SOME NEW PARAMETRIC FAMILIES OF MULTIVARIATE COPULAS

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Abstract

Recently, Rodríguez-Lallena and Úbeda-Flores [8] have introduced a class of bivariate copulas which generalizes some known families such as the Farlie-Gumbel-Morgenstern distributions. In this paper we present multivariate generalizations of this class studying symmetry and dependence concepts, measuring the dependence among the components of each class, and providing several examples.

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1 Introduction

Rodríguez-Lallena and Úbeda-Flores [8] describe a (one-parametric) family of bivariate copulas of the form

$$C_{\lambda}(u, v) = uv + \lambda f(u)g(v), \quad (u, v) \in [0, 1]^2, \quad (1)$$

where f and g are two non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$, and the admissible range of the parameter λ is $[-1/\max(\alpha\gamma, \beta\delta), -1/\min(\alpha\delta, \beta\gamma)]$, where $\alpha = \inf\{f'(u) : u \in A\}$,

$\beta = \sup\{f'(u) : u \in A\}$, $\gamma = \inf\{g'(v) : v \in B\}$ and $\delta = \sup\{g'(v) : v \in B\}$, with $A = \{u \in [0, 1] : f'(u) \text{ exists}\}$ and $B = \{v \in [0, 1] : g'(v) \text{ exists}\}$. This class of copulas provides a method for constructing bivariate distributions with a variety of dependence structures and generalizes several known families such as the Farlie-Gumbel-Morgenstern (FGM) distributions and other (see, for instance, [1], [2], [3] and [8]). In this paper we provide procedures to construct parametric families of multivariate distributions which generalize (1). For statistical applications, we refer to [4] and [5].

Let $n \geq 2$ be a natural number. An n -copula is the restriction to $[0, 1]^n$ of a continuous multivariate distribution function whose univariate margins are uniform on $[0, 1]$. The importance of copulas is described in the following result: Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector with joint distribution function H and respective margins $F_1, \dots, F_n$. Then there exists an n -copula C (which is uniquely determined on $\text{Range}F_1 \times \dots \times \text{Range}F_n$) such that $H(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$ for all $\mathbf{x} = (x_1, \dots, x_n)$ in $[-\infty, \infty]^n$. If $F_1, \dots, F_n$ are continuous, then C is unique. In the sequel, we shall suppose that $\mathbf{X}$ is continuous. For a complete review, see [6].

If $\mathbf{u} = (u_1, \dots, u_n)$ is a point in $[0, 1]^n$, $\Pi^n(\mathbf{u}) = \prod_{i=1}^n u_i$ denotes the n -copula of independent continuous random variables. Let $\mathbf{X} > \mathbf{x}$ denote the component-wise inequality, and let $\mathbf{1}$ be an n -dimensional vector of 1's. If $\mathbf{U}$ is a vector of uniform $[0, 1]$ random variables with n -copula C , then $\hat{C}(\mathbf{u}) = P[\mathbf{1} - \mathbf{U} < \mathbf{u}]$ and $\bar{C}(\mathbf{u}) = P[\mathbf{U} > \mathbf{u}]$ are, respectively, the *survival copula* and the *survival function* associated with C . Given two n -copulas C_1 and C_2 , let $C_1 \leq C_2$ denote the inequality $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$ for all $\mathbf{u}$. C_1 is *more positive lower* (respectively, *upper*) *orthant dependent* (PLOD) (respectively, PUOD) than C_2 if $C_1 \geq C_2$ (respectively, $\bar{C}_1 \geq \bar{C}_2$). C_1 is *more positive orthant dependent* (POD) than C_2 if both $C_1 \geq C_2$ and $\bar{C}_1 \geq \bar{C}_2$ hold (the corresponding negative concepts are defined by reversing the sense of the inequalities). A parametric family of n -copulas $\{C_\theta\}$, $\theta \in \mathbb{R}$, is *positively ordered* when $\theta_1 \leq \theta_2$ implies that $C_{\theta_1} \leq C_{\theta_2}$. If C is an n -copula, C is *radially symmetric* if $C = \hat{C}$; and $C(\pi\mathbf{u})$ denotes the n -copula given by any permutation π of $\mathbf{u}$.

