**Sectional curvature of QR -submanifolds
of $(p - 1)$ QR -dimension in a
quaternionic projective space $QP^{(n+p)/4}$**

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Abstract

In this paper we study n -dimensional compact QR -submanifolds of $(p - 1)$ QR -dimension immersed in a quaternionic projective space $QP^{(n+p)/4}$. Especially we provide a necessary condition in order for

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such a submanifold to be a geodesic hypersphere in $QP^{(n+p)/4}$ in terms with sectional curvature.

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1. Introduction

Let M be a connected real n -dimensional submanifold of real codimension p immersed in a real $(n+p)$ -dimensional quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists a subbundle ν of the normal bundle $TM^\perp$ such that

$$\begin{aligned} F\nu_x &\subset \nu_x, & G\nu_x &\subset \nu_x, & H\nu_x &\subset \nu_x, \\ F\nu_x^\perp &\subset T_x M, & G\nu_x^\perp &\subset T_x M, & H\nu_x^\perp &\subset T_x M \end{aligned} \quad (1.1)$$

for each x in M , where $\nu^\perp$ is the complementary orthogonal subbundle to ν in $TM^\perp$, then the submanifold is called a QR -submanifold of $\overline{M}$ and the dimension of ν QR -dimension of the QR -submanifold (cf. [1,6,9]). A real hypersurface is a QR -submanifold of zero QR -dimension. When the ambient manifold $\overline{M}$ is a quaternionic projective space, real hypersurfaces have been investigated by many authors (cf. [2,10,11]) under conditions concerning with shape operator.

In this paper we shall study n -dimensional QR -submanifolds M of $(p-1)$ QR -dimension immersed in $QP^{(n+p)/4}$ and, especially, provide necessary conditions in order that $\pi^{-1}(M)$ is locally isometric to a Riemannian product of $M_1 \times M_2$ in terms with sectional curvature, where M_1 and M_2 belong to some $(4n_1+3)$ - and $(4n_2+3)$ -dimensional spheres and π denotes the Hopf-fibration $S^{n+p+3} \rightarrow QP^{(n+p)/4}$. Moreover, those conditions give a quaternionic version of Kon's result [7, Theorem at p. 285].

2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$ -dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c) :

(a) In any coordinate neighborhood $\overline{U}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} F^2 &= -I, & G^2 &= -I, & H^2 &= -I, \\ FG &= -GF = H, & GH &= -HG = F, & HF &= -FH = G. \end{aligned} \quad (2.1)$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H .

(c) For the Riemannian connection $\bar{\nabla}$ with respect to g

$$\begin{pmatrix} \bar{\nabla}F \\ \bar{\nabla}G \\ \bar{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (2.2)$$

where p, q and r are local 1-forms defined in $\bar{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\bar{\mathcal{U}}$ (cf. [5]).

For canonical local bases $\{F_1, G_1, H_1\}$ and $\{F_2, G_2, H_2\}$ of V in coordinate neighborhoods $\bar{\mathcal{U}}_1$ and $\bar{\mathcal{U}}_2$, it follows that in $\bar{\mathcal{U}}_1 \cap \bar{\mathcal{U}}_2$

$$\begin{pmatrix} F_2 \\ G_2 \\ H_2 \end{pmatrix} = (s_{xy}) \begin{pmatrix} F_1 \\ G_1 \\ H_1 \end{pmatrix} \quad (x, y = 1, 2, 3) \quad (2.3)$$

with differentiable functions s_{xy} , where the matrix $S = (s_{xy})$ is contained in $SO(3)$ as a consequence of (2.1). As is well known (cf. [5]), every quaternionic Kähler manifold is orientable.

From now on we consider a real n -dimensional QR -submanifold M of $(p-1)$ QR -dimension immersed in $\bar{M}$. Then it is clear from (1.1) that there is a distinguished unit normal vector field ξ to M such that $F\xi, G\xi, H\xi \in TM$. We denote those unit vector fields tangent to M by

