2 \sim t \), which is a contradiction.

Again by using (1), it follows that if \( n \geq 4 \), then \( q^2 - 1 \) divides the order of a maximal torus \( T \) of \( G \). Hence if \( \pi_1(G) \) is 2–regular, then \( q = 2, 3, 4, 5, 7, 8, 9, 17 \), by Lemma 2.10.

We recall that \( \Gamma(PSL(4, q)) \) is non-connected if and only if \((q - 1) \mid 4\). Therefore if \( q = 4, 7, 8, 9 \) or 17, then \( \pi_1(G) \) is not 2–regular, by Lemma 3.3. If \( q = 2 \) or 3, then by using [3] it follows that \( \pi_1(G) \) is not 2–regular. So let \( G = PSL(4, 5) \). By Lemma 3.4, it follows that \( 2 \sim 5 \). Also \( q^2 - 1 \) and \((q^4 - 1)/(q - 1)(4, q - 1)\) divide the orders of some maximal tori of \( G \). Therefore \( 2 \sim 3 \) and \( 3 \sim 13 \). But by using Lemma 2.6, it follows that \( 13 \sim 5 \), which is a contradiction.

If \( n \geq 5 \), then as we mentioned above \( q = 2, 3, 4, 5, 7, 8, 9, 17 \). If \( q = 2 \), then \((q^3 - 1)(q^2 - 1) = 21\) and \( q^4 - 1 = 15 \) divide the orders of some maximal tori of \( G \). So \( 3 \sim 5 \) and \( 3 \sim 7 \). Also \((q^3 - 1)(q - 1)^{n-2} = 3\) is the order of a maximal tori of \( G \). Hence \( 3 \sim 2 \), by Lemma 3.4. Therefore \( \pi_1(G) \) is not 2–regular. In other cases by using the orders of maximal tori of \( G \) and Lemma 3.4, we conclude that \( \pi_1(G) \) is not 2–regular. For example if \( q = 3 \), then \( q^3 - 1 = 26 \) and \((q^4 - 1)/2 = 40\) divide the orders of some maximal tori of \( G \). Hence \( 2 \sim 13 \) and 2 \sim 5. Also 2 \sim 3, by Lemma 3.4, which is a contradiction. If \( q = 4 \), then \( 3 \sim 2, 3 \sim 5 \) and \( 3 \sim 7 \), which is a contradiction. For convenience we omit the details of other cases.

**Case 2.** Let \( G \cong PSU(n, q) \), where \( q = p^a \). The orders of maximal tori of \( PSU(n, q) \) are

\[
\prod_{i=1}^{k}(q^{r_i} - 1) \prod_{j=1}^{m}(q^{s_i} + 1) / (q + 1)(n, q + 1),
\]

(4)
\((r_1, \ldots, r_k; s_1, \ldots, s_m) \in \text{Par}(n), \ r_i \text{ even}, \ s_j \text{ odd.}\)

If \(n = 2\), then \(PSU(2, q) = PSL(2, q)\), which is discussed in Case 1.

If \(n = 3\), then for \(d = (q + 1, 3)\), we have the following result ([17]). If \(q\) is odd, then

\[
\mu(PSU(3, q)) = \begin{cases} 
q + 1, p(q + 1)/3, (q^2 - 1)/3, (q^2 - q + 1)/3 & \text{if } d = 3 \\
(p(q + 1), q^2 - 1, q^2 - q + 1) & \text{if } d = 1
\end{cases}
\]

and if \(q = 2^\alpha\), then

\[
\mu(PSU(3, q)) = \begin{cases} 
4q + 1, 2(q + 1)/3, (q^2 - 1)/3, (q^2 - q + 1)/3 & \text{if } d = 3 \\
4, 2(q + 1), q^2 - 1, q^2 - q + 1 & \text{if } d = 1
\end{cases}
\]

We can see that \((q^2 - 1)/(3, q + 1)\) divides the order of a maximal torus \(T\), and since \(p \in \pi_1(G)\), it follows that if \(\pi_1(G)\) is 2-regular, then \(\pi_1((q^2 - 1)/(3, q + 1))| \leq 2\). By using Lemma 2.10 we have \(q = 3, 9\) or \(q = 2^\alpha\), where \(\alpha\) is a prime number, or \(q = p\), where \(p + 1 = 3 \cdot 2^\beta\). Now we consider these cases separately.