The population versions of two non-parametric measures of multivariate association between components of a continuous random vector $\mathbf{U}$ with associated n -copula C are two multivariate generalizations of Kendall's tau and Spearman's rho, which are given by

$$\tau_n(C) = \frac{1}{2^{n-1} - 1} \left(2^n \int_{[0,1]^n} C(\mathbf{u}) dC(\mathbf{u}) - 1 \right) \quad \text{and}$$

$$\rho_n(C) = \frac{n+1}{2^n - (n+1)} \left[2^{n-1} \left(\int_{[0,1]^n} C(\mathbf{u}) d\Pi^n(\mathbf{u}) + \int_{[0,1]^n} \Pi^n(\mathbf{u}) dC(\mathbf{u}) \right) - 1 \right],$$

respectively (see [7]).

2 New parametric families of n -copulas

2.1 A first family

A first one-parameter multivariate extension of the class of copulas given by (1) is

$$C_\theta(\mathbf{u}) = \prod_{i=1}^n u_i + \theta \prod_{i=1}^n f_i(u_i), \quad \mathbf{u} \in [0, 1]^n, \quad (2)$$

where $\theta \in \mathbb{R}$ and f_i , $1 \leq i \leq n$, are n non-zero absolutely continuous functions such that $f_i(0) = f_i(1) = 0$. Note that all the k -dimensional margins, $2 \leq k < n$, are Π^k . The density function of (2) is $c_\theta(\mathbf{u}) = 1 + \theta \prod_{i=1}^n f'_i(u_i)$, whose parameter θ has the admissible range $-1 / \sup_{\mathbf{u} \in D^+} \left(\prod_{i=1}^n f'_i(u_i) \right) \leq \theta \leq -1 / \inf_{\mathbf{u} \in D^-} \left(\prod_{i=1}^n f'_i(u_i) \right)$, where $D^- = \{\mathbf{u} \in [0, 1]^n : \prod_{i=1}^n f'_i(u_i) < 0\}$ and $D^+ = \{\mathbf{u} \in [0, 1]^n : \prod_{i=1}^n f'_i(u_i) > 0\}$.

The survival function and the survival n -copula associated with the n -copula C_θ are given by $\overline{C}_\theta(\mathbf{u}) = \prod_{i=1}^n (1 - u_i) + (-1)^n \theta \prod_{i=1}^n f_i(u_i)$ and $\hat{C}_\theta(\mathbf{u}) = \prod_{i=1}^n u_i + (-1)^n \theta \prod_{i=1}^n f_i(1 - u_i)$, respectively, for every $\mathbf{u}$ in $[0, 1]^n$.

The expressions $\tau_n(C_\theta)$ and $\rho_n(C_\theta)$ for a member of the family (2) are given by $\tau_n(C_\theta) = 2 I_1 / (2^{n-1} - 1)$ and $\rho_n(C_\theta) = (n+1) I_1 / [2^n - (n+1)]$, where $I_1 = (1 + (-1)^n) 2^{n-1} \theta \prod_{i=1}^n \int_0^1 f_i(t) dt$. Observe that for this class of n -copulas we have the following relationship:

$$\rho_n(C_\theta) = \frac{(n+1)(2^{n-1} - 1) \tau_n(C_\theta)}{2(2^n - n - 1)}. \quad (3)$$

Let $\{C_\theta\}$ be the corresponding family of n -copulas given by (2). Then, $\{C_\theta\}$ is positively ordered if and only if $\prod_{i=1}^n f_i(u_i) \geq 0$ for all $\mathbf{u}$ in $[0, 1]^n$.