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi. \quad (2.4)$$

On the other hand, each tangent space $T_x M$ is decomposed as

$$T_x M = \mathcal{D}_x \oplus \mathcal{D}_x^\perp,$$

where $\mathcal{D}_x$ is the maximal quaternionic invariant subspace of $T_x M$ defined by

$$\mathcal{D}_x = T_x M \cap FT_x M \cap GT_x M \cap HT_x M$$

and $\mathcal{D}_x^\perp$ its orthogonal complement in $T_x M$. In this case it follows from (2.1) and (2.4) that $\mathcal{D}_x^\perp = \text{Span}\{U, V, W\}$, and so $\mathcal{D} : x \mapsto \mathcal{D}_x$ defines an $(n-3)$ -dimensional distribution on M (for details, see [1,6,9]) and $n = 4m + 3$ for some integer m . Furthermore we can see that

$$FT_x M, GT_x M, HT_x M \subset T_x M \oplus \text{Span}\{\xi\},$$

and consequently, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha ; \alpha = 1, \dots, p\}$ ($\xi_1 = \xi$) of normal vectors to M , we have the following decomposition in tangential and normal components:

$$FX = \phi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \quad (2.5)$$

$$\begin{aligned}
HX &= \theta X + w(X)\xi, \\
F\xi_\alpha &= -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \\
H\xi_\alpha &= -W_\alpha + P_3\xi_\alpha
\end{aligned} \tag{2.6}$$

($\alpha = 1, \dots, p$). Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on $T_x M$ and $T_x M^\perp$, respectively. Moreover, the hermitian property of $\{F, G, H\}$ implies

$$\begin{aligned}
g(X, \phi U_\alpha) &= -u(X)g(\xi_1, P_1\xi_\alpha), \\
g(X, \psi V_\alpha) &= -v(X)g(\xi_1, P_2\xi_\alpha), \quad \alpha = 1, \dots, p, \\
g(X, \theta W_\alpha) &= -w(X)g(\xi_1, P_3\xi_\alpha),
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
g(U_\alpha, U_\beta) &= \delta_{\alpha\beta} - g(P_1\xi_\alpha, P_1\xi_\beta), \\
g(V_\alpha, V_\beta) &= \delta_{\alpha\beta} - g(P_2\xi_\alpha, P_2\xi_\beta), \quad \alpha, \beta = 1, \dots, p, \\
g(W_\alpha, W_\beta) &= \delta_{\alpha\beta} - g(P_3\xi_\alpha, P_3\xi_\beta).
\end{aligned} \tag{2.8}$$

Also, from $g(FX, \xi_\alpha) = -g(X, F\xi_\alpha)$, $g(GX, \xi_\alpha) = -g(X, G\xi_\alpha)$ and $g(HX, \xi_\alpha) = -g(X, H\xi_\alpha)$, it follows that

$$\begin{aligned}
g(X, U_\alpha) &= u(X)\delta_{1\alpha}, \quad g(X, V_\alpha) = v(X)\delta_{1\alpha}, \\
g(X, W_\alpha) &= w(X)\delta_{1\alpha}
\end{aligned}$$

and hence

$$\begin{aligned}
g(U_1, X) &= u(X), \quad g(V_1, X) = v(X), \quad g(W_1, X) = w(X), \\
U_\alpha &= 0, \quad V_\alpha = 0, \quad W_\alpha = 0, \quad \alpha = 2, \dots, p.
\end{aligned} \tag{2.9}$$

On the other hand, comparing (2.4) and (2.6) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which and (2.9) imply

$$\begin{aligned}
g(U, X) &= u(X), \quad g(V, X) = v(X), \quad g(W, X) = w(X), \\
u(U) &= 1, \quad v(V) = 1, \quad w(W) = 1.
\end{aligned} \tag{2.10}$$

In the sequel we shall use the notations U, V, W instead of U_1, V_1, W_1 .

Next, applying F to the first equation of (2.5) and using (2.6), (2.9) and (2.10), we have

$$\phi^2 X = -X + u(X)U, \quad u(X)P_1\xi = -u(\phi X)\xi.$$

Similarly we have

$$\phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \quad (2.11)$$

$$\begin{aligned} \theta^2 X &= -X + w(X)W, \\ u(X)P_1\xi &= -u(\phi X)\xi, \quad v(X)P_2\xi = -v(\psi X)\xi, \\ w(X)P_3\xi &= -w(\theta X)\xi, \end{aligned} \quad (2.12)$$

from which, taking account of the skew-symmetry of P_1 , P_2 and P_3 and using (2.7), we also have

$$\begin{aligned} u(\phi X) &= 0, \quad v(\psi X) = 0, \quad w(\theta X) = 0, \\ \phi U &= 0, \quad \psi V = 0, \quad \theta W = 0, \\ P_1\xi &= 0, \quad P_2\xi = 0, \quad P_3\xi = 0. \end{aligned} \quad (2.13)$$

So (2.6) can be rewritten in the form

$$\begin{aligned} F\xi &= -U, \quad G\xi = -V, \quad H\xi = -W, \\ F\xi_\alpha &= P_1\xi_\alpha, \quad G\xi_\alpha = P_2\xi_\alpha, \quad H\xi_\alpha = P_3\xi_\alpha, \end{aligned} \quad (2.14)$$

where $\alpha = 2, \dots, p$.