If \(q = 2^\alpha\), where \(\alpha = 2\), then \(d = 1\) and hence \(\mu(PSU(3, 4)) = \{4, 10, 15, 13\}\), which implies that \(2 \sim 5\) and \(3 \sim 5\), but \(2 \sim 3\), which is a contradiction.

If \(q = 2^\alpha\), where \(\alpha\) is an odd prime, then \(3 \mid (q + 1)\). So \(d = 3\) and hence

\[
\mu(PSU(3, q)) = \{4, q + 1, 2(q + 1)/3, (q^2 - 1)/3, (q^2 - q + 1)/3\}.
\]

If \(r, s > 3\) are two distinct prime divisors of \(q + 1\), then we get a contradiction, since \(2 \sim r \sim s \sim 2\) and \(2 \sim 3\). So \(|\pi(q + 1)| \leq 2\) and \(3 \in \pi(q + 1)\).

Now we consider two cases: If \(q + 1 = 3^m\), then \(2^\alpha + 1 = 3^m\) and \(\alpha \geq 1\). This equation has only one solution \((q, m) = (8, 2)\), by Lemma 2.7. Then \(\mu(PSU(3, 8)) = \{4, 9, 6, 15, 57\}\), and so \(3 \sim 2, 3 \sim 5\) and \(3 \sim 19\), which is a contradiction. If \(q + 1 = 3^\beta t^\gamma\), where \(t > 3\) is a prime number, then consider a prime number \(s\) where \(s \mid (q - 1)\). Since \(q - 1\) is odd, it follows that \((q - 1, q + 1) = 1\) and hence \(s \not\in \{2, 3, t\}\). Therefore \(t \sim 2, t \sim 3\) and \(t \sim s\), which is a contradiction.

If \(q = 3\), then \(\mu(PSU(3, 3)) = \{12, 8, 7\}\) and hence \(|\pi_1(G)| = 2\), which is a contradiction.

If \(q = 9\), then \(\mu(PSU(3, 9)) = \{30, 80, 73\}\). Therefore \(\pi_1(PSU(3, 9))\) is 2-regular.

If \(q = p\) and \(p + 1 = 3 \cdot 2^\beta\), then \(\mu(PSU(3, p)) = \{3 \cdot 2^\beta, p \cdot 2^\beta, 2^\beta(p - 1), (p^2 - p + 1)/3\}\). Let \(t\) be a prime divisor of \(p - 1\). If \(t\) is odd, then \(2 \sim t, 2 \sim 3\) and \(2 \sim p\), which is a contradiction. If \(p - 1 = 2^m\), then \(\pi_1(G) = \{2, 3, p\}, 2 \sim 3\) and \(2 \sim p\) but \(3 \sim p\), which is a contradiction.

If \(n = 4\), then the prime graph of \(G\) is non-connected if and only if \((q + 1) \mid 4\), i.e. \(q = 3\). If \(q = 3\), then \(\pi_1(PSU(4, 3)) = \{2, 3, 5\}\) and \(\Gamma(G)\) is complete. Hence \(\pi_1(PSU(4, 3))\) is 2-regular.
If \( n \geq 5 \), then by (4), it follows that \( q^2 - 1 \) divides the order of a maximal torus \( T \) of \( G \). Also \( p \in \pi_1(G) \), which implies that \( |\pi(q^2 - 1)| \leq 2 \). Hence \( q = 2, 3, 4, 5, 7, 8, 9, 17 \). We get a contradiction, using Lemma 3.4 and the orders of maximal tori, as in case (1).