Let $C_{\theta_1}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_1 \prod_{i=1}^n f_i(u_i)$ and $C_{\theta_2}(\mathbf{u}) = \prod_{i=1}^n u_i + \theta_2 \prod_{i=1}^n g_i(u_i)$ be two n -copulas. Then, C_{θ_1} is more PLOD (respectively, PUOD) than C_{θ_2} if and only if $\theta_1 \prod_{i=1}^n f_i(u_i) \geq \theta_2 \prod_{i=1}^n g_i(u_i)$ (respectively, $(-1)^n \theta_1 \prod_{i=1}^n f_i(1 - u_i) \geq (-1)^n \theta_2 \prod_{i=1}^n g_i(1 - u_i)$).

Let A and B be two non-empty disjoint subsets of $\{1, \dots, n\}$ such that $|A \cup B| = n$, and let $\mathbf{U}_A = (U_i : i \in A)$ and $\mathbf{U}_B = (U_j : j \in B)$ denote subsets of the coordinates of the random vector $\mathbf{U}$. $\mathbf{U}_B$ is *left tail decreasing* in $\mathbf{U}_A$ ($\text{LTD}(\mathbf{U}_B \mid \mathbf{U}_A)$) if $P(\mathbf{U}_B \leq \mathbf{u}_B \mid \mathbf{U}_A \leq \mathbf{u}_A)$ is non-increasing in each component of $\mathbf{u}_A$ for all $\mathbf{u}_B$; $\mathbf{U}_B$ is *right tail increasing* in $\mathbf{U}_A$ ($\text{RTI}(\mathbf{U}_B \mid \mathbf{U}_A)$) if $P(\mathbf{U}_B > \mathbf{u}_B \mid \mathbf{U}_A > \mathbf{u}_A)$ is nondecreasing in each component of $\mathbf{u}_A$ for all $\mathbf{u}_B$; and $\mathbf{U}_B$ is *stochastically increasing* in $\mathbf{U}_A$ ($\text{SI}(\mathbf{U}_B \mid \mathbf{U}_A)$) if $P(\mathbf{U}_B > \mathbf{u}_B \mid \mathbf{U}_A = \mathbf{u}_A)$ is nondecreasing in each component of $\mathbf{u}_A$ for all $\mathbf{u}_B$. For the sake of simplicity of the next result, we will suppose that the functions f_i and the parameter θ in (2) are non-negative.

Theorem 2.1 *Let $\mathbf{U}$ be a random vector whose associated n -copula C_θ is defined by (2) and such that the functions f_i , $i = 1, \dots, n$, and θ are non-negative. Let $\mathbf{U}_A$ and $\mathbf{U}_B$ be two subsets of $\mathbf{U}$ as in the preceding paragraph. Then:*

- (i) *$\text{LTD}(\mathbf{U}_B \mid \mathbf{U}_A)$ if and only if $f_i(u) \geq u f'_i(u)$ for all $u \in [0, 1]$ and for every $i \in A$.*
- (ii) *$\text{RTI}(\mathbf{U}_B \mid \mathbf{U}_A)$ if and only if $f_i(u) \geq (u - 1) f'_i(u)$ for all $u \in [0, 1]$ and for every $i \in A$.*
- (iii) *$\text{SI}(\mathbf{U}_B \mid \mathbf{U}_A)$ if and only if $(-1)^n f''_i(u) \prod_{h \in A - \{i\}} f'_h(u_h) \geq 0$ for all $i \in A$, and u, u_h in $[0, 1]$.*