Applying G and H to the first equation of (2.5) and using (2.1), (2.5) and (2.14), we have

$$\begin{aligned} \theta X + w(X)\xi &= -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ \psi X + v(X)\xi &= \theta(\phi X) + w(\phi X)\xi - u(X)W, \end{aligned}$$

and consequently

$$\begin{aligned} \psi(\phi X) &= -\theta X + u(X)V, \quad v(\phi X) = -w(X), \\ \theta(\phi X) &= \psi X + u(X)W, \quad w(\phi X) = v(X). \end{aligned} \quad (2.15)$$

From the other equations of (2.5) we have by quite similar method

$$\begin{aligned} \phi(\psi X) &= \theta X + v(X)U, \quad u(\psi X) = w(X), \\ \theta(\psi X) &= -\phi X + v(X)W, \quad w(\psi X) = -u(X), \end{aligned} \quad (2.16)$$

$$\begin{aligned} \phi(\theta X) &= -\psi X + w(X)U, \quad u(\theta X) = -v(X), \\ \psi(\theta X) &= \phi X + w(X)V, \quad v(\theta X) = u(X). \end{aligned} \quad (2.17)$$

From the first three equations of (2.14), we also have

$$\begin{aligned}\psi U &= -W, & v(U) &= 0, & \theta U &= V, & w(U) &= 0, \\ \phi V &= W, & u(V) &= 0, & \theta V &= -U, & w(V) &= 0, \\ \phi W &= -V, & u(W) &= 0, & \psi W &= U, & v(W) &= 0.\end{aligned}\tag{2.18}$$

Now let ∇ be the Levi-Civita connection on M and let $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M . Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.19}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla_X^\perp \xi_\alpha, \quad \alpha = 1, \dots, p \tag{2.20}$$

for X, Y tangent to M . Here h denotes the second fundamental form and A_α the shape operator corresponding to ξ_α . They are related by $h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y) \xi_\alpha$. Furthermore, put

$$\nabla_X^\perp \xi_\alpha = \sum_{\beta=1}^p s_{\alpha\beta}(X) \xi_\beta, \tag{2.21}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

When the ambient manifold $\overline{M}$ is of constant Q -sectional curvature c , using (2.5), (2.6), (2.9) and (2.10), we can obtain from the equations of Gauss, Codazzi and Ricci that

$$R(X, Y)Z = \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y \tag{2.22}$$

$$\begin{aligned}&+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ g(\psi Y, Z)\psi X - g(\psi X, Z)\psi Y - 2g(\psi X, Y)\psi Z \\ &+ g(\theta Y, Z)\theta X - g(\theta X, Z)\theta Y - 2g(\theta X, Y)\theta Z\} \\ &+ \sum_{\alpha} g(A_\alpha Y, Z)A_\alpha X - \sum_{\alpha} g(A_\alpha X, Z)A_\alpha Y,\end{aligned}$$

$$g((\nabla_X A_1)Y - (\nabla_Y A_1)X, Z) \tag{2.23}_{(a)}$$

$$\begin{aligned}&= \frac{c}{4} \{g(\phi Y, Z)u(X) - g(\phi X, Z)u(Y) - 2g(\phi X, Y)u(Z) \\ &+ g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\ &+ g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\} \\ &+ \sum_{\beta} \{g(A_\beta X, Z)s_{\beta 1}(Y) - g(A_\beta Y, Z)s_{\beta 1}(X)\},\end{aligned}$$

$$g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) \tag{2.23}_{(b)}$$

$$= \sum_{\beta=1}^p \{g(A_\beta X, Z)s_{\beta\alpha}(Y) - g(A_\beta Y, Z)s_{\beta\alpha}(X)\}, \quad \alpha = 2, \dots, p,$$

$$g(R^\perp(X, Y)\xi_1, \xi_\beta) = g([A_1, A_\beta]X, Y), \quad \beta = 2, \dots, p \quad (2.24)$$

for any vector fields X, Y tangent to M , where $\overline{R}$ and R denote the Riemannian curvature tensor of $\overline{M}$ and M , respectively and $R^\perp$ is the curvature tensor of the connection $\nabla^\perp$ (cf. [3,12]).

3. Some properties of the shape operator A_1

In this section, we assume that the ambient manifold $\overline{M}$ is a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ of constant Q -sectional curvature c .

