A characterization of closures and interiors

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Abstract

We prove that any closure operator and any interior operator on semilattices can be characterized using a single equation axiom.

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1 Introduction

Let $(L, \vee)$ be a join semilattice, and $C : L \rightarrow L$ a map. C is called a closure operator (cf. [2]) on L if it satisfies the following axioms for all $x, y \in L$: (1) $C(x) \geq x$, (2) $C(C(x)) = C(x)$, (3) $x \leq y \Rightarrow C(x) \leq C(y)$, and (4) $C(C(x) \vee C(y)) = C(x \vee y)$. Note that, on a topological space (τ, T) , a closure operator (that is, a set-valued map $K : T \rightarrow T$ with Kuratowski's three axioms (a) $x \subset K(x)$, (b) $K^2(x) = K(x)$, and (c) $K(x \cup y) = K(x) \cup K(y)$) is a special closure operator defined above. Actually, (c) implies (3) and (4). Furthermore, it is easily seen that the axiom (4) is superfluous (cf. [3]), and thus the axioms (1), (2), (3), and (4) are equivalent to (1), (2) and (3), that is, the definition for a closure operator of semilattice above is equivalent to the usual one. In the present note, we show that any closure operator and any interior operator on a semilattice can be characterized using a single equation axiom.

2 The characterization

Theorem 2.1 *Let $(L, \vee)$ be a join semilattice, and let $C : L \rightarrow L$ be a map. Then C is a closure operator on L if and only if for all $x, y \in L$,*

$$x \vee C(C(x) \vee C(y)) = C(x \vee y).$$

Corollary 2.2 *(Iseki [1]) $Y \cup K(y) \cup K^2(y) = K(x \cup y)$ is equivalent to the Kuratowski's three axioms (a), (b) and (c).*

Dually, we have

Corollary 2.3 *Let $(L, \wedge)$ be a meet semilattice, and let $I : L \rightarrow L$ be a map. Then I is an interior operator on L (i. e. $I(x) \leq x$, $I(I(x)) = I(x)$, $x \leq y \Rightarrow I(y) \leq I(x)$ for all $x, y \in L$) if and only if for all $x, y \in L$,*

$$x \wedge I(I(x) \wedge I(y)) = I(x \wedge y).$$

Remark 2.4 *By the above corollary it follows that Theorems 5 and 6 in [3] can be simplified.*

Corollary 2.5 (Iseki [1]) *Let (τ, T) be a topological space, and let $I : T \rightarrow T$ be a map. Then $I(x \cap y) = I^2(x) \cap I(y) \cap y$ is equivalent to the following three axioms (i) $I(x) \leq x$, (ii) $I(I(x)) = I(x)$, and (iii) $I(x \cap y) = I(x) \cap I(y)$.*

Proof of Theorem 2.1. “ $\Rightarrow$ ” : By (1), we have $x \vee (C(x) \vee C(y)) = (x \vee C(x)) \vee C(y) = C(x) \vee C(y)$, and hence $x \vee (y \vee (C(x) \vee C(y))) = C(x) \vee C(y)$, which implies that $C(x \vee y) \vee C(C(x) \vee C(y)) = C(C(x) \vee C(y))$. On the other hand, from $x \vee (x \vee y) = x \vee y$ we obtain that $C(x) \vee C(x \vee y) = C(x \vee y)$, so $C(x \vee y) \vee (C(x) \vee C(y)) = C(x \vee y)$, which implies that $C^2(x \vee y) \vee C(C(x) \vee C(y)) = C(x \vee y)$. Therefore, $x \vee C(C(x) \vee C(y)) = x \vee C(x \vee y) \vee C(C(x) \vee C(y)) = C(x \vee y) \vee C(C(x) \vee C(y)) = C(x \vee y)$.

“ $\Leftarrow$ ” : Let $x = y$. Then $x \vee C(C(x) \vee C(x)) = C(x \vee x) = C(x) = x \vee C^2(x)$. So we have $C(x) \vee C^2(x) = C^2(x)$, and $C(x) \vee C^2(x) = (x \vee C^2(x)) \vee C^2(x) = x \vee C^2(x) = C(x)$. Thus, $C(x) = C^2(x)$.

If $x \vee y = y$, then $x \vee C(C(x) \vee C(y)) = C(x \vee y) = C(y)$. By $x \vee C(x) = C(x)$ it follows that $C(x) \vee C(C(x) \vee C(y)) = C(x) \vee ((C(x) \vee C(y)) \vee C(C(x) \vee C(y))) = C(C(x) \vee C(y))$. Hence, $C(x) \vee C(y) = C(x) \vee (x \vee C(C(x) \vee C(y))) = x \vee (C(x) \vee C(C(x) \vee C(y))) = x \vee C(C(x) \vee C(y)) = C(y)$. $\square$

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