Proof. Suppose the non-trivial case $0 < u_i \leq 1$, $i = 1, \dots, n$. Since all the k -dimensional margins of the C_θ are Π^k for $2 \leq k < n$, we have that $P(\mathbf{U}_B \leq \mathbf{u}_B \mid \mathbf{U}_A \leq \mathbf{u}_A) = \prod_{k \in B} u_k + \theta \prod_{k \in B} f_k(u_k) \prod_{j \in A} f_j(u_j)/u_j$, and $P(\mathbf{U}_B > \mathbf{u}_B \mid \mathbf{U}_A > \mathbf{u}_A) = \prod_{k \in B} (1 - u_k) + \theta \prod_{k \in B} f_k(u_k) \prod_{j \in A} f_j(u_j)/(1 - u_j)$. If $f_i(u) \geq u f'_i(u)$ for all $i \in A$ and $u \in (0, 1]$, we have $\frac{\partial}{\partial u_i} P(\mathbf{U}_B \leq \mathbf{u}_B \mid \mathbf{U}_A \leq \mathbf{u}_A) \leq 0$; and conversely. Thus part (i) holds. A similar argument yields part (ii). Now, since $P(\mathbf{U}_B > \mathbf{u}_B \mid \mathbf{U}_A = \mathbf{u}_A) = \prod_{k \in B} (1 - u_k) + (-1)^{|B|} \prod_{k \in B} f_k(u_k) \prod_{j \in A} f'_j(u_j)$, part (iii) follows, which completes the proof.

Let C_θ be an n -copula given by (2). If $f_1(u) = \dots = f_n(u)$ for all u in $[0, 1]$, then $C(\pi \mathbf{u}) = C(\mathbf{u})$ for any permutation π of $\mathbf{u}$; and if $f_i(u) = (-1)^n f_i(1 - u)$ for all $i = 1, \dots, n$ and $u \in [0, 1]$, then C_θ is radially symmetric.

We now provide a trivariate example of a 3-copula that belongs to (2).

Example 2.2 *Let $f_i(u) = u^b(1 - u)^a$, $1 \leq i \leq 3$, with $a, b \geq 1$. Then, for all (u_1, u_2, u_3) in $[0, 1]^3$, the function*

$$C_\theta(u_1, u_2, u_3) = u_1 u_2 u_3 + \theta u_1^b (1 - u_1)^a u_2^b (1 - u_2)^a u_3^b (1 - u_3)^a \quad (4)$$

is a 3-copula if and only if

$$-\min\left\{\frac{1}{(K_{ab}^+)^2 K_{ab}^-}, \frac{1}{(K_{ab}^-)^3}\right\} \leq \theta \leq \min\left\{\frac{1}{(K_{ab}^-)^2 K_{ab}^+}, \frac{1}{(K_{ab}^+)^3}\right\},$$

where $K_{ab}^+ = R^{b-1}(1-R)^{a-1}(R-b)$, $K_{ab}^- = S^{b-1}(1-S)^{a-1}(S-b)$, and

$$R = \frac{b(a+b-1) + \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)} \quad \text{and} \quad S = \frac{b(a+b-1) - \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}.$$

To check this, note that the density of (4) is given by $c_\theta(u_1, u_2, u_3) = 1 + \theta \prod_{i=1}^3 u_i^{b-1}(1-u_i)^{a-1}(b-(a+b)u_i)$, and the overall constraint for θ is