Differentiating the first equation of (2.5) covariantly and using (2.2), (2.5), (2.6), (2.9), (2.22) and (2.23), we have

$$(\nabla_Y \phi)X = r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y - g(A_1 Y, X)U,$$

$$(\nabla_Y u)X = r(Y)v(X) - q(Y)w(X) + g(\phi A_1 Y, X). \quad (3.1)$$

From the other equations of (2.5) we also have

$$(\nabla_Y \psi)X = -r(Y)\phi X + p(Y)\theta X + v(X)A_1 Y - g(A_1 Y, X)V,$$

$$(\nabla_Y v)X = -r(Y)u(X) + p(Y)w(X) + g(\psi A_1 Y, X), \quad (3.2)$$

$$(\nabla_Y \theta)X = q(Y)\phi X - p(Y)\psi X + w(X)A_1 Y - g(A_1 Y, X)W,$$

$$(\nabla_Y w)X = q(Y)u(X) - p(Y)v(X) + g(\theta A_1 Y, X). \quad (3.3)$$

Next, differentiating the first equation of (2.14) covariantly and comparing the tangential and normal parts, we have

$$\nabla_Y U = r(Y)V - q(Y)W + \phi A_1 Y,$$

$$g(A_\alpha U, Y) = - \sum_{\beta=2}^p s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2, \dots, p, \quad (3.4)$$

where we have put $P_1 N_\alpha = \sum_{\beta=2}^p P_{1\alpha\beta} N_\beta$ for $2 \leq \alpha \leq p$. From the other equations of (2.14), we have similarly

$$\nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y,$$

$$g(A_\alpha V, Y) = - \sum_{\beta=2}^p s_{1\beta}(Y)P_{2\beta\alpha}, \quad \alpha = 2, \dots, p, \quad (3.5)$$

$$\nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y,$$

$$g(A_\alpha W, Y) = - \sum_{\beta=2}^p s_{1\beta}(Y) P_{3\beta\alpha}, \quad \alpha = 2, \dots, p, \quad (3.6)$$

where we have put $P_2 N_\alpha = \sum_{\beta=2}^p P_{2\alpha\beta} N_\beta$ and $P_3 N_\alpha = \sum_{\beta=2}^p P_{3\alpha\beta} N_\beta$ for $2 \leq \alpha \leq p$.

In what follows we assume that the distinguished normal vector field ξ is parallel with respect to the normal connection, that is, $\nabla_X^\perp \xi = 0$. Hence it follows from (2.24) that $s_{\beta 1} = 0$, $\beta = 2, \dots, p$, and consequently

$$A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2, \dots, p \quad (3.7)$$

because of (3.4)-(3.6).

We now put

$$T := \nabla_U U + \nabla_V V + \nabla_W W + (\operatorname{div} U)U + (\operatorname{div} V)V + (\operatorname{div} W)W$$

and take an orthonormal basis $\{e_i\}_{i=1, \dots, n=4m+3}$ of tangent vectors to M such that

$$e_{m+1} := \phi e_1, \dots, e_{2m} := \phi e_m, \quad e_{2m+1} := \psi e_1, \dots, e_{3m} := \psi e_m,$$

$$e_{3m+1} := \theta e_1, \dots, e_{4m} := \theta e_m, \quad e_{4m+1} := U, \quad e_{4m+2} := V, \quad e_{4m+3} := W.$$

Then it follows from (2.8), (2.9), (2.12) and (2.14)-(2.17) that

$$T = \phi A_1 U + \psi A_1 V + \theta A_1 W, \quad (3.8)$$

$$g(T, U) = g(T, V) = g(T, W) = 0. \quad (3.9)$$

We note that T is a global vector field defined on M . For later use we compute $\operatorname{div}(T) = \sum_{i=1}^n g(e_i, \nabla_{e_i} T)$. Differentiating (3.8) covariantly and using (3.1)-(3.3), we have

$$\nabla_X T = \{u(A_1 U) + v(A_1 V) + w(A_1 W)\} A_1 X \quad (3.10)$$

$$\begin{aligned} & -g(A_1^2 U, X)U - g(A_1^2 V, X)V - g(A_1^2 W, X)W \\ & + \phi A_1 \phi A_1 X + \psi A_1 \psi A_1 X + \theta A_1 \theta A_1 X \\ & + \phi(\nabla_X A_1)U + \psi(\nabla_X A_1)V + \theta(\nabla_X A_1)W, \end{aligned}$$

from which, taking account of (2.8)-(2.10), (2.12) and (2.14)-(2.17),

$$\operatorname{div}(T) = \operatorname{tr} A_1 \{u(A_1 U) + v(A_1 V) + w(A_1 W)\} - u(A_1^2 U) - v(A_1^2 V) - w(A_1^2 W)$$