**Case 3.** If \( G \cong E_7(2) \) or \( E_7(3) \), then \( |\pi_1(G)| > 5 \), which is a contradiction. Also \( G \not\cong F_4(2) \), since \( \pi_1(G) = \{2, 3, 5\} \) and \( \Gamma(G) \) is not complete by Lemma 2.4. We know that \( \pi_1(Sz(q)) = \{2\} \) and so \( G \not\cong Sz(q) \).

**Case 4.** Let \( G \cong G_2(q) \), where \( q = p^3 > 2 \).

By using [10], it follows that \( q^2 - 1 \) is the order of a maximal torus of \( G \). Also \( p \in \pi_1(G) \), which implies that \( |\pi(q^2 - 1)| \leq 2 \). Therefore \( q = 3, 4, 5, 7, 8, 9, 17 \).

If \( q = 3 \) or \( q = 9 \), then \( \pi_1(G) \) is complete by Lemma 2.4. Therefore \( \pi_1(G) \) is 2–regular if and only if \( |\pi_1(G)| = 3 \). But if \( q = 3 \), then \( |\pi_1(G)| = 2 \). So \( \pi_1(G_2(9)) \) is 2–regular. If \( q = 4 \), then \( q^2 - 1 = 15, q^2 + q + 1 = 21 \) and \( (q - 1)^2 \) are the orders of some maximal tori of \( G \). Therefore \( 3 \sim 2, 3 \sim 5 \) and \( 3 \sim 7 \), which is a contradiction.

It is proved that if \( q \equiv 1 \pmod{3} \), then \( q_3 \in \pi_1(G) \) and \( p \sim q_3 \). Also if \( q \equiv -1 \pmod{3} \), then \( q_6 \in \pi_1(G) \) and \( p \sim q_6 \) (see [18]). Now by using these facts we have:

If \( q = 5 \), then \( q^2 - 1 = 24 \) and \( q^2 - q + 1 = 21 \) are the orders of some maximal tori of \( G \). Therefore \( 3 \sim 2 \) and \( 3 \sim 7 \). Also by using Lemma 3.4, it follows that \( 5 \sim 2 \). But \( 7 = q_6 \sim 5 \), which is a contradiction. Similarly if \( q = 7 \), then \( q^2 - 1 = 48 \) and \( q^2 + q + 1 = 57 \) and so \( 3 \sim 2, 3 \sim 19 \) and \( 2 \sim 7 \). But \( 19 = q_3 \sim 7 \), which is a contradiction. If \( q = 8 \), then \( q^2 - 1 = 63, q^2 - q + 1 = 57 \) and \( (q + 1)^2 = 9^2 \). Therefore \( 3 \sim 7, 3 \sim 19 \) and \( 3 \sim 2 \), which is a contradiction. If \( q = 17 \), then \( q^2 - q + 1 = 273 \) and so \( 3 \sim 7 \sim 13 \sim 3 \), which is a contradiction.

**Case 5.** Let \( G \cong E_8(q) \), where \( q = p^a \).

If \( q \neq 2 \), then by using the order components of \( E_8(q) \) we know that \( q(q^4 - 1)(q^5 - 1)(q^6 - 1)(q^7 - 1)(q^9 - 1) \) divides \( m_1 \). Hence by using Zsigmondy’s Theorem (Lemma 2.9) it follows that

\[
\{p, q_4, q_5, q_6, q_7, q_9\} \subseteq \pi_1(G),
\]

and so \( |\pi_1(G)| > 5 \), which is a contradiction. Similarly \( |\pi_1(E_8(2))| > 5 \), which is a contradiction.

If \( G \cong E_6(q) \) or \( 2E_6(q) \), then similarly we get a contradiction.