$$\theta \prod_{i=1}^3 u_i^{b-1}(1-u_i)^{a-1}(b-(a+b)u_i) \geq -1. \quad (5)$$

It is clear that if $u_1 = u_2 = u_3 = b/(a+b)$, we have that $c_\theta(u_1, u_2, u_3) = 1$. Let $g(u) = u^{b-1}(1-u)^{a-1}(b-(a+b)u)$. Then the solutions of $g'(u) = 0$ are $u_* = S < b/(a+b)$ and $u^* = R > b/(a+b)$. Furthermore, we have that $g''(u_*) < 0$ and $g''(u^*) > 0$. Thus, the function $-g(u)$ (respectively, $g(u)$) attains its maximum value at the point u^* (respectively, u_*). Now, consider the orthants $O_1 = \{(u_1, u_2, u_3) \in [0, 1]^3 : g(u_1) > 0, g(u_2) > 0, \text{ and } g(u_3) > 0\}$, $O_2 = \{(u_1, u_2, u_3) \in [0, 1]^3 : g(u_1) < 0, g(u_2) < 0, \text{ and } g(u_3) < 0\}$, $O_3 = \{(u_1, u_2, u_3) \in [0, 1]^3 : \text{one of } g(u_1), g(u_2) \text{ and } g(u_3) \text{ is positive}\}$ and $O_4 = \{(u_1, u_2, u_3) \in [0, 1]^3 : \text{one of } g(u_1), g(u_2) \text{ and } g(u_3) \text{ is negative}\}$. In the orthant O_1 , we have that $(u_1, u_2, u_3) \in [0, b/(a+b))^3$. Hence (5) requires to have $\theta \geq -1 / \prod_{i=1}^3 u_i^{b-1}(1-u_i)^{a-1}(b-(a+b)u_i)$. In this case, the maximum is attained at the point u_* . Thus $\theta \geq -1/(K_{ab}^-)^3$. In O_2 , we have that $b/(a+b) \leq u_1, u_2, u_3 < 1$. Hence (5) requires to have $\theta \leq 1 / \prod_{i=1}^3 u_i^{b-1}(1-u_i)^{a-1}[(a+b)u_i - b]$. In this case, the maximum is attained at the point u^* . Thus $\theta \leq 1/(K_{ab}^+)^3$. In O_3 , two of u_i 's are greater than $b/(a+b)$. For instance, if $(u_1, u_2) \in (b/(a+b), 1)^2$ and $0 \leq u_3 < b/(a+b)$, we are required to have $\theta \geq -1/u_3^{b-1}(1-u_3)^{a-1}[b - (a+b)u_3] \prod_{i=1}^2 u_i^{b-1}(1-u_i)^{a-1}[(a+b)u_i - b]$. Hence $\theta \geq -1/(K_{ab}^+)^2 K_{ab}^-$. A similar argument shows that $\theta \leq 1/(K_{ab}^-)^2 K_{ab}^+$ for the orthant O_4 .

Observe that if $a = b = 1$, we have a (one-parametric) trivariate extension of the FGM family with $\theta \in [-1, 1]$. Moreover, $\tau_3(C_\theta) = \rho_3(C_\theta) = 0$. Let C_{θ_1} and C_{θ_2} be two copulas of the type given by (4). Then C_{θ_1} is more PLOD than C_{θ_2} if $\theta_1 \geq \theta_2$. Finally, suppose $\theta > 0$; then, if $\mathbf{U}$ is a random vector with 3-copula C_θ , from Theorem 2.1 we have that C_θ is $LTD(\mathbf{U}_B | \mathbf{U}_A)$ if and only if $b = 1$, C_θ is $RTI(\mathbf{U}_B | \mathbf{U}_A)$ if and only if $a = 1$, and C_θ is $SI(\mathbf{U}_B | \mathbf{U}_A)$ if and only if $a = b = 1$.

2.2 A second family

Our next purpose is to define a new class of one-parametric n -copulas that generalizes (1) and such that the k -margins are in the class. Let f and g be two functions as in Section 1. Let α be a real number, and consider the function C_α^* given by

$$C_\alpha^*(\mathbf{u}) = \prod_{i=1}^n u_i + \alpha \sum_{1 \leq i < j \leq n} f(u_i)g(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n u_k, \quad \mathbf{u} \in [0, 1]^n. \quad (6)$$

Then each k -margin, $2 \leq k < n$, of (6) is again in this family. C_α^* is an absolutely continuous function with density $c_\alpha^*(\mathbf{u}) = 1 + \alpha \sum_{1 \leq i < j \leq n} f'(u_i)g'(u_j)$.

Thus, the function defined by (6) is an n -copula if and only if

$$\frac{-1}{\max\left\{\sum_{1 \leq i < j \leq n} f'(u_i)g'(u_j)\right\}} \leq \alpha \leq \frac{-1}{\min\left\{\sum_{1 \leq i < j \leq n} f'(u_i)g'(u_j)\right\}}. \quad (7)$$

The values of Kendall's tau and Spearman's rho associated with C_α^* are given by $\tau_n(C_\alpha^*) = 2 I_2 / (2^{n-1} - 1)$ and $\rho_n(C_\alpha^*) = (n+1) I_2 / [2^n - (n+1)]$, where $I_2 = 2\alpha n(n-1) \int_0^1 f(u)du \int_0^1 g(u)du$. Observe that for this class of n -copulas we have (again) the relationship in (3) with C_α^* .