$$\begin{aligned}
& + \sum_{i=1}^n \{g(\phi A_1 \phi A_1 e_i, e_i) + g(\psi A_1 \psi A_1 e_i, e_i) + g(\theta A_1 \theta A_1 e_i, e_i)\} \\
& - \sum_{i=1}^m \{g((\nabla_{e_i} A_1) \phi e_i - (\nabla_{\phi e_i} A_1) e_i + (\nabla_{\psi e_i} A_1) \theta e_i - (\nabla_{\theta e_i} A_1) \psi e_i, U) \\
& + g((\nabla_{e_i} A_1) \psi e_i - (\nabla_{\psi e_i} A_1) e_i + (\nabla_{\theta e_i} A_1) \phi e_i - (\nabla_{\phi e_i} A_1) \theta e_i, V) \\
& + g((\nabla_{e_i} A_1) \theta e_i - (\nabla_{\theta e_i} A_1) e_i + (\nabla_{\phi e_i} A_1) \psi e_i - (\nabla_{\psi e_i} A_1) \phi e_i, W)\} \\
& - g((\nabla_V A_1) W - (\nabla_W A_1) V, U) - g((\nabla_W A_1) U - (\nabla_U A_1) W, V) \\
& - g((\nabla_U A_1) V - (\nabla_V A_1) U, W),
\end{aligned}$$

or equivalently

$$\operatorname{div}(T) = \operatorname{tr} A_1 \{u(A_1 U) + v(A_1 V) + w(A_1 W)\} \quad (3.11)$$

$$\begin{aligned}
& -u(A_1^2 U) - v(A_1^2 V) - w(A_1^2 W) + \frac{3(n-3)}{4}c \\
& + \sum_{i=1}^n \{g(\phi A_1 \phi A_1 e_i, e_i) + g(\psi A_1 \psi A_1 e_i, e_i) + g(\theta A_1 \theta A_1 e_i, e_i)\}
\end{aligned}$$

because of (2.26) with $s_{\beta 1} = 0$.

On the other hand, by using (2.8)-(2.10), (2.12) and (2.14)-(2.17) we can easily show that

$$\begin{aligned}
& \|\phi A_1 - A_1 \phi\|^2 + \|\psi A_1 - A_1 \psi\|^2 + \|\theta A_1 - A_1 \theta\|^2 \\
& = 6\operatorname{tr} A_1^2 - 2\{u(A_1^2 U) + v(A_1^2 V) + w(A_1^2 W)\} \\
& + 2 \sum_{i=1}^n \{g(\phi A_1 \phi A_1 e_i, e_i) + g(\psi A_1 \psi A_1 e_i, e_i) + g(\theta A_1 \theta A_1 e_i, e_i)\},
\end{aligned} \quad (3.12)$$

from which, combining with (3.11),

$$\begin{aligned}
\operatorname{div}(T) & = \frac{1}{2} \{\|\phi A_1 - A_1 \phi\|^2 + \|\psi A_1 - A_1 \psi\|^2 + \|\theta A_1 - A_1 \theta\|^2\} \\
& + \frac{3(n-3)}{4}c - 3\operatorname{tr} A_1^2 + \operatorname{tr} A_1 \{u(A_1 U) + v(A_1 V) + w(A_1 W)\}.
\end{aligned} \quad (3.13)$$

Finally we introduce some lemmas provided in [9] for later use.

Lemma 3.1 *Let M be an n -dimensional QR -submanifold of $(p-1)$ QR -dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ of constant Q -sectional curvature c and let the distinguished normal vector field ξ be parallel with respect to the normal connection. Then*

$$\|\nabla A_1\|^2 \geq \{3(n-3)/8\}c^2.$$

Lemma 3.2 *Let M be as in Lemma 3.1. If*

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1,$$

then $\|\nabla A_1\|^2 = \{3(n-3)/8\}c^2$ and

$$A_1^2 X = \lambda A_1 X + \frac{c}{4}\{X - u(X)U - v(X)V - w(X)W\}. \quad (3.14)$$

Moreover, in this case

$$A_1 U = \lambda U, \quad A_1 V = \lambda V, \quad A_1 W = \lambda W \quad (3.15)$$

and the function $\lambda = u(A_1 U) = v(A_1 V) = w(A_1 W)$ is locally constant.