**Case 6.** If \( G \cong F_4(q), 3D_4(q), 2G_2(q) \), where \( q = 3^{2m+1} \) and \( m > 0 \), or \( 2F_4(q) \) where \( q = 2^{2m+1} \) and \( m > 0 \), then we get a contradiction, similarly.

Since the proof is similar for these groups, we give the details of the proof of \( 3D_4(q) \). So let \( G \cong 3D_4(q) \). We know that \( q^2 - 1 \) divides the order of a maximal torus of \( G \). Hence \( q = 2, 3, 4, 5, 7, 8, 9, 17 \). If \( q = 2 \), then \( |\pi_1(G)| = 3 \) and the graph of \( G \) is complete, by Lemma 2.4. So \( \pi_1(G) \) is 2–regular. If \( q = 3 \), then \( q^2 - 1 = 26 \) and \( q^3 + 1 = 28 \) divide the order of some maximal tori of \( G \) (see
and so 2 ∼ 13, 2 ∼ 7 and 2 ∼ 3, by Lemma 3.4. Therefore π₁(G) is not 2-regular. If q ≥ 4, then (q^3 − 1)(q + 1) divides the order of a maximal torus of G and |π((q^3 − 1)(q + 1))| ≥ 3. Since |π₁(G)| ̸= 3, we get a contradiction.

**Case 7.** If G ∼= Bₙ(q), Cₙ(q), Dₙ(q) or 2Dₙ(q), then we get a contradiction, similarly to the above cases. For convenience we give the details of the proof for Cₙ(q). So let G ∼= Cₙ(q) = PSp(2n, q), where q = pᵃ.

First let n = 2. We know that PSp(4, 2) ∼= S₆, which is not a simple group, also we have

$$\mu(G) = \begin{cases} 
\{(q^2 + 1)/2, (q^2 - 1)/2, p(q + 1), p(q - 1)\} & \text{if } p \neq 2, 3 \\
\{(q^2 + 1)/(2, q - 1), (q^2 - 1)/(2, q - 1), p(q + 1), p(q - 1), p^2\} & \text{if } p = 2, 3 
\end{cases}$$

(7)

By using [17], it follows that (q^2 − 1)/(2, q − 1) divides the order of a maximal torus of G. Also by Lemma 2.4, we know that Γ(G) is complete. Hence |π((q^2 − 1)/(2, q − 1))| = 2 and since π((q^2 − 1)/(2, q − 1)) = π(q^2 − 1), it follows that |π(q^2 − 1)| = 2, and so q = 4, 5, 7, 8, 9, 17.

If n ≥ 3, then similar to the above cases it follows that q = 2, 3, 4, 5, 7, 8, 9, 17. If n = 3 and q = 2, then Γ(G) is complete and |π₁(G)| = 3. Therefore π₁(G) is 2-regular. If q = 2 and n ≥ 4, then q^4 − 1 = 15 and (q^3 − 1)(q + 1) = 21 divide the order of some maximal torus of G and so 3 ∼ 5 and 3 ∼ 7. Also (q − 1)^n−2(q + 1)^2 = 9 is the order of a maximal torus of PSp(2n, 2). Hence by using Lemma 3.4, it follows that 3 ∼ 2, which is a contradiction.

If q = 3, then q^2 + 1 = 10 and (q^3 + 1)/(2, q − 1) = 14 divide the order of some maximal tori of G. So 2 ∼ 7, 2 ∼ 5 and 2 ∼ 3, which is a contradiction.

If q = 4, then (q + 1)^n = 5^n is the order of a maximal torus of G, and so 5 ∼ 2. Also q^2 − 1 = 15 and (q^2 + 1)/(2, q + 1) = 65 divide the orders of some maximal tori of G, and so 5 ∼ 3 and 5 ∼ 13, which is a contradiction.

Similarly for other values of q, we get a contradiction by using Lemma 3.4 and the order of maximal tori of G.

Now the proof of the main theorem is complete.

\[\diamond\]

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