Let $\{C_\alpha^*\}$ be the corresponding family of n -copulas given by (6). Then, $\{C_\alpha^*\}$ is positively ordered if and only if $\sum_{1 \leq i < j \leq n} f(u_i)g(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n u_k \geq 0$.

Let $C_{\alpha_l}^*(\mathbf{u}) = \prod_{i=1}^n u_i + \alpha_l \sum_{1 \leq i < j \leq n} f_l(u_i)g_l(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n u_k$ ($l = 1, 2$) be two n -copulas. $C_{\alpha_1}^*$ is more PLOD than $C_{\alpha_2}^*$ when $\alpha_1 \sum_{1 \leq i < j \leq n} f_1(u_i)g_1(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n u_k \geq \alpha_2 \sum_{1 \leq i < j \leq n} f_2(u_i)g_2(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n u_k$. On the other hand, since $\overline{C}_\alpha^*(\mathbf{u}) = \prod_{i=1}^n (1 - u_i) + \alpha_1 \sum_{1 \leq i < j \leq n} f_1(u_i)g_1(u_j) \prod_{\substack{k=1 \\ k \neq i,j}}^n (1 - u_k)$, a similar result holds for PUOD.

Let C_α^* be an n -copula given by (6). If $f(u) = g(u)$ for all u in $[0, 1]$ then $C(\pi\mathbf{u}) = C(\mathbf{u})$ for any permutation π of $\mathbf{u}$; and if $f(u) = f(1 - u)$ and $g(u) = g(1 - u)$ (or $f(u) = -f(1 - u)$ and $g(u) = -g(1 - u)$), $u \in [0, 1]$, then C is radially symmetric.

Example 2.3 Let f and g be two functions such that $f(u) = g(u) = u(1 - u)$ for all $u \in [0, 1]$, and let $\mathbf{a} = (a_1, \dots, a_n)$ be a point in $[0, 1]^n$ such that $a_i = \pm 1$

for all $i = 1, \dots, n$. It is easy to check that $\max_{u \in [0,1]} f'(u) = \max_{u \in [0,1]} g'(u) = 1$ and $\min_{u \in [0,1]} f'(u) = \min_{u \in [0,1]} g'(u) = -1$. Thus, we must obtain the maximum and the minimum value of $\sum_{1 \leq i < j \leq n} a_i a_j$ for which the inequalities in (7) are satisfied.

It is clear that $\max(\sum_{1 \leq i < j \leq n} a_i a_j)$ in $Q^+ = \{\mathbf{a} \in [0, 1]^n : \sum_{1 \leq i < j \leq n} a_i a_j \geq 0\}$ is obtained when $a_i = 1$ for all $i = 1, \dots, n$ or $a_i = -1$ for all $i = 1, \dots, n$, and equal to $n(n-1)/2$. Therefore $\theta \geq -2/[n(n-1)]$. For the upper bound of θ , we must find the minimum value of $\sum_{1 \leq i < j \leq n} a_i a_j$ in $Q^- = \{\mathbf{a} \in [0, 1]^n :$

$\sum_{1 \leq i < j \leq n} a_i a_j \leq 0\}$. But $\sum_{1 \leq i < j \leq n} a_i a_j = (1/2) \sum_{i=1}^n a_i \sum_{\substack{j=1 \\ j \neq i}}^n a_j$. If k (respectively,

$n-k$) is the number of a_i 's such that are equal to 1 (respectively, -1), then $\sum_{i=1}^n a_i = k - (n-k) = 2k - n$. So $\sum_{1 \leq i < j \leq n} a_i a_j = (1/2) \sum_{i=1}^n a_i (2k - n - a_i) =$