Lemma 3.3 *Let M be as in Lemma 3.1 with $c \neq 0$. If*

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1,$$

then

$$A_\alpha\phi + \phi A_\alpha = 0, \quad A_\alpha\psi + \psi A_\alpha = 0, \quad A_\alpha\theta + \theta A_\alpha = 0, \quad (3.16)$$

$$\text{tr} A_\alpha = 0, \quad \alpha = 2, \dots, p. \quad (3.17)$$

Proof. Differentiating the first equation of (3.7) covariantly and using (3.4) and (3.7) itself, we can easily obtain

$$(\nabla_X A_\alpha)U + A_\alpha\phi A_1 X = 0,$$

or equivalently

$$g((\nabla_X A_\alpha)Y, U) + g(A_\alpha\phi A_1 X, Y) = 0 \quad (3.18)$$

for any vector fields X, Y tangent to M . By means of (2.23)_(b), (3.7) and the assumption $\phi A_1 = A_1\phi$, it can be easily verified from (3.18) that

$$(A_\alpha\phi + \phi A_\alpha)A_1 X = 0. \quad (3.19)$$

Inserting $A_1 X$ back into (3.19) and using (3.7), (3.14) and (3.19) itself, we have the first equation of (3.16). Similarly we can get the second and third equations of (3.16). (3.17) can be easily followed from (3.7) and the fact that

$$A_\alpha = \phi A_\alpha \phi, \quad A_\alpha = \phi A_\alpha \phi, \quad A_\alpha = \phi A_\alpha \phi,$$

which is a direct consequence of (2.11), (3.7) and (3.16). $\diamond$

4. Main theorems

In order to prove our main theorem, we need the following theorem ([9, Theorem 5.2, p.114]) which is a direct consequence of Lemma 3.2 :

Theorem K-P. *Let M be an n -dimensional QR-submanifold of $(p-1)$ QR-dimension in $QP^{(n+p)/4}$. Suppose*

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1$$

and that the distinguished normal vector field N is parallel with respect to the normal connection. Then M is locally isometric to

$$M_{n_1, n_2}^Q = \pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2))$$

for some portion (n_1, n_2) of $(n-3)/4$ and some r_1, r_2 such that $r_1^2 + r_2^2 = 1$, where $S^{4n_i+3}(r_i)$ denotes $(4n_i+3)$ -dimensional sphere with radius r_i and π is the Hopf-fibration $S^3 \rightarrow S^{n+p+3} \rightarrow QP^{(n+p)/4}$.

From now on we assume that the ambient manifold is a quaternionic projective space $QP^{(n+p)/4}$ of constant Q -sectional curvature 4. Suppose that the distinguished normal vector field ξ is parallel with respect to the normal connection and that the trace of the shape operator A_1 corresponding to ξ vanishes, that is,

$$\text{tr} A_1 = 0. \quad (4.1)$$

Then, from (2.23)_(a) with $c = 4$ and $s_{1\beta} = 0$, we have

$$\sum (\nabla_i A_1) e_i = 0, \quad (4.2)$$

where $\{e_i\}_{i=1, \dots, n}$ is the orthonormal basis given in §3 of tangent vectors to M and $\nabla_i := \nabla_{e_i}$.

Using (3.1)-(3.6) and (4.2), we can easily obtain

$$\begin{aligned} \sum (\nabla_i \nabla_i A_1) X &= \sum (R(e_i, X) A_1) e_i - 3\{g(A_1 U, X) U + g(A_1 V, X) V \\ &\quad + g(A_1 W, X) W - \phi A_1 \phi X - \psi A_1 \psi X - \theta A_1 \theta X\} \end{aligned}$$

for any vector X tangent to M . Hence we have

$$\begin{aligned} g(\nabla^2 A_1, A_1) &= \sum_{i,j} g((R(e_i, e_j) A_1) e_i, A_1 e_j) - 3\{u(A_1^2 U) + v(A_1^2 V) + w(A_1^2 W)\} \\ &\quad + 3 \sum \{g(A_1 \phi A_1 \phi e_i, e_i) + g(A_1 \psi A_1 \psi e_i, e_i) + g(A_1 \theta A_1 \theta e_i, e_i)\}. \end{aligned} \quad (4.3)$$

Combining those results and Lemma 3.3, we have

Theorem 4.1 *Let M be an n -dimensional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and let*

the distinguished normal vector field ξ be parallel with respect to the normal connection. Suppose that the trace of the shape operator A_1 in the direction of ξ vanishes and that the minimum of sectional curvature of M is $3/n$. Then

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1 \quad (4.4)$$

at any point in M . Hence M is minimal and

$$A_\alpha\phi + \phi A_\alpha = 0, \quad A_\alpha\psi + \psi A_\alpha = 0, \quad A_\alpha\theta + \theta A_\alpha = 0 \quad (4.5)$$

at every point in M .

Proof. Taking account of the Laplacian of $tr A_1^2$, we have

$$\int_M \|\nabla A_1\|^2 * 1 = - \int_M g(\nabla^2 A_1, A_1) * 1. \quad (4.6)$$

On the other hand, it follows from Lemma 3.1 that

$$0 \leq \int_M \{ \|\nabla A_1\|^2 - 6(n-3) + \frac{1}{2}(\|\phi A_1 - A_1\phi\|^2 + \|\psi A_1 - A_1\psi\|^2 + \|\theta A_1 - A_1\theta\|^2) \} * 1.$$

Moreover, (4.3) and (4.6) yield

$$\begin{aligned} & \int_M [\|\nabla A_1\|^2 - 6(n-3) + \frac{1}{2}(\|\phi A_1 - A_1\phi\|^2 + \|\psi A_1 - A_1\psi\|^2 + \|\theta A_1 - A_1\theta\|^2)] * 1 \\ &= \int_M [\frac{1}{2}(\|\phi A_1 - A_1\phi\|^2 + \|\psi A_1 - A_1\psi\|^2 + \|\theta A_1 - A_1\theta\|^2) - 6(n-3) \\ &\quad - 3 \sum \{ g(A_1\phi A_1\phi e_i, e_i) + g(A_1\psi A_1\psi e_i, e_i) + g(A_1\theta A_1\theta e_i, e_i) \} \\ &\quad + 3 \{ u(A_1^2 U) + v(A_1^2 V) + w(A_1^2 W) \} - \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1 e_j)] * 1, \end{aligned}$$

from which together with (3.11) and (3.12) with $c = 4$, we obtain

$$\begin{aligned} 0 &\leq \int_M \{ \|\nabla A_1\|^2 - 6(n-3) + \frac{1}{2}(\|\phi A_1 - A_1\phi\|^2 + \|\psi A_1 - A_1\psi\|^2 \\ &\quad + \|\theta A_1 - A_1\theta\|^2) \} * 1 = \int_M \{ 3tr A_1^2 - \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1 e_j) \} * 1. \quad (4.7) \end{aligned}$$

Now we choose an orthonormal frame $\{e_j\}$ of M such that

$$A_1 e_j = \lambda_j e_j \quad (j = 1, \dots, n).$$

Then it is clear that

$$\sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1 e_j) = \sum_{i,j} \{ g((R(e_i, e_j)A_1)e_i, A_1 e_j) - g(A_1 R(e_i, e_j)e_i, A_1 e_j) \}$$

$$= \frac{1}{2} \sum_{i,j} (\lambda_i - \lambda_j)^2 K_{ij},$$

where K_{ij} denotes the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Hence, if the minimum of sectional curvatures of M is $3/n$, the above equation implies

$$\sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j) \geq \frac{3}{2n} \sum_{i,j} (\lambda_i - \lambda_j)^2 = 3tr A_1^2. \quad (4.8)$$

Combining (4.7) and (4.8), we have (4.4), which and Lemma 3.3 imply (4.5) and the minimality of M . $\diamond$

By means of Theorem 4.1 we can prove the following theorem under additional condition:

Theorem 4.2. *Let M be as in Theorem 4.1 and assume that there exists an orthonormal basis $\{\xi, \xi_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If the trace of the shape operator A_1 in the direction of ξ vanishes and if the minimum of sectional curvatures of M is $3/n$, then there exists a totally geodesic quaternionic projective space $QP^{(n+1)/4}$ of $QP^{(n+p)/4}$ such that $M \subset QP^{(n+1)/4}$.*

Proof. Under our assumptions it follows from Theorem 4.1 that

$$tr A_\alpha = 0$$

for any $2 \leq \alpha \leq p$. Moreover, it is clear from (2.23)_(b) that, for any vector fields X, Y tangent to M ,

$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = 0$$

since $s_{\alpha\beta} = 0$, $2 \leq \alpha, \beta \leq p$, and consequently

$$\sum (\nabla_i A_\alpha) e_i = 0,$$

where $\{e_i\}_{i=1,\dots,n}$ is the orthonormal basis given in §3 of tangent vectors to M . Hence we have

$$\sum (\nabla_i \nabla_i A_\alpha) X = \sum (R(e_i, X) A_\alpha) e_i$$

for any vector field X tangent to M , and so

$$g(\nabla^2 A_\alpha, A_\alpha) = \sum_{i,j} g((R(e_i, e_j) A_\alpha) e_i, A_\alpha e_j).$$