$(1/2)[(2k - n) \sum_{i=1}^n a_i - \sum_{i=1}^n a_i^2] = ((2k - n)^2 - n)/2$, and $\min_{\mathbf{a} \in Q^-} \sum_{1 \leq i < j \leq n} a_i a_j = -\lfloor n/2 \rfloor$, where $\lfloor t \rfloor$ denotes the integer part of the real number t . In short, the

function $C_\alpha^*(\mathbf{u}) = \prod_{i=1}^n u_i + \alpha \sum_{1 \leq i < j \leq n} u_i u_j (1 - u_i)(1 - u_j) \prod_{\substack{k=1 \\ k \neq i, j}}^n u_k$ is an n -copula if

and only if $-2/[n(n-1)] \leq \alpha \leq 1/\lfloor n/2 \rfloor$. The expressions for τ_n and ρ_n are given by $\tau_n(C_\alpha^*) = n(n-1)\alpha/[9(2^{n-1} - 1)]$ and $\rho_n(C_\alpha^*) = n(n^2 - 1)\alpha/[18(2^n - n - 1)]$. Furthermore, C_α^* is radially symmetric; C_α^* is more POD than Π^n if and only if $\alpha \geq 0$; and $C_\alpha^*(\pi \mathbf{u}) = C_\alpha^*(\mathbf{u})$ for any permutation π of $\mathbf{u}$.

2.3 A third family

Let $\Theta = \{\theta_{i_1 \dots i_k} : 1 \leq i_1 < \dots < i_k \leq n, 2 \leq k \leq n\}$ be a set of $2^n - n - 1$ parameters. A third class of distributions which generalizes (1) can be the following $(2^n - n - 1)$ -parameter family:

$$C_\Theta^{**}(\mathbf{u}) = \prod_{i=1}^n u_i + \sum_{k=2}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} \theta_{i_1 \dots i_k} \prod_{j=i_1}^{i_k} f_j(u_j) \prod_{\substack{l=1 \\ l \neq i_1, \dots, i_k}}^n u_l, \quad \mathbf{u} \in [0, 1]^n, \quad (8)$$

where f_i , $1 \leq i \leq n$, are n non-zero absolutely continuous functions such that $f_i(0) = f_i(1) = 0$. Using the density function, we have that C_Θ^{**} is an n -copula if and only if $1 + \sum_{k=2}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} \theta_{i_1 \dots i_k} \prod_{j=i_1}^{i_k} f_j'(u_j) \geq 0$ for all $u_i \in [0, 1]$, $1 \leq i \leq n$. Each k -margin, $2 \leq k < n$, of a member given

by (8) is again an k -copula of the same type. We obtain that $\tau_n(C_{\Theta}^{**}) = 2I_3/(2^{n-1} - 1)$ and $\rho_n(C_{\Theta}^{**}) = (n+1)I_3/(2^n - n - 1)$, where $I_3 = \sum_{k=2}^n 2^{k-1}(1 + (-1)^k) \sum_{1 \leq i_1 < \dots < i_k \leq n} \theta_{i_1 \dots i_k} \prod_{j=i_1}^{i_k} \int_0^1 f_j(u_j) du_j$. Again, we have the relationship in (3) with C_{Θ}^{**} .

Finally, if $f_i(u) = -f_i(1 - u)$ for all $i = 1, \dots, n$ in (8), then we have that C_{Θ}^{**} is radially symmetric.

The n -copulas given by (8) are a “natural” extension of the (known) FGM family of n -copulas (see [5]) taking $f_j(u) = u(1 - u)$, $u \in [0, 1]$, for all $j = 1, \dots, n$ (for applications, see [4]). Denoting this family by E_{Θ}^{**} , we have that $\tau_n(E_{\Theta}^{**}) = I_4/(2^n - n - 1)$ and $\rho_n(E_{\Theta}^{**}) = (n+1)I_4/[2(2^n - n - 1)]$, where $I_4 = \sum_{k=2}^n [1 + (-1)^k]/3^k \sum_{1 \leq i_1 < \dots < i_k \leq n} \theta_{i_1 \dots i_k}$.

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