Taking account of the Laplacian of $tr A_\alpha^2$, we have

$$\int_M \|\nabla A_\alpha\|^2 * 1 = - \int_M g(\nabla^2 A_\alpha, A_\alpha) * 1,$$

which implies

$$0 \leq \int_M \|\nabla A_\alpha\|^2 * 1 = - \int_M \sum_{i,j} g((R(e_i, e_j)A_\alpha)e_i, A_\alpha e_j) * 1. \quad (4.9)$$

As shown in the proof of Theorem 4.1, we choose an orthonormal frame $\{e_j\}$ of M such that

$$A_\alpha e_j = \mu_j e_j \quad (j = 1, \dots, n).$$

Then we have

$$\sum_{i,j} g((R(e_i, e_j)A_\alpha)e_i, A_\alpha e_j) = \frac{1}{2} \sum_{i,j} (\mu_i - \mu_j)^2 K_{ij}.$$

Hence, if the minimum of sectional curvatures of M is $3/n$, the above equation and (4.9) yield

$$\nabla_X A_\alpha = 0, \quad \alpha = 2, \dots, p \quad (4.10)$$

for any vector field X tangent to M .

On the other hand, differentiating the first equation of (4.5) covariantly and using (3.1), (4.5) itself and (4.10), we have

$$u^1(X)A_\alpha A_1 Y - g(A_1 A_\alpha X, Y)U_1 = 0$$

for any vector fields X, Y tangent to M . Putting $X = U$ in this equation and using (2.10) and (3.7), we have

$$A_\alpha A_1 Y = 0,$$

from which, inserting $A_1 Y$ back into and taking account of (3.14) with $c = 4$,

$$A_\alpha = 0, \quad \alpha = 2, \dots, p.$$

Now we put $N_0(x) := \{\eta \in T_x^\perp M \mid A_\eta = 0\}$ and let $Q_0(x)$ be the maximal quaternionic invariant subspace of $N_0(x)$, that is, $Q_0(x) = FN_0(x) \cap GN_0(x) \cap HN_0(x) \cap N_0(x)$. Then the orthogonal complement $Q_1(x)$ of $Q_0(x)$ in $T_x^\perp M$ is $\text{Span}\{xi\}$, which is invariant under parallel translation with respect to the normal connection under our assumption. Therefore we can apply the codimension reduction theorem [8, Theorem 3.4, p. 115], which completes the proof of our theorem. $\diamond$

In S^{n+4} ($n = 4m + 3$ for some integer m) we have the family of generalized Clifford surfaces whose spheres lie in quaternionic subspaces (cf. [10]) :

$$M_{4n_1+3, 4n_2+3} = S^{4n_1+3}(((4n_1+3)/(n+3))^{\frac{1}{2}}) \times S^{4n_2+3}(((4n_2+3)/(n+3))^{\frac{1}{2}}),$$

where $4(n_1 + n_2) = n - 3$. Then we have $\pi(M_{4n_1+3,4n_2+3})$, which will be also denoted by M_{n_1,n_2}^Q , as model spaces of minimal real hypersurfaces of $QP^{(n+1)/4}$ satisfying the assumptions stated in Theorem 4.1. In the special case $n_1 = 0$, $M_{0,(n-3)/4}^Q$ is called a *geodesic minimal hypersphere* of $QP^{(n+1)/4}$, and is a homogeneous, positively curved manifold diffeomorphic to the sphere (for details, see [2,10,11]). Moreover, by means of Lemma 3.2 we can also see that the minimum of the sectional curvature of $M_{0,(n-3)/4}^Q$ is $3/n$ and that of M_{n_1,n_2}^Q ($n_1, n_2 \geq 1$) is zero.

Combining Theorem 4.2 and Theorem K-P, we have

Theorem 4.3. *Let M be as in Theorem 4.1 and assume that there exists an orthonormal basis $\{\xi, \xi_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If the trace of the shape operator A_1 in the direction of ξ vanishes and if the minimum of sectional curvatures of M is $3/n$, then M is isometric to the geodesic minimal hypersphere $M_{0,(n-3)/4}^Q$ of $QP^{(n+1)/4}$.*

Corollary 4.4 ([4]) *Let M be a compact, minimal real hypersurface of a quaternionic projective space $QP^{(n+1)/4}$. If the minimum of sectional curvatures of M is $3/n$, then M is isometric to the geodesic minimal hypersphere $M_{0,(n-3)/4}^Q$.*

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